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Short Note

A comment on the computation of non-conservative products

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ABSTRACT

We are interested in the solution of non-conservative hyperbolic systems, and consider in particular the so-called path-conservative schemes (see e.g. [2,3]) which rely on the theoretical work in [1]. The example of the standard Euler equations for a perfect gas is used to illuminate some computational issues and shortcomings of this approach.

1. Introduction

Non-conservative hyperbolic systems arise in a wide range of applications, which makes their theoretical study and numerical approximation a very important topic. We are interested in the numerical solution of non-conservative hyperbolic systems

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0, \tag{1}$$

subject to initial conditions. Here $U \in \Omega \subset \mathbb{R}^p$.

The challenge here is twofold: first, to generalize the notion of weak solutions to the case where the underlying hyperbolic system is not in conservation form, by giving an acceptable definition of shock waves. Second, once the theoretical framework has been established, to compute solutions to those systems. The difficulty lies in the fact that while for conservative systems, shock relations depend on the solution states to the immediate left/right of the shock, in the non-conservative case they depend not only on those states but also on the path that connects them

$$\sigma(U_R - U_L) = \int_{U_L}^{U_R} A(U)\, dU,$$

here σ denotes the shock speed, and $U_{L,R}$ the left/right states. In the conservative case, $A(U)$ is the Jacobian of a flux function $F(U)$ and the above relation recovers the standard Rankine–Hugoniot relations.

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There has been many contributions on the topic during recent years. From the theoretical point of view, most notably the work of Dal Maso et al. [1] where the notion of path is introduced to define generalized shock relations: given a family of paths, i.e. C^1 function from $[0,1]\times\Omega^2\mapsto\Omega$

$$(x,u,v)\mapsto\Phi(z;u,v),$$

such that $\Phi(0,u,v)=u$ and $\Phi(1;u,v)=v$. and given two states U_L and U_R, the two states are said to define a shock moving at speed σ if

$$\sigma(U_L-U_R)=\int_0^1 A(\Phi(s,;U_L,U_R))\frac{\partial\Phi(s;U_L,U_R)}{\partial s}ds. \tag{2}$$

With this relation, a mathematical theory for weak solutions for hyperbolic systems in non-conservation form is developed, and the solution to the Riemann problem may be constructed [1].

Once the Rankine–Hugoniot relations have been encoded by (2), a numerical approximation of (1), called path-conservative, have been proposed by Parés and collaborators (see for example [2,3]). Given $\Delta x>0$ and the mesh $\{x_j\}_{j\in\mathbb{Z}}$ with $x_j=j\Delta x$, the path-conservative schemes are defined as schemes of the form

$$U_j^{n+1}=U_j^n-\frac{\Delta t}{\Delta x}\Big(D^-_{j+1/2}(U_j^n,U_{j+1}^n)+D^+_{j-1/2}(U_{j-1}^n,U_j^n)\Big) \tag{3}$$

where the residuals $D^{\pm}_{j+1/2}(U_j^n,U_{j+1}^n)$ satisfy the conservation relation

$$D^-_{j+1/2}(U_j^n,U_{j+1}^n)+D^+_{j+1/2}(U_j^n,U_{j+1}^n)=\int_0^1 A\big(\Phi(s,;U_j,U_{j+1})\big)\frac{\partial\Phi(s;U_L,U_R)}{\partial s}ds. \tag{4}$$

In [2], the theoretical framework for path-conservative schemes was presented, and a Lax-Wendroff theorem was conjectured: if the numerical solution $(U_j^n)_{j,n}$ obtained by a path-conservative scheme converges, its limit is the weak solution in the sense defined by the theory of [1]. A more recent paper [3] found that in fact those schemes generate convergence error source-term which is supported on shock trajectories and that the error measure is usually 'small'. The paper presented a thorough numerical investigation to evaluate the range of validity of certain schemes.

Several questions seem legitimate: (i) how does one go about choosing a path; (ii) what influence does the choice of path and discretization scheme have on the computed solution; (iii) once a path is specified and a consistent path-conservative scheme designed, does the numerical solution converge to the assumed path; and (iv) in cases where the correct jump conditions are known unambiguously, can a path-conservative scheme be designed so that it converges to the correct solution?

The answers to these questions are in general quite difficult, so we proceed by considering the following illuminating example.

2. A simple example

Consider the Euler equations of fluid dynamics in Lagrangian coordinates

$$\begin{aligned} &v_t-u_m=0,\\ &u_t+p_m=0,\\ &e_t+(pu)_m=0. \end{aligned} \tag{5}$$

Here, v is the specific volume, u the velocity, e the specific total energy and p is the pressure. We also use $e=\varepsilon+u^2/2$, where ε denotes the specific internal energy. The equation of state for perfect gas is given by

$$\varepsilon=(\gamma-1)pv$$

with γ the specific heat ratio, taken to be 1.4 in the numerical experiments.

We also write (5) in a non-conservative manner, in terms of (v,u,ε)

$$\begin{aligned} &v_t-u_m=0,\\ &u_t+p_m=0,\\ &\varepsilon_t+pu_m=0. \end{aligned} \tag{6}$$

We observe that (6) is 'minimally' non-conservative in that it only has *one* non-conservative product, and otherwise has a conservative sub-system in the first two equations. We consider both systems (5) and (6) and try to shed light on the questions raised in the previous section. Of course, in this case, the correct shock relations are given by the conservative system (5). In designing numerical approximations to solve (6), the ultimate task is to compute solutions of (6) which recover the shock relations of (5).

2.1. A numerical example

We write (6) in quasi-linear form

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ u \\ \varepsilon \end{pmatrix} + A\frac{\partial}{\partial m}\begin{pmatrix} v \\ u \\ \varepsilon \end{pmatrix} = 0,$$

where A is given by

$$A = \begin{pmatrix} 0 & -1 & 0 \\ p_v & 0 & p_\varepsilon \\ 0 & p & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -p/v & 0 & (\gamma-1)/v \\ 0 & p & 0 \end{pmatrix}.$$

In this case, the shock relations defined by the theory in [1] are

$$\sigma(U_R - U_L) = \int_0^1 A(\Phi(s,;U_L,U_R))\frac{\partial \Phi(s;U_L,U_R)}{\partial s}ds = \begin{pmatrix} -\Delta v \\ \Delta u \\ \int_0^1 p\frac{\partial u(s)}{\partial s}ds \end{pmatrix}. \tag{7}$$

We define a path which is linear in in v, u and p

$$v(s) = sv_L + (1-s)v_R, \qquad u(s) = su_L + (1-s)u_R, \qquad p(s) = p_L + (1-s)p_R,$$

and note that for this path (i) $\frac{\partial u(s)}{\partial s} = \Delta u$ and (ii) $p(s)$ is linear over the path and its integral can be evaluated exactly. Here we have used the standard notation $\Delta f = f_L - f_R$ where f is any of $u,\ v,\ p,\ \varepsilon$. In this case, the non-conservative product may be integrated *exactly*

$$\int_0^1 p\frac{\partial u(s)}{\partial s}ds = \Delta u\int_0^1 p(s)ds = \overline{p}\Delta u$$

with $\overline{p} = (p_L + p_R)/2$.

We further note that the shock conditions for this choice of path, according to (2), are

$$\begin{aligned} &\sigma\Delta v + \Delta u = 0, \\ &\sigma\Delta u - \Delta p = 0, \\ &\sigma\Delta\varepsilon - \bar{p}\Delta u = 0, \end{aligned} \tag{8}$$

Which are, in fact, the *exact (!)* shock relations for the gas dynamics equations. The above choice of path, linear in v, u and p therefore reproduces the *exact* shock relations. In that sense, this is a correct choice of path and at least the ambiguity of how

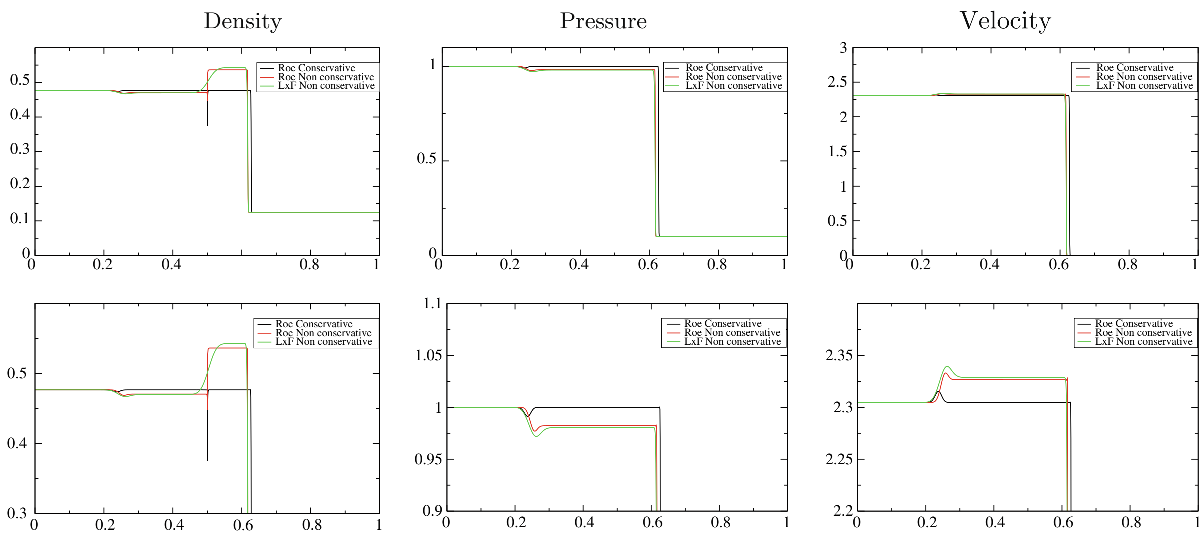

Fig. 1. Computed solutions by Roe-type and Lax-Friedrich (LxF)-type path-conservative schemes. and by Roe's scheme for the conservative system (5). Bottom line is a zoom of the top.

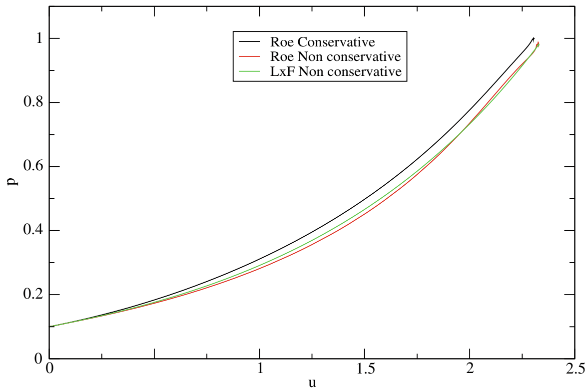

Fig. 2. Numerical viscous path in $u-p$ plane, corresponding to Roe-type and LxF-Type path-conservative schemes, and to Roe's scheme for the conservative system (5).

Table 1
Exact and computed pressure behind a shock by a Roe-type path-conservative scheme. The right state is $(\rho, u, p) = (0.125, 0, 0.1)$, the left state is controlled by p_e. The relative error is given by $|(p_e - p_c)/p_e|$.

p_e	p_c	Rel. error
0.2	0.1997547	0.00122
0.3	0.2988168	0.00394
0.4	0.397263	0.00684
0.5	0.4953561	0.00928
0.6	0.5931094	0.01143
0.7	0.6906313	0.01338
0.8	0.7879680	0.01504
0.9	0.8851661	0.01648

to choose a path is resolved here. We turn to the next question: does the corresponding path-conservative scheme converge to the correct solution?

We have used a Roe-type path-conservative scheme [2], here the eigenvalues of $\overline{A}$ are 0 and $\pm\overline{C}$ with $\overline{C}^2 = \gamma\bar{p}/\bar{v}$ and the Roe averages are [5] $\bar{v} = \frac{v_L+v_R}{2}, \ \bar{u} = \frac{u_L+u_R}{2}, \ \bar{p} = \frac{p_L+p_R}{2}$.

2.2. Numerical test

We have conducted a series of runs with initial data corresponding to right moving shocks of various strengths. The computation was done using 1,500 grid cells, and to the best of our judgement, the results are converged. The right state in all tests is $U_R = (v, u, p)_R = (8, 0, 0.1)$ corresponding to the right state of the standard Sod's shock tube problem. The left state is chosen to yield a single right running shock, and the parameter controlling the strength of the shock is p_L (or Δp). In the first example, shown in Fig. 1 the left state was chosen so that $p_L = 1.0$. Results by a Roe-type and a Lax-Friedrichs (LxF)-type path-conservative schemes are shown in Fig. 1. It is clear from the figure that numerical solutions by the path-conservative schemes do not recover the correct solution, here computed by a Roe-type scheme of the conservative formulation (5) and shown by a black curve. This is despite the fact that the chosen path in this example is correct, at least in the sense that it reproduces the correct jump conditions. The fact that the system has a conservative sub-system appears to be of little help here. We also note that solutions obtained by Roe's and by LxF path-conservative scheme converge to close but different solutions, despite the fact that they are both based on the same path definition.

We ask further whether the computed solutions converge to the assumed path that underlies the scheme, in this case linear in v, u and p. Fig. 2 shows the numerical viscous path corresponding to the various schemes. It is clear that there is little connection between the viscous path towards which the computed solution converges, and the assumed path, which in this case would be represented as a straight line in the $u-p$ plane. This finding is particularly disheartening. In computations of non-conservative products, the discussion often revolves around the question of what is the correct choice of path, and usually proceeds by saying 'suppose we know what the correct path is, here is what we do'. The present computation shows that even if we knew how to define the path correctly, the numerical solution *will not*, in general, converge to the assumed path. Rather the computation is dominated by viscous terms that arise due to truncation errors, and those have little to do with the assumed path. While the same is true for the computed solution for the conservative system (5), in the conservative case, the numerical method does not rely on getting the path right because of the conservative nature of the underlying system and scheme.

Finally, we repeated the computation for data corresponding to right moving shocks of increasing strengths, and measured the resulting relative errors (in pressure). Results are summarized in Table 1. It is clear that the error is inherent to the scheme, and increases with the strength of the shock. This behaviour is shared by other schemes for non-conservative systems (see for example [4], where errors are of similar size to those in Table 1).

3. Concluding remarks

Our study shows that the so-called path-conservative schemes are not, in general, able to compute correctly the solution of non-conservative hyperbolic problems. The difficulty goes beyond determining what is the correct path. Even if the correct path is assumed to be known, it is in general not possible to design a scheme that converges to the assumed path. The fact that the problem may have a conservative sub-system is of little help. These points were illustrated using the Euler equations in Lagrangian form, where the choice of linear path in fact gives the correct jump conditions, but the computed solution is not able to recover them.

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