A residual distribution method using discontinuous elements for the computation of possibly non smooth flows.

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Abstract

In this paper, we describe a residual distribution (RD) method where, contrarily to "standard" this type schemes, the mesh is not necessarily conformal. It also enable to use discontinuous elements, contrarily to the "standard" case where continuous elements are requested. More over, if continuity is forced, the scheme becomes similar to the standard RD case. Hence, the situation becomes comparable with the Discontinuous Galerkin (DG) method, but it is simpler to implement than DG and has guaranteed L^∞ bounds. We focus on the second order case, but the method can be easily generalized to higher degree polynomials.

1 Introduction

This paper is devoted to the design of an approximation method for steady hyperbolic problems by means of a scheme which enjoys the most possible compact stencil. There exist many similar methods, for example the Discontinuous Galerkin method, or the continuous Residual Distribution schemes. In the first case, the solution is represented in each element of the mesh by polynomial functions where no continuity is enforced at the element boundaries. Hence, the method is very flexible since the mesh does not need to be conformal, nor the polynomial degree be the same in each element. Other approximation techniques than local polynomial representations can be chosen In our opinion, one of its disadvantages is its complexity, especially when one considers mixed hyperbolic/elliptic problems such as the Navier Stokes equations. Moreover, and this is the point we are interested in here, when discontinuous solutions are computed, the non linear non oscillatory stabilization mechanisms are not completely satisfactory because they depend on parameters or are quite complex to design, see [1, 2, 3, 4] for example. Either they are very complex to set up, or they introduce too much dissipation.

In the case of the residual distribution (RD) methods, the solution is also approximated by piecewise polynomial functions, but here the approximation is globally continuous. Hence, the algorithmic complexity is lower (in term of memory especially). Another property is that there exists a very general and systematic method that enables us to guaranty accuracy formal $O(h^{k+1})$ accuracy, even at local extrema, and L^∞ stability. However, the mesh must be conformal, see [5, 6, 7] among several others.

In this note, we describe a residual distribution method where the functional representation does not need to be continuous across edges. The method is general and could be extended to any order of accuracy, following the lines of [8], but here, we have only developed it for a local P^1 interpolation in each element to present the ideas. Contrary to the "classical" RD schemes, the continuity across edges is no longer enforced. This method is simpler than the one described in [9]. Indeed, the scheme reduces to the one of [5, 10] and [6] if continuity is enforced across edges. Compared with standard DG methods, the scheme non oscillatory properties are obtained *without* any parameter.

The paper is organized as follows. We first describe the method for a scalar problem. Then the method is extended to the Euler equations for fluid dynamics. The extension to 3D is straightforward as well as on non conformal meshes. This paper opens the road for $h-p$ adaptation for RD schemes.

This paper is a translation of a 2007 report written in french, [11], with some improvements. In the meantime, M. Hubbard [12] has published a similar technique. However, the similarity starts and ends in that we both use discontinuous elements. Hubbard then develops his method using an extension of the N scheme. We have used Lax Friedrichs method, but following [13], any standard finite volume scheme can be rewritten as a RD scheme, and hence can be plugged into our framework. The method is also much simpler than [9].

2 The scalar case

Let us consider the following problem, defined in $\Omega \subset \mathbb{R}^2$ to make the presentation simpler

$$\begin{array}{ll} \text{div } f(u) = 0 & \text{if } x \in \Omega \\ u = g & \text{if } x \in \Gamma^-, \end{array} \tag{1}$$

Γ^- is the inflow boundary

$$\Gamma^- = \{x \in \partial\Omega \text{ such that } \nabla_u f \cdot \vec{n}(x) < 0\}$$

and $\vec{n}(x)$ is the outward unit normal $x \in \partial\Omega$.

In a first step, we consider a conformal triangulation of Ω using triangles. We explain the method, and in a second step, we show how to generalize it to non conformal triangulations and for non triangular meshes. The 3D case can be dealt with in a similar way.

Let us denote by K a generic element of $\mathcal{T}_h$. The real number h represents the maximum of the diameters of the elements of $\mathcal{T}_h$.

In K, we say that the degrees of freedom are located at the vertices, and we represent the approximated solution in K by the degree one interpolant polynomial at the vertices of K. Let us denote by u^h this piecewise linear approximation, that is in principle discontinuous at across edges. In the following, we use the notations described in Figure 1.

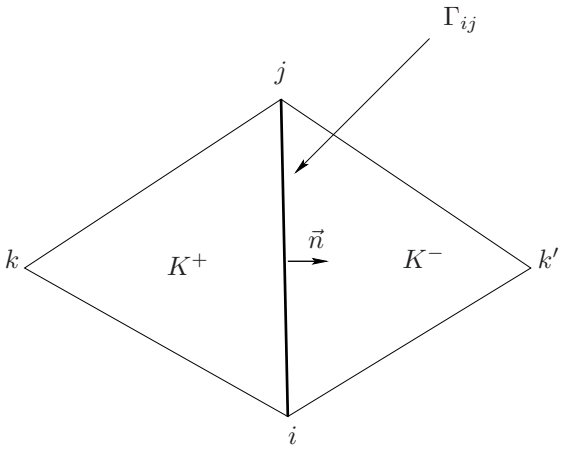


Figure 1: Geometrical elements for defining the scheme.

In [9], the degrees of freedom are located at the midpoint of the edges that connect the centroid of K and its vertices. This choice was motivated by the fact that the P^1 basis functions associated to these nodes are orthogonal in $L^2(K)$. This property enables us to reinterpret the DG schemes as RD schemes, and hence to adapt the stabilization techniques of RD to DG. In particular, we are able to enforce a L^∞ stability property. However, this method was a bit complex, and it is not straightforward to generalize it to more general elements than triangles.

The geometrical idea behind the new version of the method is to forget the RD interpretation of the DG scheme and to let the geometrical localization of the degrees of freedom move to the vertices of the element.

With this in mind, we define two types of total residuals:

- A total residual per element K
$$\Phi^K = \int_{\partial K} f(u)\cdot\vec{n}dl$$
which is evaluated thanks to a quadrature formula,

- A total residual per edge Γ, i.e.
$$\Phi_\Gamma = \int_\Gamma \big[f(u)\cdot\vec{n}\big]dl$$
where $[f(u)\cdot\vec{n}]$ represents the jump of the function $f(u)\cdot\vec{n}$ across Γ. Here, if $\vec{n}$ is the outward unit normal to K (see figure 1), this enables us to define a right side and a left side. Hence we set
$$[f(u)\cdot\vec{n}] = (f(u_R) - f(u_L))\cdot\vec{n}.$$
We notice that Φ_Γ only depends on the values of u on each side of Γ.

The idea is to split the total residuals into sub-residuals so that a monotonicity preserving scheme can be defined. Here, we choose the (local) Lax–Friedrichs scheme, but other choices could be possible, see [13] for rephrasing finite volumes into the RD framework. Thus we consider

- For the element K which vertices are $\{i,j,k\}$ and $l\in\{i,j,k\}$,
$$\Phi_l^K = \frac{\Phi^K}{3} + \alpha_K(u_l - \overline{u}) \tag{2a}$$
with
$$\overline{u} = \frac{u_i+u_j+u_k}{3},$$
and $\alpha_K \geq \max_{x\in K}||f'(u^h(x))||$ where $||\ .\ ||$ is any norm in $\mathbb{R}^2$, for example the Euclidian norm.

- and for the edge Γ,
$$\Phi_l^\Gamma = \frac{\Phi^\Gamma}{4} + \alpha_\Gamma(u_l - \overline{u}) \tag{2b}$$
with
$$\overline{u} = \frac{u_i+u_j+u_k+u_p}{3}$$
where u_i,u_j,u_k,u_p are the values on each side of Γ and $\alpha_\Gamma \geq \max_{K=K^+,K^-}\max_{x\in\partial K\cup\Gamma}||f'(u^h(x))||$, see Figure 1 for a definition of $K^\pm$.

We have the following conservation relations
$$\begin{aligned}\sum_{i\in K}\Phi_i^K &= \Phi^K,\\ \sum_{i\in\Gamma}\Phi_i^\Gamma &= \Phi^\Gamma\end{aligned} \tag{3}$$

The choice $\alpha_K \geq \max_{x\in K}||f'(u^h(x))||$ and $\alpha_\Gamma \geq \max_{K=K^+,K^-}\max_{x\in\partial K\cup\Gamma}||f'(u^h(x))||$ are justified by the following standard argument. If we set $Q=K$ or Γ, we can rewrite the two residuals as
$$\Phi_l^Q = \sum_{j\in Q} c_{ij}^Q(u_i - u_j)$$
with $c_{ij}^Q \geq 0$ under the above mentioned conditions. Indeed, if we introduce the value $\overline{u}$ that appears in the formulas, we get (for $Q=K$ for example)
$$\begin{aligned}\Phi_l^K &= \frac{\Phi^K}{3} + \alpha_K(u_l-\overline{u})\\ &= \frac{1}{3}\int_{\partial K}\big(f(u)-f(\bar{u})\big)\cdot\vec{n}dl + \alpha_K(u_l-\overline{u})\\ &= \sum_{j\in K}\frac{1}{3}\left(\int_{\partial K}\left(\int_0^1 f'(su+(1-s)\bar{u})ds\cdot\vec{n}dl\right)ds - \alpha_K\right)(u_i-u_j)\end{aligned}$$

which proves the result.

We get a first order scheme by determining u^h the solution of: find u^h linear in each triangle K such that for any degree of freedom i (i.e. vertex of the triangulation),

$$\sum_{K,i\in K} \Phi_i^K + \sum_{\Gamma,i\in\Gamma} \Phi_i^\Gamma = 0. \tag{4}$$

We specify later the boundary conditions.

Using standard arguments, as defining u^h as the limit of the solution of

$$u_i^{n+1} = u_i^n - \omega_i \left(\sum_{K,i\in K} \Phi_i^K + \sum_{\Gamma,i\in\Gamma} \Phi_i^\Gamma \right)$$

with

$$\omega_i \left(\sum_{K,i\in K} c_{ij}^K + \sum_{\Gamma,i\in\Gamma} c_{ij}^\Gamma \right) \leq 1,$$

we see that we have a maximum principle.

It is possible to construct a scheme that is formally second order accurate by setting

$$\Phi_i^{K,\star} = \beta_i^K \Phi^K \text{ and } \Phi_i^{\Gamma,\star} = \beta_i^\Gamma \Phi^K \tag{5}$$

with, setting

$$x_i^K = \frac{\Phi_i^K}{\Phi^K}, \qquad x_i^\Gamma = \frac{\Phi_i^\Gamma}{\Phi^\Gamma},$$

and

$$\beta_i^K = \frac{\max(x_i^K, 0)}{\sum\limits_{j\in K} \max(x_j^K, 0)}, \qquad \beta_i^\Gamma = \frac{\max(x_i^\Gamma, 0)}{\sum\limits_{j\in K} \max(x_j^\Gamma, 0)}. \tag{6}$$

As in the "classical" RD framework, the coefficients β are well defined thanks to the conservation relations (3). The scheme writes as (4) where the residuals Φ_i^K (resp. Φ_i^Γ) are replaced by $\Phi_i^{K,\star}$ (resp. $\Phi_i^{\Gamma,\star}$.

Boundary conditions. If Γ is an inflow boundary edge, we need to set weakly the boundary condition $u = g$. Consider a numerical flux, say an upwind flux, denoted by $\mathcal{F}(u^h, g, \vec{n}(x))$. We consider the boundary residual

$$\Phi^\Gamma = \int_\Gamma (\mathcal{F}(u^h, g, \vec{n}(x)) - f(u^h)\cdot\vec{n}) dl$$

that we split into two parts following the same procedure as above. If l and l' are the two vertices of Γ, we have defined Φ_l^Γ and $\Phi_{l'}^\Gamma$, and

$$\Phi_l^\Gamma + \Phi_{l'}^\Gamma = \Phi^\Gamma.$$

The scheme, when we take into account the boundary conditions, is again (4) where the list of edges takes into account the boundary edges, if needed.

Conservation and accuracy issues. In [9], we have shown that a scheme of the type (4) where the residual satisfies the conservation constraints (3) (including on the boundary) and standard stability assumptions (as in the Lax Wendroff theorem) is convergent and the limit solution is a weak solution of the PDE (1).

The accuracy constraint (5) and (6) are also analyzed in the same reference [9]. In that case, the assumption that the problem is steady is essential in showing that the residuals (including the boundary residuals) satisfies

$$\Phi^Q(u^h) = O(h^{d+1})$$

where u^h is the interpolant of the exact solution (assuming it is smooth) and d is the dimension of Q: $d = 2$ for a triangle and $d = 1$ for an edge.

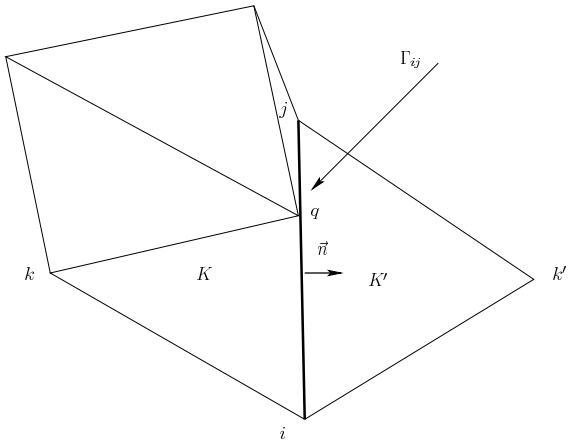


Figure 2: Example of a non conformal mesh.

Extension to non conformal meshes. In what follows, we refer to figure 2. The scheme (4) stays the same. Only the evaluation of the the residuals need to be precised. The total residuals per elements or edges remain identical. By edge Γ, we mean an edge seen from a given element. In the case of figure 2, for the element K, we get the edge $[i, q]$, while for K', we take $[i, j]$. Notice that the degree of freedom q, again referring to 2, is active of each of the elements of the figure except K': this explains the definition we have taken for edges. The rest is identical.

3 Application to the scalar case.

We test the scheme on a standard benchmark: the Burgers equations which is non linear. The Burgers problem is

$$\begin{aligned} &\tfrac{1}{2}\frac{\partial u^2}{\partial x} + \frac{\partial u}{\partial y} = 0 && (x,y) \in]0,1[^2 \\ &u(x,y) = \begin{cases} 1-2x & x \in [0,1], y = 0 \\ 1.5 & x = 0, y \in [0,1] \\ -0.5 & x = 1, y \in [0,1]. \end{cases} \end{aligned} \tag{7}$$

One of the problem we had to deal with is the visualization of the results. We have used a software that is only able to represent point value data ones, not cell centered data as here. Hence to transform our data into cell centered data, we had to compute, for any vertex M_i,

$$u_i = \frac{\sum\limits_{K, M_i \in K} u_i^K}{\sum\limits_{K, M_i \in K} 1}$$

where u_i^K represents the value at M_i when M_i is seen as belonging to K.

The figure 3 represents the isolines obtained for r (7). The results are non oscillatory and similar to those obtained by other methods, for example in [9, 6].

4 Extension to the Euler equations

For a system, the scheme remains formally identical: we can rephrase word to word the definitions of the total residuals, as well as that of the Lax Friedrichs scheme. The parameter α in (2) becomes

$$\max_{||n||=1} \rho\big(An_x + Bb_y\big)$$

where A (resp. B) is the x- (resp. y-) Jacobian of the Euler flux and $\rho(M)$ is the spectral radius of the matrix M. In the system case, the scheme is formally identical. The only difference is in the definition of the "limited" residuals (5) and (6), i.e. in the definition of the matrices β_i^K and β_i^Γ needs to be precised.

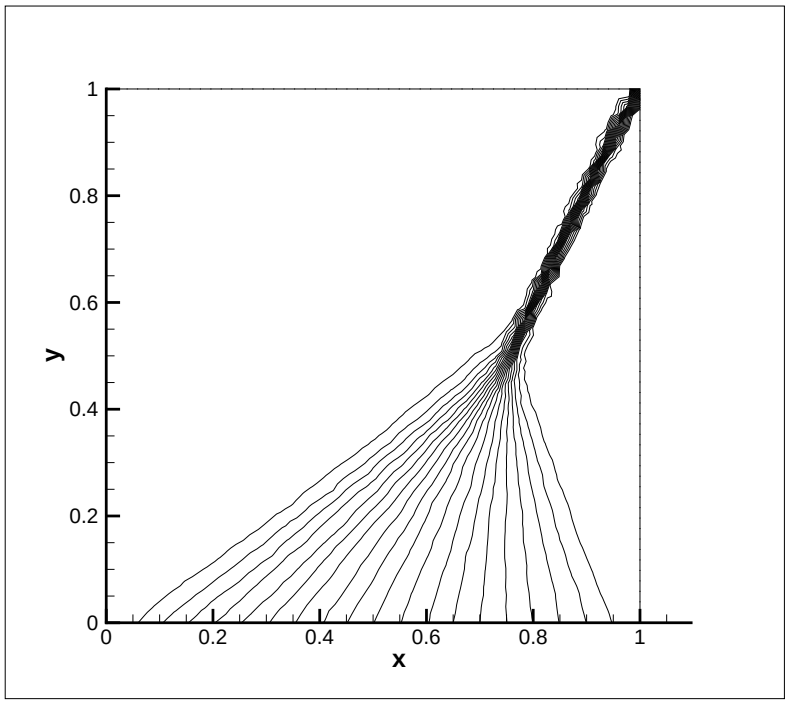


Figure 3: Results obtained for (7).

The methods is the one described by [5] that we recall. The Euler equations write

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0$$

where the vector of conserved variables is

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix},$$

and the fluxes F and G are

$$F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}, \qquad G(U) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}.$$

Here, as usual, ρ represents the density, u and v are the two components of the velocity vector, E is the total energy and p is the pressure. The system is closed by an equation of state, here we assume that the fluid is a caloricaly perfect gas,

$$p = (\gamma - 1)\Big(E - \frac{1}{2}\rho(u^2 + v^2)\Big).$$

The ratio of specific heats γ is set to 1.4.

Let us consider a direction (in practice the velocity vector) $\vec{n}$ which components are n_x and n_y. Denoting A and B the Jacobian matrices of the flux F and G with respect to U, we know that the matrix

$$An_x + Bny$$

is diagonalizable with distinct real eigenvalues: the system is strictly hyperbolic. The eigenvalues are $\lambda_1 = \vec{u}\cdot\vec{n}$ which is double and $\lambda_\pm = \vec{u}\cdot\vec{n} \pm c$. As usual, c represents the speed of sound,

$$c^2 = \gamma\frac{p}{\rho}.$$

Let us denote by r_1, r_2 the eigenvectors associated to λ_1 and $r_{3,4}$ those associated to $\lambda_\pm$. More precisely, if H represents the total enthalpy, $u_n = \vec{u}\cdot\vec{n}$ and $u_t = -n_y u + n_x v$, we have

$$r_1 = \begin{pmatrix} 1 \\ u \\ v \\ \frac{u^2+v^2}{2} \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 0 \\ -n_y \\ n_x \\ u_t \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 \\ u - cn_x \\ v - cn_y \\ H - u_n c \end{pmatrix}, \qquad r_4 = \begin{pmatrix} 1 \\ u + cn_x \\ v + cn_y \\ H + u_n c \end{pmatrix}.$$

By itself, the choice of the eigenvectors is not important, what is important is that these eigenvectors are orthonormal for the quadratic form defined by the Hessian of the entropy. Here, the quantities involved in the definition of the eigenvectors, i.e. the speed of sound, the velocity, the enthalpy, are evaluated at an average state. Many choices have been tested, and these experiences have revealed that the choice is not very important. We have taken a state defined by the primitive variables that are the arithmetic averages of the states at the vertices of K or Γ, the elements for which we are computing the second order residuals.

Once this is done, we proceed as follows, for the element $Q = K$ or Γ.

1. We decompose Φ_l^Q, $l = 1, \ldots, N$ (N=3 for a triangle, 4 for an edge), in the eigen-basis

$$\Phi_l^Q = \sum_{\ell=1,4} (\Phi_l^Q)_\ell r_\ell,$$

2. For each parameter ℓ (hence for any eigenvector r_ℓ), we notice that

$$\sum_{l=1}^{N} (\Phi_l^Q)_\ell = (\Phi^Q)_\ell$$

   and we define $(\Phi_l^Q)_\ell^\star$ by

$$(\Phi_l^Q)_\ell^\star = \frac{\left((\Phi_l^Q)_\ell/(\Phi^Q)_\ell\right)^+}{\sum\limits_{j=1}^{N}\left((\Phi_j^Q)_\ell/(\Phi^Q)_\ell\right)^+}\,(\Phi^Q)_\ell,$$

   with $x^+ = \max(x, 0)$.

3. Then

$$(\Phi_l^Q)^\star = \sum_{\ell=1}^{4} (\Phi_l^Q)_\ell^\star r_\ell.$$

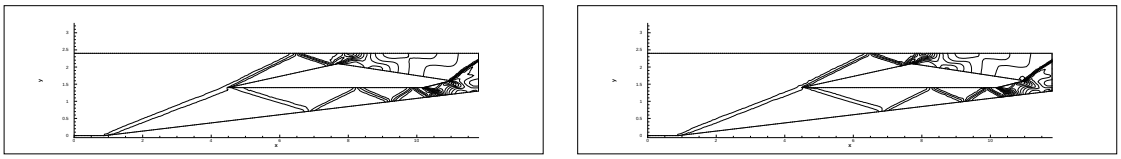
In any of the results that we have obtained, we have not added any filtering term as it was necessary in [6]. For the moment, it is not possible to tell if such a term is needed or not for the following reason: The graphic software we have used needs data at the vertices of the mesh. Here, a vertex carries several degrees of freedom (one per element), and we have made an arithmetic average. This certainly smoothes the results.

We have run a quite complex case, that has been already documented in [6]. It is a scramjet which conditions are

- Left and right boundary: supersonic inflow and outflow conditions. The inflow conditions are

$$\rho = 1.4,\ u = 3.6,\ v = 0,\ p = 1.$$

- The other boundaries are solid walls.

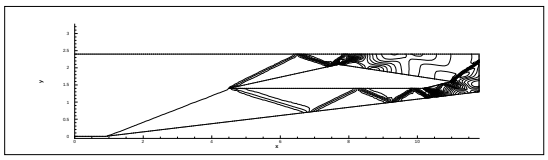


Continuous elements Discontinuous element.

Figure 4: Density isolines for the "classical" scheme (left) and the scheme described in this paper (right). 30 isolines are represented.

The boundary conditions at the solid walls are obtained by mirror conditions.

The density isolines (Figure 4), pressure isolines (Figure 5) and Mach number isolines(Figure 6) are given. They are compared with the results obtained by the continuous residual distribution method of [6], which is also second order in space. The isolines are almost identical for the two schemes. Figure 7 represents a zoom of the Mach number isolines at the exit of the scramjet. Once again, the quality of the results is similar.

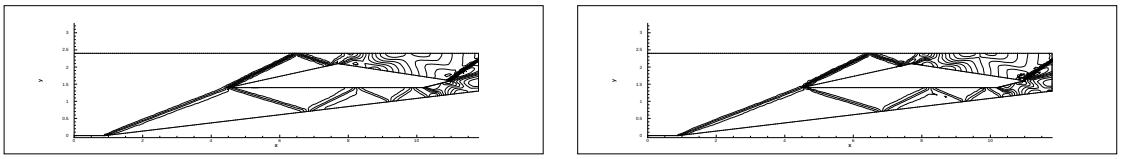


Continuous elements Discontinuous element.

Figure 5: Pressure field for the "classical" scheme (left) and the scheme described in this paper (right). 30 isolines are represented.

5 Conclusions and perspectives

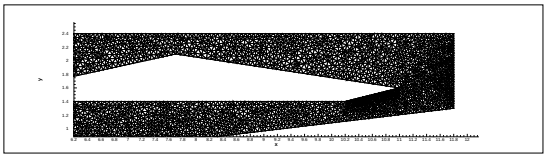
We have described an extension of the Residual distribution schemes using discontinuous elements. The main difference between these schemes and the discontinuous Galerkin ones is in the stabilization mechanism. For scalar problems we are able to prove L^∞ stability. Extension to more than second order accuracy, following the lines of [14] should be straightforward as well as for meshes using non triangular elements. After this work (a preliminary version is in [11]) and [9] was completed, the reference [12] has been published. Though some similarities, we believe that our approach is more general and more



Continuous elements Discontinuous element.

Figure 6: Mach number for the "classical" scheme (left) and the scheme described in this paper (right). 20 isolines are represented.

Continuous elements Discontinuous element.



Zoom of the mesh, exit zone.

Figure 7: Zoom of the Mach number for the "classical" scheme (left) and the scheme described in this paper (right). Exit of the scramjet. 20 isolines are represented.

suitable to high order extension : it does not rely on a specific choice of a residual distribution mechanism (here we have chosen Lax Friedrichs, but other choices could have been done).

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