KODAIRA DIMENSION OF ABSTRACT MODULAR SURFACES

ANNA CADORET

Abstract. Let k be a field of characteristic 0 and S a normal scheme, separated, of finite type and geometrically connected over k; let η denote the generic point of S. Given an abelian scheme $A \to S$, the étale fundamental group $\pi_1(S)$ of S acts continuously on the discrete module $A_\eta[n]$ of n-torsion points of the geometric generic fibre of $A \to S$. We denote by:

$$\rho_{A,n} : \pi_1(S) \to \mathrm{GL}(A_\eta[n]) \simeq \mathrm{GL}_{2\dim(A_\eta)}(\mathbb{Z}/n)$$

the corresponding continuous linear representation.To such data one can associate 'abstract modular schemes' $S_{A,1}(n)$ and $S_A(n)$ which, in this setting, are the modular analogues of the usual modular curves $Y_1(n)$ and $Y(n)$. We conjecture that, under some natural isotriviality assumption on $A_{\overline{\eta}}$, the abstract modular schemes $S_{A,1}(n)$ and $S_A(n)$ are of general type for n large enough. The case when S is a curve follows from previous works of A. Tamagawa and the author. In this paper, we consider the case when S is a surface. Our main result is that, under the natural isotriviality assumption mentioned above, $S_A(\ell^n)$ is of general type as well for n large enough and $S_{A,1}(\ell^n)$ is of general type for n large enough, provided S is not rational.There is an arithmetic motivation for these geometric results. Indeed, if k is a number field and $S_{A,1}(n)$ is of general type, Lang conjecture predicts that $S_{A,1}(n)(k)$ is not Zariski-dense in $S_{A,1}(n)$. This observation shows that Lang conjecture implies the ℓ-primary torsion conjecture for abelian varieties provided one can show that $S_{A,1}(\ell^n)$ is of general type for n large enough. In particular, our result shows that Lang conjecture for surfaces implies that the k-rational ℓ-primary torsion in a family of (higher dimensional) abelian varieties parametrized by a surface S which is not rational is uniformly bounded.

2010 *Mathematics Subject Classification.* Primary: 14E20; Secondary: 14J10, 14K15.

1. Introduction

Let k be a field of characteristic 0 and let $\mathcal{P}(k)$ (resp. $\mathcal{B}(k)$) denote the category of smooth, projective and geometrically connected schemes over k (resp. the category of normal schemes, of finite type, separated and geometrically connected over k). Fix $S \in \mathcal{B}(k)$ with generic point η and let $A \to S$ be an abelian scheme. For any integer n, the kernel $A[n]$ of the multiplication-by-n morphism is a finite étale group scheme over S hence the étale fundamental group $\pi_1(S)$ of S acts continuously on its (discrete) geometric generic fibre:

$$A_\eta[n] := A_\eta[n](k(\overline{\eta})).$$

We denote by:

$$\rho_{A,n} : \pi_1(S) \to \mathrm{GL}(A_\eta[n]) \simeq \mathrm{GL}_{2g}(\mathbb{Z}/n)$$

the corresponding continuous linear representation (here $g = \dim(A_\eta)$).

Using the theory of étale fundamental groups, one can associate to such data projective systems of *abstract modular schemes* (see subsection 2.2.4):

$$(S_{A,1}(n) \to S_{A,1}(m))_{m,n\in\mathbb{Z}_{\geq 1},\, m|n} \text{ and } (S_A(n) \to S_A(m))_{m,n\in\mathbb{Z}_{\geq 1},\, m|n}$$

which, in this setting, are the modular analogues of the projective systems of usual modular curves:

$$(Y_1(n) \to Y_1(m))_{m,n\in\mathbb{Z}_{\geq 1},\, m|n} \text{ and } (Y(n) \to Y(m))_{m,n\in\mathbb{Z}_{\geq 1},\, m|n}$$

classifying n-torsion points and full level-n structures of elliptic curves respectively. More precisely, a closed point $s \in S$ lifts to a $k(s)$-rational point on $S_{A,1}(n)$ if and only if A_s has a $k(s)$-rational torsion point of order exactly n. In particular, if for $X \in \mathcal{B}(k)$ and an integer $d \geq 1$ we write $S^{\leq d}$ for the set of all closed points $s \in S$ such that $[k(s) : k] \leq d$, then:

$$S_{A,1}(n)^{\leq d} = \emptyset \Longleftrightarrow A_s(k(s))_{tors} \subset A[n],\ s \in S^{\leq d}$$

This is in this sense that the $S_{A,1}(n)$ are regarded as natural analogues of the $Y_1(n)$. The $S_A(n)$ play the part of the Galois closure of the $S_{A,1}(n) \to S$ (but, in general, they are strict quotients of the abstract modular schemes $S_A[n]$ classifying 'full level-n' structure that is with the property that a closed point $s \in S$ lifts to a $k(s)$-rational point on $S_A[n]$ if and only if $A_s(k(s))[n] = A[n]$ - see remark 2.3).

Thus, the main motivation to study the diophantine properties of the abstract modular schemes $S_{A,1}(n)$ and $S_A(n)$ is to obtain uniform boundedness result for the rational torsion in the family A_s, $S^{\leq d}$.

More precisely, if k is finitely generated, we consider the following properties:

(A-1) $S_{A,1}(n)(k)$ is not Zariski dense in $S_{A,1}(n)$ for $n \gg 0$;

(A-2) (1) $|S_{A,1}(n)(k)| < +\infty$ for $n \gg 0$;
(2) For all integer $d \geq 1$, $|S_{A,1}(n)^{\leq d}| < +\infty$ for $n \gg 0$;

(A-3) (1) $S_{A,1}(n)(k) = \emptyset$ for $n \gg 0$;
(2) For all integer $d \geq 1$, $S_{A,1}(n)^{\leq d} = \emptyset$ for $n \gg 0$;

Via the classical theorems or conjectures relating arithmetic and geometry, these arithmetic considerations yield geometric ones. Namely, if k is algebraically closed, we consider the following properties:

(G-1) All the connected component of $S_{A,1}(n)$ are of general type for $n \gg 0$;

(G-2) If S is a curve,
(1) All the connected components of $S_{A,1}(n)$ are of genus ≥ 2 for $n \gg 0$;
(2) For all integer $d \geq 1$, all the connected components of $S_{A,1}(n)$ are of k-gonality $\geq d$ for $n \gg 0$;

Indeed, one has the following implications:

(G-2)(1) $\Rightarrow$ (A-2)(1)	: Mordell conjecture ([FW92]);
(G-2)(2) $\Rightarrow$ (A-2)(2)	: Lang conjecture for subvarieties of abelian varieties ([F94], [Fr94]);
(G-1) $\Rightarrow$ (A-1)	: Lang conjecture (see conjecture 4.1).

Actually, thanks to the projective system structure on the abstract modular schemes $S_{A,1}(n)$, $n \geq 1$ it is enough to consider the above properties in the two following situations:

ℓ-primary case: Fix a prime ℓ and consider the projective system:

$$(S_{A,1}(\ell^{n+1}) \to S_{A,1}(\ell^n))_{n\geq 0}.$$

This amounts to studying the ℓ-adic representation:

$$\rho_{A,\ell^\infty} : \pi_1(S) \to \mathrm{GL}(T_\ell(A_\eta)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$$

of $\pi_1(S)$ on the ℓ-adic Tate module $T_\ell(A_\eta) := \varprojlim A_\eta[\ell^n]$.

Prime case: Consider the family $S_{A,1}(\ell)$, ℓ: prime.

We will write (X/ℓ^∞) and (X/ℓ) when we want to refer to one of the properties X above only in the ℓ-primary and prime case respectively.

Using the fact that the projective limit of a projective system of non-empty finite sets is non-empty and Lang-Néron theorem [LNe59], it is straightforward to check the following implications:

(A-2/ℓ^∞)(1)$\Rightarrow$ (A-3/ℓ^∞)(1)
(A-2/ℓ^∞)(2)$\Rightarrow$ (A-3/ℓ^∞)(2)

And, more generally, one can derive (A-3/ℓ^∞) from (A-1/ℓ^∞) by induction on the dimension of S. In contrast, it does seem that one can directly deduce (A-3/ℓ)(1) or (2) from the other conditions.

When S is a curve, the questions above have been investigated thoroughly by A. Tamagawa and the author (*Cf.* [CT08] and [CT09a] for (G-2/ℓ^∞)(1), (A-2/ℓ^∞)(1) (hence (A-3/ℓ^∞)(1)), [CT09b] for (G-2/ℓ^∞)(2), (A-2/ℓ^∞)(2) (hence (A-3/ℓ^∞)(1)), [CT10a] and [CT10b] for (G-2/ℓ)(1), (A-2/ℓ)(1)). In summary, one has:

Theorem 1.1. ([CT08, Thm. 1.1], [CT09a, Thm. 3.4], [CT09b, Thm. 3.3], [CT10a][Thm. 2.1]) *Assume that k is algebraically closed, S is a curve and $A_{\overline{\eta}}$ contains no non-trivial k-isotrivial abelian subvariety. Then:*

<u>ℓ-primary case:</u> *the minimum of the k-gonalities of the connected components of $S_{A,1}(\ell^n)$ goes to $+\infty$ with n;*

<u>Prime case:</u> *the minimum of the genera of the connected components of $S_{A,1}(\ell)$ goes to $+\infty$ with ℓ.*

(This implies, in particular, that the minimum of the k-gonalities (hence the genera[1]) of the connected components of $S_A(\ell^n)$ goes to $+\infty$ with n and that the minimum of the genera of the connected components of $S_A(\ell)$ goes to $+\infty$ with ℓ.).

This theorem and the previous discussion motivate the following conjectural generalization to higher dimensional $S \in \mathcal{B}(k)$.

Conjecture 1.2. *Assume that k is algebraically closed and that $A_{\overline{\eta}}$ contains no non-trivial weakly k-isotrivial abelian subvariety. Then, all the connected components of $S_{A,1}(n)$ are of general type for n large enough.*

Here, if k is an algebraically closed field of characteristic 0, K/k a field extension of strictly positive transcendence degree and $\mathfrak{a}$ an abelian variety over K, we say that $\mathfrak{a}$ is *weakly k-isotrivial* if it can be defined over a field subextension F/k of K/k of strictly smaller transcendence degree over k. (See subsection 4.2 for more details about this notion).

In this paper, we investigate conjecture 1.2 in the ℓ-primary case when S is a surface and prove:

Theorem 1.3. (Main Theorem) *Assume that k is algebraically closed, that $S \in \mathcal{B}(k)$ is a surface and that $A_{\overline{\eta}}$ contains no non-trivial weakly k-isotrivial abelian subvariety. Then,*

(1) *All the connected components of $S_A(\ell^n)$ are of general type for n large enough;*
(2) *For n large enough, any connected component of $S_{A,1}(\ell^n)$ is either of general type or rational.*

In the final section of this paper, we discuss in more details the connection between conjecture 1.2 and two of the most classical conjectures in arithmetic geometry, namely Lang conjecture (conjecture 4.1) and the (ℓ-primary) torsion conjecture for abelian varieties (conjecture 4.2).

In particular, we show that the arithmetic form of the torsion conjecture over a finitely generated field k of characteristic 0 is equivalent to the fact that for any $S \in \mathcal{B}(k)$ and abelian scheme $A \to S$ one has $S_{A,1}(n)(k) = \emptyset$ for n large enough (lemma 4.3). And, assuming Lang conjecture, we show that the ℓ-primary form of conjecture 1.2 implies the ℓ-primary arithmetic form of the torsion conjecture (lemma 4.4). As a by product of the arguments involved in these considerations and theorem 1.3 (2), we obtain that Lang conjecture for surfaces implies that the k-rational ℓ-primary torsion in a family of (higher dimensional) abelian varieties

[1] Recall that given an irreducible curve C over a field k, the k-gonality γ_C of C is the minimal degree of a non-constant rational map $C \to \mathbb{P}^1_k$ and that it is related to the genus g_C of (the smooth proper model of) C by the following inequality:

$$\frac{\gamma_C + 2}{2} \leq g_C.$$

parametrized by a surface S which is not rational is uniformly bounded (corollary 4.7).

Conversely, we show that the geometric form of the torsion conjecture implies conjecture 1.2 for surfaces. Since a weak analogue of the geometric form of the torsion conjecture is available for full-level-n structures [HwT06], we obtain in particular a uniform variant of conjecture 1.2 for the $S_A[n]$ when S is a surface (corollary 4.11).

The paper is organized as follows. Section 2 provides the definitions and basic facts required to understand the statements and arithmetic motivations of theorem 1.3. More precisely, in subsection 2.1, we recall how to extend the definition of birational invariants from $\mathcal{P}(k)$ to $\mathcal{B}(k)$ and, in subsection 2.2, we define abstract modular schemes *via* the theory of étale fundamental group and review specific properties of the ℓ-adic representation ρ_{A,ℓ^∞}, which will be used in the sequel (especially theorem 2.4). In section 3, we carry out the proof of theorem 1.3. In subsection 3.1, we recall basic facts about the Enriques-Kodaira birational classification of surfaces, which is a key ingredient in our proofs. We then describe briefly the strategy of the proof of theorem 1.3 in subsection 3.2 and go over the technical details of the proof of part (2) in subsections 3.3.1, 3.3.2 and 3.3.3. The proof of part (1) is performed in subsection 3.4. Eventually, in the concluding section 4, we discuss the connection of conjecture 1.2 with Lang conjecture and the (ℓ-primary) torsion conjecture for abelian varieties.

Acknowlegments: I would like to thank Akio Tamagawa for lots of constructive discussions during the preparation of this paper. I am also grateful to Brendan Hassett for explaining me the argument of lemma 4.8 for K3-surfaces.

2. Preliminaries

We retain the conventions of section 1 for k, $\mathcal{P}(k)$ and $\mathcal{B}(k)$. Observe that, since k is of characteristic 0, schemes in $\mathcal{B}(k)$ are automatically geometrically integral (*e.g.* [EGAIV2, 6.7.4]).

2.1. Extending birational invariants.

Let $\sim$ denote the birational equivalence on $\mathcal{B}(k)$.

In our arguments, we will use birational geometric invariants which are classically defined for schemes in $\mathcal{P}(k)$. As we are interested in schemes in the wider class $\mathcal{B}(k)$, we need to extend the definition of these invariants from schemes in $\mathcal{P}(k)$ to schemes in $\mathcal{B}(k)$. This can be done by means of the following lemma.

Lemma 2.1. *The canonical map:*

$$\phi : \mathcal{P}(k)/\sim \to \mathcal{B}(k)/\sim$$

is bijective.

Proof. Let $S \in \mathcal{B}(k)$. Since S is separated over k, it follows from Nagata compactification theorem [N62], [N63] that there exists a scheme S^{cpt} proper over k and an open immersion $i : S \hookrightarrow S^{cpt}$ over k. Up to replacing S^{cpt} by the reduced subscheme associated with the Zariski closure of $i(S)$ in S^{cpt}, one can always assume that S^{cpt} is integral. Then, it follows from Hironaka desingularization theorem [Hi64] that there exists a regular scheme $\tilde{S}^{cpt}$ and a proper, birational morphism $\pi : \tilde{S}^{cpt} \to S^{cpt}$ inducing an isomorphism above each regular point of S^{cpt}. Then, by construction $\tilde{S}^{cpt} \in \mathcal{P}(k)$ and $\tilde{S}^{cpt} \sim S$. □

Remark 2.2. Note that Hironaka desingularization theorem is only known for fields of characteristic 0, whence the restriction to this setting in lemma 2.1.

Let $I : \mathcal{P}(k) \to \mathbb{Z} \cup \{-\infty\}$ be a numerical invariant which is constant on the equivalence class of $\sim$, that is factors *via*:

$$\begin{tikzcd} \mathcal{P}(k) \arrow[r, "I"] \arrow[d] & \mathbb{Z} \cup \{-\infty\} \\ \mathcal{P}(k)/\sim \arrow[ur, "\overline{I}"'] & \end{tikzcd}$$

Then we extend I on $\mathcal{B}(k)$ by setting:

$$I : \mathcal{B}(k) \twoheadrightarrow \mathcal{B}(k)/\sim \overset{\phi^{-1}}{\tilde{\to}} \mathcal{P}(k)/\sim \xrightarrow{\overline{I}} \mathbb{Z} \cup \{-\infty\}.$$

2.2. **Abstract modular schemes.** Fix $S \in \mathcal{B}(k)$ with generic point η and let $A \to S$ be an abelian scheme. For any integer n, the kernel $A[n]$ of the multiplication-by-n morphism is a finite étale group scheme over S hence the étale fundamental group $\pi_1(S)$ of S acts continuously on its (discrete) geometric generic fibre:

$$A_\eta[n] := A_\eta[n](k(\overline{\eta})).$$

We denote by:

$$\rho_{A,n} : \pi_1(S) \to \mathrm{GL}(A_\eta[n]) \simeq \mathrm{GL}_{2g}(\mathbb{Z}/n)$$

the corresponding continuous linear representation (here $g = \dim(A_\eta)$). Given any integers $m, n \geq 1$ with $m|n$, multiplication-by-$\frac{n}{m}$ induces a $\pi_1(S)$-equivariant epimorphism:

$$A_\eta[n] \twoheadrightarrow A_\eta[m].$$

In particular, one obtains a ℓ-adic representation (that is a continuous morphism of profinite groups):

$$\rho_{A,\ell^\infty} : \pi_1(S) \to \mathrm{GL}(T_\ell(A_\eta)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$$

of $\pi_1(S)$ on the ℓ-adic Tate module:

$$T_\ell(A_\eta) := \varprojlim A_\eta[\ell^n].$$

Note that, if $A_\eta^\vee$ denotes the dual abelian variety, one has a $\pi_1(S)$-equivariant isomorphism:

$$T_\ell(A_\eta) \simeq \mathrm{H}^1(A_{\overline{\eta}}^\vee, \mathbb{Z}_\ell)(1).$$

2.2.1. *Specialization.* One arithmetic motivation to study ρ_{A,ℓ^∞} is that it contains the ℓ-adic Galois representation at each closed special fibre; this follows from the Galois-equivariant specialization theory for ℓ-adic cohomology.

Indeed, given a closed point $s \in S$, one has:

- a quasi-splitting of the 'fundamental sort exact sequence' for $\pi_1(S)$:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(S_{\overline{k}}) \arrow[r] & \pi_1(S) \arrow[r] & \Gamma_k \arrow[r] & 1 \\ & & & \Gamma_{k(s)} \arrow[ul, "\sigma_s"] \arrow[u, hook] & \end{tikzcd}$$

- a specialization isomorphism:

$$sp_s : T_\ell(A_\eta) \tilde{\to} T_\ell(A_s),$$

which is Galois-equivariant in the following sense:

$$\tau \cdot sp_s(v) = sp_s(\sigma_s(\tau) \cdot v),\ v \in T_\ell(A_\eta),\ \tau \in \Gamma_{k(s)}.$$

In other words, the ℓ-adic Galois representation at the special fibre:

$$\rho_{A_s,\ell^\infty} : \Gamma_{k(s)} \rightarrow \mathrm{GL}(T_\ell(A_s))$$

decomposes as:

$$\rho_{A_s,\ell^\infty} = \rho_{A,\ell^\infty} \circ \sigma_s,$$

where the 'generic part' - ρ_{A,ℓ^∞}, is independent of s whereas the quasi-splitting - σ_s, only depends on s. So, once the generic representation ρ_{A,ℓ^∞} is given, describing the ρ_{A_s,ℓ^∞} when s varies among the closed points of S amounts to describing the quasi-splittings σ_s. But, precisely, the formalism of Galois categories provides a description of the image of the quasi-splitting σ_s in terms of the lifting of s to $k(s)$-rational points on étale covers of S.

2.2.2. *Control of the Galois image.* Indeed, since the category of étale covers of S is Galois with Galois group $\pi_1(S)$, there is a functorial correspondance between open subgroups $U \subset \pi_1(S)$ and connected étale covers $S_U \rightarrow S$. Moreover, $S_U \rightarrow S$ is defined over k_U, where k_U is the finite extension of k corresponding to the image of $U = \pi_1(S_U)$ in Γ_k *via* the structural epimorphism:

$$\pi_1(S) \twoheadrightarrow \Gamma_k = \pi_1(\mathrm{spec}(k)).$$

A straightforward consequence of the 'Galois dictionnary' (*e.g.* [SGA1, V, Prop. 6.4]) is that $\sigma_s(\Gamma_{k(s)}) \subset U$ if and only if $s \in S$ lifts to a $k(s)$-rational point $s_U \in S_U$ that is:

$$\begin{tikzcd}
S_U \arrow[d] & \\
S & \mathrm{spec}(k(s)). \arrow[l, "s"] \arrow[ul, dotted, "s_U"']
\end{tikzcd}$$

So, to understand the image of the Galois representations ρ_{A_s,ℓ^∞}, one is led to study the diophantine properties of the S_U, properties which are partly encoded in their geometric invariants. To determine those geometric invariants, it can be more convenient to work over an algebraically closed base field. In terms of étale fundamental groups, the following observation relates the arithmetic and geometric situation. Write $U^{geo} := U \cap \pi_1(S_{\overline{k}})$. Then, one has:

$$\begin{tikzcd}
(S_{\overline{k}})_{U^{geo}} \arrow[r, equal] \arrow[d] & S_U \times_S S_{\overline{k}} \arrow[dl] \\
S_{\overline{k}}. &
\end{tikzcd}$$

The S_U are the bricks abstract modular schemes are made of.

2.2.3. *Abstract modular schemes.* For $n \geq 1$, set $G_n := \mathrm{im}(\rho_{A,n})$ and for any subgroup $U \subset G_n$, write, to simplify:

$$S_U := S_{\rho_{A,n}^{-1}(U)} \rightarrow S, \ \ k_U := k_{\rho_{A,n}^{-1}(U)}$$

and let κ_{S_U} denote the Kodaira dimension of S_U. (Be aware of the change of notation for S_U).

Let $\mathcal{S}(G_n)$ denote the set of all subgroups of G_n. The projective system structure $(G_n \twoheadrightarrow G_m)_{m,n\in\mathbb{Z}_{\geq 1},\, m|n}$ (induced by the multiplication-by-$\frac{n}{m}$ morphisms $A_\eta[n] \twoheadrightarrow A_\eta[m]$, $m,n \in \mathbb{Z}_{\geq 1}$, $m|n$) endows the $\mathcal{S}(G_n)$, $n \geq 1$ with a projective system structure. Let

$$\mathcal{F} = (\mathcal{F}_n)_{m,n\in\mathbb{Z}_{\geq 1},\, m|n} \in \varprojlim \mathcal{S}(G_n).$$

We will say that the (non connected) étale cover:

$$S_{A,\mathcal{F}}(n) := \bigsqcup_{U\in\mathcal{F}_n} S_U \rightarrow S$$

is the *abstract modular scheme associated with* $\mathcal{F}_n$ and define:

$$d_{A,\mathcal{F}}(n) := \min\{[G_n : U]\}_{U\in\mathcal{F}_n},\ \kappa_{A,\mathcal{F}}(n) := \min\{\kappa_{S_U}\}_{U\in\mathcal{F}_n},$$

which we call - by abuse of language - the degree and Kodaira dimension of the abstract modular scheme $S_{A,\mathcal{F}}(n)$. Also, we will say that $S_{A,\mathcal{F}}(n)$ is of general type if $\kappa_{A,\mathcal{F}}(n) = \dim(S)$.

By functoriality, the projective system structure on $\mathcal{F}$ induces a projective system structure on the corresponding abstract modular schemes:

$$(S_{A,\mathcal{F}}(n) \to S_{A,\mathcal{F}}(m))_{m,n\in\mathbb{Z}_{\geq 1},\, m|n}.$$

2.2.4. *Abstract modular schemes associated with torsion.* For any integer $n \geq 1$, let:

$$A_\eta[n]^\times \subset A_\eta[n]$$

denote the subset of torsion points of order exactly n.

(1) $\underline{S_{A,1}(n)\ (\leftrightarrow Y_1(n))}$: Let $S_{A,1}(n)$, denote the abstract modular scheme associated with:

$$\mathcal{F}_n = \{\mathrm{Stab}_{G_n}(v) \mid v \in A_\eta[n]^\times\}.$$

We will write $d_{A,1}(n)$ and $\kappa_{A,1}(n)$ for its degree over S and Kodaira dimension respectively. Given $v \in A_\eta[n]^\times$, we will also use the simplified notation:

$$S_v := S_{\mathrm{Stab}_{G_n}(v)} \to S,\ k_v := k_{\mathrm{Stab}_{G_n}(v)}.$$

From the above, a closed point $s \in S$ lifts to a $k(s)$-rational point on $S_{A,1}(n)$ if and only if:

$$A_s[n]^\times(k(s)) \neq \emptyset.$$

This is why one can think of the $S_{A,1}(n)$, $n \geq 0$ as the abstract modular analogues of the $Y_1(n)$, $n \geq 0$.

(2) $\underline{S_A(\ell^n)\ (\leftrightarrow Y(n))}$: To define the natural abstract modular analogue of $Y(n)$ - that we denote by $S_A(n)$, recall that $Y(n) \to Y(0)$ is the Galois closure of $Y_1(n) \to Y(0)$. So we would like $S_A(n)$ to play the part of a Galois closure for $S_{A,1}(n)$. For this, observe that, for a $\pi_1(S)$-submodule $M \subset A_\eta[n]$, free as $\mathbb{Z}/n$-module, if we write $\rho_M : \pi_1(S) \to \mathrm{GL}(M)$ for the induced representation and:

$$G_M := \mathrm{im}(\rho_M),\ \ \mathrm{Fix}(M) := \{g \in G_n \mid g|_M = Id_M\}$$

then the connected étale cover:

$$S_M := S_{\mathrm{Fix}(M)} \to S$$

is Galois with group G_M. Furthermore, for $v \in A_\eta[n]^\times$ and:

$$M = M(v) := \mathbb{Z}[G_n \cdot v] \subset A_\eta[n],$$

the cover $S_{M(v)} \to S$ is the Galois closure of $S_v \to S$. So, we define $S_A(n)$ to be the abstract modular scheme associated with:

$$\mathcal{F}_n := \{\mathrm{Fix}(M(v)) \mid v \in A_\eta[n]^\times\}.$$

We will write $d_A(n)$ and $\kappa_A(n)$ for its degree over S and Kodaira dimension respectively.

Remark 2.3. Note that, in general, the covers $S_{M(v)} \to S$ are *strict* quotients of the connected étale cover $S_A[n] \to S$ associated with the open subgroup $\ker(\rho_{A,n}) \subset \pi_1(S)$. Actually, for $S_A[n]$, one can prove a uniform version of conjecture 1.2 for surfaces (see corollary 4.11).

(3) $\underline{S_{A,0}(n)\ (\leftrightarrow Y_0(n))}$: Let $S_{A,0}(n)$, denote the abstract modular scheme associated with:

$$\mathcal{F}_n = \{\mathrm{Stab}_{G_n}(\langle v\rangle) \mid v \in A_\eta[n]^\times\},$$

where $\langle v\rangle$ denotes the cyclic subgroup generated by v. Given $v \in A_\eta[n]^\times$, we will also use the simplified notation:

$$S_{\langle v\rangle} := S_{\mathrm{Stab}_{G_n}(\langle v\rangle)} \to S.$$

Note that, by definition, $S_v \to S_{\langle v\rangle}$ is Galois with group:

$$\mathrm{Aut}(S_v/S_{\langle v\rangle}) = \mathrm{Stab}_{G_n}(\langle v\rangle)/\mathrm{Stab}_{G_n}(v) \hookrightarrow \mathrm{Aut}(\langle v\rangle) \simeq (\mathbb{Z}/\ell^n)^\times.$$

2.2.5. *Properties of ρ_{A,ℓ^∞}.* We consider the following technical geometric conditions on:

$$\rho_{A,\ell^\infty} : \pi_1(S) \to \mathrm{GL}(T_\ell(A_\eta)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell).$$

(A/ℓ^∞) (i) For any open subgroup $\Pi \subset \pi_1(S_{\overline{k}})$, one has $|\rho_{A,\ell^\infty}(\Pi)^{ab}| < +\infty$;
($\Longleftrightarrow$ (ii) $(Lie(\rho_{A,\ell^\infty}(\pi_1(S_{\overline{k}})))^{ab} = 0$.)

(I/ℓ^∞) For any open subgroup $\Pi \subset \pi_1(S_{\overline{k}})$, the submodule $T_\ell(A_\eta)^\Pi$ of Π-invariant vectors is trivial.

It is easy to see that condition (I/ℓ^∞) and the fact that $\pi_1(S_{\overline{k}})$ is topologically finitely generated imply:

$$\lim_{n\to+\infty} d_{A,1}(\ell^n) = +\infty$$

(hence, *a fortiori*, $\lim_{n\to+\infty} d_A(\ell^n) = +\infty$).

Basically, conditions (A/ℓ^∞) and (I/ℓ^∞) are the minimal conditions to make our proof work in the case when S is a curve.

The fact that $\rho_{A,\ell^\infty} : \pi_1(S) \to \mathrm{GL}(T_\ell(A_\eta))$ satisfies condition (A/ℓ^∞) is a special case of the following theorem.

Theorem 2.4. *Let $f : X \to S$ be a smooth, proper morphism with geometrically connected fibres. Then the representations:*

$$\rho_f^i : \pi_1(S) \to \mathrm{GL}(\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)),\ i \geq 1$$

satisfy condition (A/ℓ^∞).

Proof. First, we may assume that k is an algebraically closed field of characteristic 0. Also, we may replace freely $\pi_1(S)$ by any of its open subgroups. Since all the schemes are finitely generated, we may assume that k has finite transcendence degree over $\mathbb{Q}$ and fix an embedding $k \hookrightarrow \mathbb{C}$. This induces a comparison isomorphism of profinite groups:

$$\pi_1^{top}(S^{an};\overline{s})^\vee \tilde{\to} \pi_1(S_{\overline{k}};\overline{s}),$$

where $(-)^\vee$ denotes the profinite completion and $\pi_1^{top}(S^{an};\overline{s})$ the topological fundamental group of the topological space underlying the analytic space S^{an} associated with $S \times_k \mathbb{C}$.

Set $H_B^i := \mathrm{H}^i_{\mathrm{Betti}}(X_{\overline{s}}^{an}, \mathbb{Z})$ and $H_{B\mathbb{Q}}^i := H_B^i \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathfrak{G}$ denote the image of $\pi_1^{top}(S^{an};\overline{s})$ in $\mathrm{GL}(H_{B\mathbb{Q}}^i)$. Up to replacing S by a connected étale cover, we may assume that the Zariski closure $\mathfrak{G}^z$ of $\mathfrak{G}$ in $\mathrm{GL}(H_{B\mathbb{Q}}^i)$ is connected hence semisimple ([D71, Cor. (4.2.9) (a)]). On the other hand, set:

$$G := \mathrm{im}(\rho_f^i) \subset \mathrm{GL}(\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Z}_\ell)).$$

and let G^z denote the Zariski-closure of G in $\mathrm{GL}(\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell))$. It follows from the comparison isomorphism between Betti cohomology with coefficient in $\mathbb{Q}$ and étale cohomology with coefficients in $\mathbb{Q}_\ell$ that the algebraic groups G^z and $(\mathfrak{G}^z)_{\mathbb{Q}_\ell}$ coincide. The conclusion thus follows from:

Lemma 2.5. *Let $G \subset \mathrm{GL}_r(\mathbb{Q}_\ell)$ be a closed (ℓ-adic) subgroup and assume that its Zariski closure G^z in $\mathrm{GL}_{r\mathbb{Q}_\ell}$ is a semisimple algebraic group over $\mathbb{Q}_\ell$. Thenthe Lie algebra of G (as a ℓ-adic Lie group) coincide with the Lie algebra of G^z (as an algebraic group over $\mathbb{Q}_\ell$). In particular, $(Lie(G))^{ab} = 0$.*

Proof of lemma 2.5. Since G^z is a connected semisimple algebraic group, it is the almost direct product:

$$G_1^z \times \cdots \times G_r^z \to G^z$$

of its minimal connected normal subgroups and, with $\mathfrak{g}_i^z := Lie(G_i^z)$, $i = 1, \ldots, r$ and $\mathfrak{g}^z := Lie(G^z)$, the following:

$$\mathfrak{g}^z = \mathfrak{g}_1^z \oplus \cdots \oplus \mathfrak{g}_r^z$$

is the decomposition of $\mathfrak{g}^z$ into the direct sum of its simple ideals.

$\mathrm{GL}_{d\mathbb{Q}_\ell}$ acts *via* the adjoint representation on $\mathrm{M}_d(\mathbb{Q}_\ell)$. As $\mathfrak{g} := Lie(G) \subset \mathrm{M}_d(\mathbb{Q}_\ell)$ is a $\mathbb{Q}_\ell$-submodule the stabilizer of $\mathfrak{g}$ in $\mathrm{GL}_{d\mathbb{Q}_\ell}$ is an algebraic subgroup N of $\mathrm{GL}_{d\mathbb{Q}_\ell}$. By definition $G \subset N(\mathbb{Q}_\ell)$ hence $G^z \subset N$. As a result, the action of G^z on $\mathfrak{g}^z := Lie(G^z)$ *via* the adjoint representation restricts to $\mathfrak{g}$. This implies that:

$$\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i^z$$

for some subset $I \subset \{1, \ldots, r\}$.

As a result, there is an open subgroup $U \subset_{op} G$ such that $U \subset \prod_{i \in I} G_i^z(\mathbb{Q}_\ell)$ hence:

$$G^z = U^z \subset \prod_{i \in I} G_i^z,$$

which is only possible if $I = \{1, \ldots, r\}$. □

Remark 2.6. In [CT09a], we give a different proof of theorem 2.4. There, we reduce first to the case where S is a curve by a Bertini-type argument. Then, by specialization theory, we reduce to the case where the base field is the algebraic closure of a finite field of characteristic $p \neq \ell$ and, eventually, we conclude by a Frobenius weight argument based on Weil conjectures. The idea of the proof presented here was suggested to me by Y. André. Lemma 2.5 is attributed to A. Weil; its proof was explained to me by B. Edixhoven. For lack of a suitable reference, I have included it in the text.

The fact that $\rho_{A,\ell^\infty} : \pi_1(S) \to \mathrm{GL}(T_\ell(A_\eta))$ satisfies condition (I/ℓ^∞) is ensured by the geometric form of Lang-Néron theorem [LNe59] provided $A_{\overline{\eta}}$ contains no non-trivial $\overline{k}$-isotrivial abelian subvariety. But, to make our arguments work for higher dimensional base schemes $S \in \mathcal{B}(k)$, we have to ensure not only that ρ_{A,ℓ^∞} satisfies condition (I/ℓ^∞) but also that for any generically finite morphism $T \to S$ (with $\dim(T) > 0$), the representation $\rho_{A\times_S T,\ell^\infty}$ satisfies condition (I/ℓ^∞) as well. According to the geometric form of Lang-Néron theorem, this is ensured by:

$(*)$ $A_{\overline{\eta}}$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety.

Here, if k is an algebraically closed field of characteristic 0, K/k a field extension of strictly positive transcendence degree and $\mathfrak{a}$ an abelian variety over K, we say that $\mathfrak{a}$ is *weakly k-isotrivial* if it can be defined over a field subextension F/k of K/k of strictly smaller transcendence degree over k.

Remark 2.7. Concerning the prime case, the families of representations:

$$\rho_{A,\ell} : \pi_1(S) \to \mathrm{Gl}(A_\eta[\ell]),\ \ell : \text{prime},$$

satisfy natural analogues - denoted by (A/ℓ) and (I/ℓ) - of conditions (A/ℓ^∞) and (I/ℓ^∞). See [CT10b] for precise formulations and proofs.

3. Proofs

As mentioned in subsection 2.2, it is enough to prove theorem 1.3 when k is algebraically closed. So, from now on and till the end of this section, k is an algebraically closed field of characteristic 0.

3.1. **Classification of surfaces.** We gather in this subsection classical results about surfaces that will be used in the proof of theorem 1.3. Given an integer $d \geq 0$, let $\mathcal{P}^d(k) \subset \mathcal{P}(k)$ (resp. $\mathcal{B}^d(k) \subset \mathcal{B}(k)$) denote the full subcategory of d-dimensional schemes in $\mathcal{P}(k)$ (resp. in $\mathcal{B}(k)$).

3.1.1. *Birational classification.* Given $S \in \mathcal{P}^2(k)$, we will write $a_S : S \to \mathrm{Alb}(S)$ for the Albanese morphism. The following numerical invariants:

- Kodaira dimension: $\kappa : \mathcal{P}^2(k) \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$
- Irregularity: $q\colon \begin{array}{ccc} \mathcal{P}^2(k) & \to & \mathbb{Z}_{\geq 0} \\ S & \mapsto & \dim_k(\mathrm{H}^0(S, \Omega^1_{S|k})) = \dim(\mathrm{Alb}(S)) \end{array}$
- Geometric genus: $p_g\colon \begin{array}{ccc} \mathcal{P}^2(k) & \to & \mathbb{Z}_{\geq 0} \\ S & \mapsto & \dim_k(\mathrm{H}^2(S, \mathcal{O}_S)) \end{array}$

factor *via* $\mathcal{P}^2(k) \to \mathcal{P}^2(k)/\sim$. So, as explained in subsection 2.1, their definition extends to $\mathcal{B}^2(k)$.

A surface $S \in \mathcal{P}^2(k)$ is said to be *relatively minimal* if any birational morphism from S to another surface in $\mathcal{P}^2(k)$ is an isomorphism. A surface $S \in \mathcal{P}^2(k)$ is said to be *minimal* if, given $S' \in \mathcal{P}^2(k)$, any birational map $S' \overset{\sim}{\dashrightarrow} S$ is a birational morphism. Any minimal surface is relatively minimal and, for a minimal surface, any birational map $S \overset{\sim}{\dashrightarrow} S$ automatically extends to an automorphism $S \tilde{\to} S$.

If $\kappa_S = -\infty$ then the birational class of S may contain several non-isomorphic relatively minimal surfaces. If $\kappa_S \geq 0$ then the birational class of S contains a unique (up to isomorphism) relatively minimal surface which, actually, is minimal. Enriques-Kodaira classification gives a geometrical description of $\mathcal{P}^2(k)/\sim$ in terms of the relatively minimal models.

κ_S	q_S	$p_{g,S}$	
2			General type
1			Honest Elliptic
0	2	2	Abelian
	1	0	Bielliptic
	0	1	K3
	0	0	Enriques
$-\infty$	$g \geq 1$	0	ruled of genus $g \geq 1$
	0	0	rational

Enriques-Kodaira classification

The terminology is the following:

- A relatively minimal *rational* surface is isomorphic either to $\mathbb{P}^2_k$ or to a geometrically ruled surfaces of genus 0:

$$S_n := \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(n)) \to \mathbb{P}^1_k,\ n \geq 1$$

(and two such surfaces $S_n \to \mathbb{P}^1_k$, $S_{n'} \to \mathbb{P}^1_k$ are not isomorphic for $n \neq n'$).

- A relatively minimal *ruled* surface *of genus* g is isomorphic to a geometrically ruled surface of genus g that is:
$$\mathrm{Proj}(\mathcal{E}) \to C,$$
where C is a smooth, proper and geometrically connected curve of genus g and $\mathcal{E}$ is a locally free sheaf of rank 2 (and two such surfaces $\mathrm{Proj}(\mathcal{E}) \to C$, $\mathrm{Proj}(\mathcal{E}') \to C'$ are isomorphic if there exists an isomorphism $u : C \tilde{\to} C'$ and an invertible sheaf $\mathcal{L}$ on C such that $u^*\mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L}$).

- Minimal *honest elliptic* surfaces are isomorphic to elliptic surfaces:
$$p : S \to C$$
whose elliptic fibration is an Iitaka fibration (see subsection 3.4);

- Minimal *abelian* surfaces are isomorphic to abelian varieties;

- Minimal *bielliptic* surfaces are isomorphic to elliptic surfaces:
$$a : S \to \mathrm{Alb}(S) = C$$
whose elliptic fibration is given by the Albanese morphism;

- Minimal *K3* and *Enriques* surfaces are more complicated to describe. We will only use that for any Enriques surface $S \in \mathcal{P}^2(k)$, one has $\pi_1(S) = \mathbb{Z}/2$ and the unique connected étale cover of S is a K3 surface, that we denote by $K_S \to S$.

We refer to the classical textbook [B78] for a detailled proof of the Enriques-Kodaira classification and an overview of relatively minimal surfaces.

3.1.2. *Generically finite morphisms.* For any two $S,\ S' \in \mathcal{P}^2(k)$ and generically finite k-morphism $S \to S'$ one has $\kappa_S \geq \kappa_{S'}$, $q_S \geq q_{S'}$ and $p_{g,S} \geq p_{g,S'}$. This imposes constraints on the existence of such morphisms. More precisely, Let $\mathcal{T} := (\kappa, q, p)$ be a triple with $q,\ p = 0, 1$ or $-$ and $\kappa = -\infty, 0, 1, 2$, where, by $-$, we mean that we impose no condition on the variable. Endow the set of such triples $\mathcal{T}$ with the order $\prec$ defined by $\tau \prec \tau'$ if and only if $\kappa \leq \kappa'$, $q \leq q'$ and $p \leq p'$ (with the convention that $- > 2 > 1 > 0 > -\infty$) then, $\mathcal{T}' \leq \mathcal{T}$ if and only if there can be generically finite morphisms from a surface with invariants $\mathcal{T}$ to a surface with invariant $\mathcal{T}'$. This is summarized in the diagram below.

$$\begin{tikzcd}
 & \text{General type} \arrow[dl] \arrow[d] & & \\
\text{Ruled } g = n+1 \arrow[d] & \text{Honest Elliptic} \arrow[l] \arrow[d] & & \\
\text{Ruled } g = n \arrow[d] & \text{Abelian} \arrow[ddl] \arrow[ddr] \arrow[drr] & & \\
\cdots \arrow[d] & & & \text{Bielliptic} \arrow[ddlll] \arrow[ddl] \\
\text{Ruled } g = 2 \arrow[d] & & \text{K3} \arrow[d] & \\
\text{Ruled } g = 1 \arrow[r] & \text{Rational} & \text{Enriques,} \arrow[l] &
\end{tikzcd}$$

Diagram of generically finite morphisms

where an arrow $A \to B$ means that there can be generically finite k-morphisms from a surface of type A to a surface of type B.

In particular, one has:

Lemma 3.1. *Let $\cdots \to S_{n+1} \to S_n \to \cdots \to S_0$ be a projective system in $\mathcal{B}^2(k)$ whose transition morphisms are generically finite morphisms. Then, one of the following occurs:*

(1) *S_n is of general type for $n \gg 0$;*
(2) *S_n is honest elliptic for $n \gg 0$;*
(3) *S_n is abelian for $n \gg 0$;*
(4) *S_n is bielliptic for $n \gg 0$;*
(5) *S_n is K3 for $n \gg 0$;*
(6) *S_n is Enriques for $n \gg 0$;*
(7) *S_n is ruled of genus ≥ 1 for $n \gg 0$;*
(8) *S_n is rational for $n \gg 0$.*

3.2. **Reduction to projective systems.** After reducing the proof of theorem 1.3 to the case of projective systems in subsection 3.2, we prove first theorem 1.3 (2) in subsection 3.3. Theorem 1.3 (2) automatically implies theorem 1.3 (1) except when S is rational. We rule this remaining case out in subsection 3.4.

Now, endow the set of triples $\mathcal{T}$ with the lexical order $\leq$. As observed in subsection 3.1.2, if there can be generically finite morphisms from a surface with invariants $\mathcal{T}'$ to a surface with invariant $\mathcal{T}$ then $\mathcal{T} \leq \mathcal{T}'$. For every $n \geq 0$ and $v \in A_\eta[\ell^n]^\times$, write $\mathcal{T}_v := (\kappa_{S_v}, q_{S_v}, p_{g,S_v})$ and $\mathcal{T}_{M(v)} := (\kappa_{S_{M(v)}}, q_{S_{M(v)}}, p_{g,S_{M(v)}})$. Consider the sets:

$$E_{A,1,\leq\mathcal{T}}(n) := \{v \in A_\eta[\ell^n]^\times \mid \mathcal{T}_v \leq \mathcal{T}\},\ \ E_{A,1,\mathcal{T}}(n) := \{v \in A_\eta[\ell^n]^\times \mid \mathcal{T}_v = \mathcal{T}\}$$

and:

$$E_{A,\leq\mathcal{T}}(n) := \{v \in A_\eta[\ell^n]^\times \mid \mathcal{T}_{M(v)} \leq \mathcal{T}\},\ \ E_{A,\mathcal{T}}(n) := \{v \in A_\eta[\ell^n]^\times \mid \mathcal{T}_{M(v)} = \mathcal{T}\}.$$

By definition, $E_{A,1,\leq\mathcal{T}}(n)$ and $E_{A,\leq\mathcal{T}}(n)$ are either finite or empty. For any $n \geq 0$ and $v \in A_\eta[\ell^{n+1}]^\times$, by fonctoriality, one has an étale cover $S_v \to S_{\ell v}$ so:

$$\mathcal{T}_{\ell v} \leq \mathcal{T}_v,\ \ \mathcal{T}_{M(\ell v)} \leq \mathcal{T}_{M(v)}.$$

As a result, the multiplication-by-ℓ maps:

$$A_\eta[\ell^{n+1}]^\times \to A_\eta[\ell^n]^\times$$

induce structures of projective system:

$$(E_{A,1,\leq\mathcal{T}}(n+1) \to E_{A,1,\leq\mathcal{T}}(n))_{n\geq 0},\ \ (E_{A,\leq\mathcal{T}}(n+1) \to E_{A,\leq\mathcal{T}}(n))_{n\geq 0}$$

For our purpose, it is enough to consider the following 8 values of $\mathcal{T}$:

$\mathcal{T}$	Surfaces with invariant $\leq \mathcal{T}$
$(2,-,-)$	all;
$(1,-,-)$	all except those general type;
$(0,-,-)$	all except those of general type or honest elliptic;
$(0,1,-)$	all except those of general type, honest elliptic, or abelian;
$(0,0,0)$	rational, ruled, Enriques or K3;
$(0,0,0)$	rational, ruled or Enriques;
$(-\infty,-,-)$	rational or ruled;
$(-\infty,0,-)$	rational.

These triples $\mathcal{T}$ are totally ordered for $\leq$. So write them as:

$$\mathcal{T}_1 = (-\infty, 0, -) < \mathcal{T}_2 < \cdots < \mathcal{T}_8 = (2, -, -).$$

Concerning theorem 1.3 (2), we are to prove that either $E_{A,1,\leq \mathcal{T}_1}(n) \neq \emptyset$ for all $n \geq 0$ or there exists $N \geq 1$ such that $E_{A,1,\leq \mathcal{T}_1}(n) = \emptyset$ for $n \geq N$ and, then, for all $\mathcal{T} \leq \mathcal{T}_7$, one has $E_{A,1,\leq \mathcal{T}}(n) = \emptyset$ for n large enough. For the second part of the assertion, we proceed as follows. Assume that we have proved that there exists $n_i \geq 1$ such that $E_{A,1,\leq \mathcal{T}_i}(n) = \emptyset$ for $n \geq n_i$ but $E_{A,1,\leq \mathcal{T}_{i+1}}(n) \neq \emptyset$, for all $n \geq 1$. Then, one gets a projective system of non-empty finite sets:

$$((E_{A,1,\mathcal{T}_{i+1}}(n+1) \to (E_{A,1,\mathcal{T}_{i+1}}(n))_{n \geq n_i}$$

and as the projective limit of a projective system of non-empty finite sets is always non-empty, one thus gets:

$$v = (v_n)_{n \geq 0} \in T_\ell(A_\eta)^\times := \varprojlim A_\eta[\ell^n]^\times (= T_\ell(A_\eta) \smallsetminus \ell T_\ell(A_\eta))$$

such that S_{v_n} has invariants $\mathcal{T}_{i+1}$ for all $n \geq n_i$. So, to prove theorem 1.3 (2), it is enough to consider projective systems of the form:

$$\cdots \to S_{v_{n+1}} \to S_{v_n} \to \cdots \to S_{v_1} \to S (= S_{v_0}),$$

where:

$$v = (v_n)_{n \geq 0} \in T_\ell(A_\eta)^\times$$

and show that cases (2)-(7) of lemma 3.1 cannot occur.

Concerning theorem 1.3 (1), we are to prove that for all $\mathcal{T} \leq \mathcal{T}_7$, one has $E_{A,\leq \mathcal{T}}(n) = \emptyset$ for n large enough. The argument above shows that it is enough to consider projective systems of the form:

$$\cdots \to S_{M(v_{n+1})} \to S_{M(v_n)} \to \cdots \to S_{M(v_1)} \to S (= S_{M(v_0)}),$$

where:

$$v = (v_n)_{n \geq 0} \in T_\ell(A_\eta)^\times$$

and show that cases (2)-(8) of lemma 3.1 cannot occur. From theorem 1.3 (2) and the fact that $\kappa_{S_{M(v_n)}} \geq \kappa_{S_{v_n}}$ for all $n \geq 0$, it follows from theorem 1.3 (2) that the only case, we already know that cases (2)-(7) cannot occur. So the only case to rule out is case (8).

3.3. **Proof of theorem 1.3 (2).** For simplicity, set $S_n := S_{v_n}$ and let η_n denote the generic point of S_n, $n \geq 0$.

3.3.1. *Case (3).* Assume that case (3) occurs. Up to replacing A by $A \times_S S_n$ for some n large enough, we may assume that S_n is birational to an abelian surface A_n over k for all $n \geq 0$. Hence, fixing birational maps $\phi_n : S_n \dashrightarrow A_n$, the projective system:

$$\cdots \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 (= S)$$

induces a profinite commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & S_{n+1} & \longrightarrow & S_n & \longrightarrow \cdots \longrightarrow & S_1 & \longrightarrow & S_0 \\
& \Big\downarrow{\scriptstyle \phi_{n+1}} & & \Big\downarrow{\scriptstyle \phi_n} & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_0} \\
\cdots \dashrightarrow & A_{n+1} & \dashrightarrow & A_n & \dashrightarrow \cdots \dashrightarrow & A_1 & \dashrightarrow & A_0,
\end{array}$$

(where $A_{n+1} \dashrightarrow A_n$ is just $A_{n+1} \overset{\phi_{n+1}^{-1}}{\dashrightarrow} S_{n+1} \to S_n \overset{\phi_n^{-1}}{\dashrightarrow} A_n$). By rigidity, a rational map between abelian varieties is automatically a morphism so $A_{n+1} \dashrightarrow A_n$ is a generically finite morphism hence an isogeny (up to translation). Furthermore, one has:

$$\mathrm{Aut}(S_n/S_0) \overset{(1)}{=} \mathrm{Aut}(k(\eta_n)/k(\eta)) \overset{(2)}{=} \mathrm{Aut}(A_n/A_0),$$

where (1) follows from the fact that S_n is the normalization of S in $k(\eta_n)/k(\eta)$ and (2) follows from rigidity. This shows that $S_n = S_{M(v_n)} \to S$ is a Galois cover and that its automorphism group $G_{M(v_n)}$ is a quotient of $\mathbb{Z}^4$ (hence, in particular, is abelian). But:

$$|G_{M(v_n)}| \geq d_A(\ell^n)$$

and, from condition (I/ℓ^∞), one has $\lim_{n\to+\infty} d_A(\ell^n) = +\infty$, which contradicts condition (A/ℓ^∞). So case (3) cannot occur.

Remark 3.2. This part of the argument works for S of arbitrary dimension.

3.3.2. *Cases (2), (4), (7).* To rule out cases (2), (4) and (7), our strategy consists in considering geometric generic fibers of canonical fibrations to reduce to the case of curves and conclude by theorem 1.1. More precisely, we show that if case (2), (4) or (7) holds, one can construct a profinite commutative diagram:

$$(\dagger)\quad \begin{tikzcd} \cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "\phi_{n+1}"'] & S_n \arrow[r] \arrow[d, dotted, "\phi_n"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "\phi_1"'] & S_0 \arrow[d, dotted, "\phi_0"'] \\ \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r] & C_0, \end{tikzcd}$$

where $\phi_n : S_n \dashrightarrow C_n$ are dominant rational maps, $C_{n+1} \to C_n$ are finite (non constant) morphisms of curves and that, if ζ_n denotes the generic point of C_n, then $S_{\overline{\zeta}_n}$ has genus 0 or 1. But this contradicts theorem 1.1 for $A \times_S S_{\overline{\zeta}_0}$ (which contains no non-trivial $k(\overline{\zeta}_0)$-isotrivial abelian subvariety by assumption (*)) since, from the commutativity of the above diagram, $S_{n\overline{\zeta}_n}$ maps to $(S_{\overline{\zeta}_0})_{v_n}$. So, it remains to construct diagram (†) in each of the cases (2), (4) and (7).

3.3.2.1. Case (7). Assume that case (7) occurs. Up to replacing A by $A \times_S S_n$ for some $n \gg 0$, we may assume that S_n is birational to a product $\mathbb{P}^1_k \times_k C_n$, where C_n is smooth proper curve of genus $g_n \geq 1$ for all $n \geq 0$. Hence, fixing birational maps $\varphi_n : S_n \dashrightarrow \mathbb{P}^1_k \times_k C_n$, the projective system:

$$\cdots \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 (= S)$$

induces a profinite commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "\varphi_{n+1}"'] & S_n \arrow[r] \arrow[d, dotted, "\varphi_n"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "\varphi_1"'] & S_0 \arrow[d, dotted, "\varphi_0"'] \\ \cdots \arrow[r, dotted] & \mathbb{P}^1_k \times_k C_{n+1} \arrow[r, dotted] & \mathbb{P}^1_k \times_k C_n \arrow[r, dotted] & \cdots \arrow[r, dotted] & \mathbb{P}^1_k \times_k C_1 \arrow[r, dotted] & \mathbb{P}^1_k \times_k C_0. \end{tikzcd}$$

Since there is no non-constant rational map from a genus 0 to a genus ≥ 1 curve, given any section $s_{n+1} : C_{n+1} \to \mathbb{P}^1_k \times_k C_{n+1}$ of the projection $p_{n+1} : \mathbb{P}^1_k \times_k C_{n+1} \to C_{n+1}$, the resulting rational map (which is automatically a morphism):

$$C_{n+1} \stackrel{s_{n+1}}{\to} \mathbb{P}^1_k \times_k C_{n+1} \dashrightarrow \mathbb{P}^1_k \times_k C_n \stackrel{p_n}{\to} C_n$$

is dominant and makes the following diagram commute:

$$\begin{tikzcd} \mathbb{P}^1_k \times_k C_{n+1} \arrow[r, dotted] \arrow[d, "p_{n+1}"'] & \mathbb{P}^1_k \times_k C_n \arrow[d, "p_n"'] \\ C_{n+1} \arrow[r] & C_n. \end{tikzcd}$$

3.3.2.2. Case (4). Assume that case (4) occurs. Up to replacing ρ by its restriction $\rho|_{\pi_1(S_n)}$ for some $n \gg 0$, we may assume that S_n is birational to a minimal bielliptic surface B_n whose elliptic fibration $a_{B_n} : B_n \to C_n := \mathrm{Alb}(B_n)$ is the Albanese morphism for all $n \geq 0$. Hence, fixing birational maps $\varphi_n : S_n \dashrightarrow B_n$, the projective system:

$$\cdots \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 (= S)$$

induces a profinite commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "\varphi_{n+1}"'] & S_n \arrow[r] \arrow[d, dotted, "\varphi_n"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "\varphi_1"'] & S_0 \arrow[d, dotted, "\varphi_0"'] \\
\cdots \arrow[r, dotted] & B_{n+1} \arrow[r, dotted] & B_n \arrow[r, dotted] & \cdots \arrow[r, dotted] & B_1 \arrow[r, dotted] & B_0.
\end{tikzcd}$$

It follows from lemma 3.3 below that the rational map:

$$B_{n+1} \dashrightarrow B_n \overset{a_{B_n}}{\to} C_n$$

is a morphism. Hence, from the universal property of Albanese, it factors through a commutative diagram:

$$\begin{tikzcd}
B_{n+1} \arrow[r, dotted] \arrow[d, "a_{B_{n+1}}"'] & B_n \arrow[d, "a_{B_n}"] \\
C_{n+1} \arrow[r] & C_n.
\end{tikzcd}$$

Lemma 3.3. *Let S be a smooth, proper surface over k and C a smooth, proper curve over k with genus ≥ 1. Then any rational map $p : S \dashrightarrow C$ is a morphism.*

Proof. By elimination of indeterminancy [B78, II, Thm. 7], there exists a commutative diagram:

$$\begin{tikzcd}
& \tilde{S} \arrow[dl, "\epsilon"'] \arrow[dr, "\pi"] & \\
S \arrow[rr, dotted, "p"'] & & C
\end{tikzcd}$$

where $\epsilon = \epsilon_n \circ \cdots \circ \epsilon_1$ is the composite of n monoidal transformations and $\pi : \tilde{S} \to C$ is a morphism. Assume that n is minimal and ≥ 1 and let $E_n \subset \tilde{S}$ denote the exceptional divisor of the monoidal transformation ϵ_n. As C has genus ≥ 1, the image of E_n by π is a point. So $\pi : \tilde{S} \to C$ factors through ϵ_n [B78, II, Rem. 13 2)], which contradicts the minimality of n. $\square$

3.3.2.3. Case (2). Let $S \in \mathcal{P}(k)$ with $\kappa_S \geq 1$. Recall that an *Iitaka fibration* for S is a rational map $S \dashrightarrow B$ such that $B \in \mathcal{P}(k)$ with dimension κ_S and the generic fiber F of $S \dashrightarrow B$ is connected with Kodaira dimension 0. Let $\omega_S := \wedge^n \Omega^1_{S/k}$ denote the canonical sheaf on S and $\nu(S,m) := \dim_k(\mathrm{H}^0(S, \omega_S^{\otimes m}))$ the mth plurigenus of S, $m \geq 0$. If $\mathcal{N}(S)$ denote the set of all $m \geq 1$ such that $\nu(S,m) \geq 1$ then, for any $m \in \mathcal{N}(S)$, $\omega_S^{\otimes m}$ defines a rational map $\phi_{S,m} : X \dashrightarrow \mathbb{P}_k^{\nu(S,m)-1}$ (called the mth canonical map). Let $d_m : B_{S,m} \to \phi_{S,m}(S)$ denote a desingularization of the Zariski closure of $\phi_{S,m}(S)$ in $\mathbb{P}_k^{\nu(S,m)-1}$. Then the rational map:

$$d_m^{-1} \circ \phi_{S,m} : S \dashrightarrow B_{S,m}$$

is an Iitaka fibration for $m \gg 0$. An Iitaka fibration is birationally unique.

Assume that case (2) occurs. Up to replacing ρ by its restriction $\rho|_{\pi_1(S_n)}$ for some $n \gg 0$, we may assume that S_n is birational to a minimal honest elliptic surface Q_n whose elliptic fibration $Q_n \to C_n$ is an Iitaka fibration for all $n \geq 0$. Hence, fixing birational maps $\varphi_n : S_n \dashrightarrow Q_n$, the projective system:

$$\cdots \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 (= S)$$

induces a profinite commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "\varphi_{n+1}"'] & S_n \arrow[r] \arrow[d, dotted, "\varphi_n"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "\varphi_1"'] & S_0 \arrow[d, dotted, "\varphi_0"'] \\
\cdots \arrow[r, dotted] & Q_{n+1} \arrow[r, dotted] & Q_n \arrow[r, dotted] & \cdots \arrow[r, dotted] & Q_1 \arrow[r, dotted] & Q_0.
\end{tikzcd}$$

Now, the conclusion follows from the general lemma below.

Lemma 3.4. *Let $\cdots \to S_{n+1} \to \cdots \to S_1 \to S_0$ be a projective system of generically finite morphims in $\mathcal{P}(k)$. Assume that $\kappa_{S_0} \geq 0$ Then one can construct a profinite commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "q_{n+1}"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "q_1"'] & S_0 \arrow[d, dotted, "q_0"'] \\
\cdots \arrow[r, dotted] & B_{n+1} \arrow[r, dotted] & \cdots \arrow[r, dotted] & B_1 \arrow[r, dotted] & B_0,
\end{tikzcd}$$

where $q_n : S_n \dashrightarrow B_n$ is an Iitaka fibration and $B_{n+1} \dashrightarrow B_n$ is a dominant rational map, $n \geq 0$.

Proof. Any generically finite morphism $\pi : S' \to S$ in $\mathcal{P}(k)$ is separable hence induces monomorphisms:

$$\mathrm{H}^0(S, \omega_S^{\otimes n}) \hookrightarrow \mathrm{H}^0(S', \omega_{S'}^{\otimes n}),\ n \geq 1.$$

Also, for every $n \geq 1$ such that $\mathrm{H}^0(S, \omega_S^{\otimes n}) \neq 0$ we have monomorphisms:

$$\mathrm{H}^0(S, \omega_S^{\otimes n}) \hookrightarrow \mathrm{H}^0(S, \omega_S^{\otimes rn}),\ r \geq 1.$$

So, let $n_0 \geq 1$ such that $\phi_{S_0,n_0} : S_0 \dashrightarrow B_0$ is an Iitaka fibration and define inductively a sequence of integers $n_i \geq 1$, $i \geq 0$ such that:

$$\phi_{S_i,n_0n_1\cdots n_i} : S_i \dashrightarrow B_i$$

is an Iitaka fibration (such a sequence always exists; see for instance the proof of [E81, Th. 3]). Then we get a sequence of monomorphisms:

$$\mathrm{H}^0(S_0, \omega_{S_0}^{\otimes n_0}) \hookrightarrow \mathrm{H}^0(S_1, \omega_{S_1}^{\otimes n_0n_1}) \hookrightarrow \cdots \hookrightarrow \mathrm{H}^0(S_i, \omega_{S_i}^{\otimes n_0n_1\cdots n_i}) \hookrightarrow \cdots$$

Thus, any compatible system of bases:

$$(\underline{\mathbf{s}}_n = (s_1, \cdots, s_{d_1}, \cdots, s_{d_i}))_{i\geq 0}$$

of the $\mathrm{H}^0(S_i, \omega_{S_i}^{\otimes n_0n_1\cdots n_i})$ produces a commutative diagram as required. $\square$

Remark 3.5. As lemma 3.4 holds for $S \in \mathcal{P}(k)$ of arbitrary dimension, our argument shows that, for $\delta \geq 2$, one has:

Conjecture 1.2 (2) for δ'-dimensional varieties of Kodaira dimension 0, $\delta' \leq \delta$
$+$ Conjecture 1.2 (2) for δ-dimensional varieties of Kodaira dimension $-\infty$
$\Rightarrow$ Conjecture 1.2 (2) for δ-dimensional varieties.

Hence, the difficult cases of conjecture 1.2 are for $\kappa_S = -\infty,\ 0$.

This argument also shows that theorem 1.3 (1) implies conjecture 1.2 (2) for all $S \in \mathcal{B}^3(k)$ with $\kappa_S \geq 1$.

3.3.3. *Cases (5) and (6).* Assume that case (6) occurs and let us show that, then, necessarily, case (5) occurs as well. Up to replacing A by $A \times_S S_n$ for some $n \gg 0$, we may assume that S_n is birational to a minimal Enriques surface E_n for all $n \geq 0$. Hence, fixing birational maps $\varphi_n : S_n \dashrightarrow E_n$, the projective system:

$$\cdots \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 (= S)$$

induces a profinite commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1} \arrow[r] \arrow[d, dotted, "\varphi_{n+1}"'] & S_n \arrow[r] \arrow[d, dotted, "\varphi_n"'] & \cdots \arrow[r] & S_1 \arrow[r] \arrow[d, dotted, "\varphi_1"'] & S_0 \arrow[d, dotted, "\varphi_0"'] \\
\cdots \arrow[r, dotted] & E_{n+1} \arrow[r, dotted] & E_n \arrow[r, dotted] & \cdots \arrow[r, dotted] & E_1 \arrow[r, dotted] & E_0.
\end{tikzcd}$$

Up to replacing S_0 by its regular locus, we may assume that the $\varphi_n : S_n \dashrightarrow E_n$ are open immersions. Also, by elimination of indeterminancy [B78, II, Thm. 7] up to replacing E_n by a birational smooth proper surface (which is not necessarily minimal), we may assume that the rational maps $E_{n+1} \dashrightarrow E_n$ are morphisms. Let $K_n \to E_n$ denote the universal covering of E_n. As $E_n \times_{E_0} K_0 \to E_n$ is a degree $\mathbb{Z}/2$ connected étale cover and $\pi_1(E_n) = \mathbb{Z}/2$, we have $K_n = E_n \times_{E_0} K_0$. But, then, base changing $A \to S$ *via* $S \times_{E_0} K_0 \to S$, we have:

$$(S \times_{E_0} K_0)_{v_n} = S_n \times_S (S \times_{E_0} K_0) = S_n \times_{E_0} K_0 \overset{(1)}{\to} E_n \times_{E_0} K_0 = K_n,$$

where (1) is an open immersion. So $(S \times_{E_0} K_0)_{v_n}$ is birational to a K3 surface and case (5) occurs.

As a result, it is enough to rule out the case where S_n is a K3-surface for all $n \geq 0$. For this, the idea is to use the following uniform boundedness result for finite subgroups of automorphism groups of K3 surfaces.

Theorem 3.6. ([X96, Cor. p. 87]) *A finite group G which acts faithfully on a K3 surface has order $\leq C (= 5760)$.*

According to theorem 3.6, it is enough to prove that S_n has a finite subgroup of its automorphism group of order $> C$ for n large enough. A way to produce finite automorphisms on a surface is to show that this surface can be realized as a Galois cover. In our situation, $S_n = S_{v_n}$ is a Galois cover of $T_n := S_{\langle v_n \rangle}$ with group:

$$\mathrm{Aut}(S_n/T_n) = \mathrm{Stab}_{G_n}(\langle v_n \rangle)/\mathrm{Stab}_{G_n}(v_n) \hookrightarrow \mathrm{Aut}(\langle v_n \rangle) \simeq (\mathbb{Z}/\ell^n)^\times.$$

So it is enough to show that:

$$\lim_{n \to +\infty} [\mathrm{Stab}_{G_n}(\langle v_n \rangle) : \mathrm{Stab}_{G_n}(v_n)] = +\infty,$$

which is equivalent to:

$$\lim_{n \to +\infty} |G_n v_n \cap \langle v_n \rangle| = +\infty.$$

And, setting:

$$G := \rho_{A,\ell^\infty}(\pi_1(S)) \subset \mathrm{GL}(T_\ell(A_\eta)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell),$$

this follows from:

Lemma 3.7. *For any $v \in T_\ell(A_{\overline{\eta}})^\times$, the set $Gv \cap \langle v \rangle$ is infinite.*

Proof. Set $W(v) := \mathbb{Q}_\ell[Gv] \subset V_\ell(A_\eta)$ and $r := \dim(W(v))$. Then $W(v)$ is a simple $\mathbb{Q}_\ell[G]$-module with basis:

$$\underline{\epsilon} = (e_1 = v, e_2, \ldots, e_r)$$

over $\mathbb{Q}_\ell$. Let $\mathcal{L} : W(v) \twoheadrightarrow \mathbb{Q}_\ell$ denotes the projection onto the line $\mathbb{Q}_\ell v$. By definition, $Gv \cap \langle v \rangle$ can be identified with $\mathcal{L}(Gv)$. Thus, we are to prove that $\mathcal{L}(Gv)$ is infinite. Consider the dual $\mathbb{Q}_\ell$-basis $e_1^\vee, \ldots, e_r^\vee$ of $W(v)^\vee := \mathrm{Hom}_{\mathbb{Q}_\ell}(W(v), \mathbb{Q}_\ell)$. Then, by definition, $\mathcal{L} = e_1^\vee$. Given $g \in G$, write $C_{g,i}$ (resp. $R_{g,i}$) for the ith column (resp. row) of the matrix of g written in $\underline{\epsilon}$, $i = 1, \ldots, r$. Then:

$$\mathcal{E} := \mathcal{L}(Gv) = \{\mathcal{L}(gv)\}_{g \in G} = \{\mathcal{L}(gg'v)\}_{g,g' \in G} = \{aR_{g,k}C_{g',1}\}_{g,g' \in G}$$

Since $W(v)$ is a simple $\mathbb{Q}_\ell[G]$-module, $W(v)^\vee$ is a simple $\mathbb{Q}_\ell[G]$-module as well. In particular, the $g^{-1}\mathcal{L} = \mathcal{L}(g-) = R_{g,k}$, $g \in G$ generate $W(v)^\vee$ as a $\mathbb{Q}_\ell$-vector space. Hence, one can fix a $\mathbb{Q}_\ell$-basis of the form $R_{g_1,k}, \ldots, R_{g_r,k}$ for $\mathbb{Q}_\ell(v)^\vee$. The matrix A whose rows are the $R_{g_i,k}$, $i = 1, \ldots, r$ is in $\mathrm{GL}_r(\mathbb{Q}_\ell)$ with the property that $AC_{g,1} \in \mathcal{E}^r$, $g \in G$. Hence:

$$Gv = \{C_{g,1}\}_{g \in G} \subset A^{-1}\mathcal{E}^r.$$

So the conclusion follows from condition (I/ℓ^∞), which ensures that Gv is infinite. □.

3.4. **Proof of theorem 1.3 (1).**

3.4.1. *End of the proof of theorem 1.3 (1).* As explained at the end of subsection 3.2, we are only to prove that, for all

$$v = (v_n)_{n\geq 0} \in T_\ell(A_\eta)^\times$$

and associated projective system:

$$\cdots \to S_{M(v_{n+1})} \to S_{M(v_n)} \to \cdots \to S_{M(v_1)} \to S(= S_{M(v_0)}),$$

the surface $S_{M(v_n)}$ is no longer rational for n large enough. Otherwise, writing $M_n := M(v_n)$, G_{M_n} would be a finite subgroup of the Cremona group $\mathrm{Cr}_2(k)$. But finite subgroups of $\mathrm{Cr}_2(k)$ are well understood and, in particular, one has:

Theorem 3.8. (See [S09, Thm. 3.1]) *There exists an integer $J \geq 1$ such that any finite subgroup G of $\mathrm{Cr}_2(k)$ contains a normal abelian subgroup $A \lhd G$ whose index $[G : A]$ divides J.*

Since $\pi_1(S)$ is topologically finitely generated, there are only finitely many isomorphism classes of étale covers of S with degree $\leq J$ hence at least one of them - say $S' \to S$ - appears infinitely many times among the connected étale covers corresponding to the open subgroups $\rho_{M_n}^{-1}(A_{M_n}) \subset \pi_1(S)$ (where A_{M_n} denotes a normal abelian subgroup of G_{M_n} of index dividing J). So, up to base-changing $A \to S$ *via* $S' \to S$, we may assume that G_{M_n} is abelian for infinitely many $n \geq 0$. But, as already mentioned, one always have:

$$|G_{M_n}| \geq d_A(\ell^n)$$

and, from condition (I/ℓ^∞):

$$\lim_{n\to+\infty} d_A(\ell^n) = +\infty,$$

which contradicts condition (A/ℓ^∞).

Remark 3.9. In subsection 3.3.3, we proved that S_{v_n} carries an automorphism u_n of order going to $+\infty$ with n. Unfortunately, this is not enough to rule out case (8) for $S_{A,1}(\ell^n)$ since, from [Bl06, Thm. 4.6], finite abelian subgroups of the Cremona group $\mathrm{Cr}_2(k)$ are of the following form: $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, for any integers $m, n \geq 1$; $\mathbb{Z}/2n\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$, for any integer $n \geq 1$; $(\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$; $(\mathbb{Z}/3\mathbb{Z})^3$ and $(\mathbb{Z}/2\mathbb{Z})^4$. However, if Δ denotes the dimension of Gv as a ℓ-adic analytic space and if we assume that *all the minimal level ℓ^{n_0}-structure are defined over S* (that is $S_{M(v)} = S$ for all $v \in A_\eta[\ell^{n_0}]^\times$, which also amounts to $S_A[\ell^{n_0}] = S$), one can show that, for any

$$v = (v_n)_{n\geq 0} \in T_\ell(A_\eta)^\times$$

and for n large enough, the surface S_{v_n} carries a finite subgroup of its automorphism group isomorphic to $(\mathbb{Z}/\ell^{n_0})^\Delta$. This essentially follows from [CT09b, Lemma 3.5]. So, under numerical assumptions on Δ, n_0 (depending on whether $\ell = 2, 3$ or not), case (8) for $S_{A,1}(\ell^n)$ can be ruled out as well.

4. Lang conjecture and the ℓ-primary torsion conjectures

The so-called Lang (or Lang-Bombieri) conjecture is the higher dimensional analogue of Mordell conjecture. It predicts that if k is a number field and $S \in \mathcal{P}(k)$ (of dimension > 0) is of general type then $S(k)$ is not Zariski-dense in S. From subsection 2.1, this is equivalent to:

Conjecture 4.1. (Lang) *Let k be a number field and let $S \in \mathcal{B}(k)$ (of dimension > 0). Assume that S is of general type. Then $S(k)$ is not Zariski-dense in S.*

There are only few known cases of Lang conjecture, among which the most significant one is that it holds for subvarieties of general type of abelian varieties [F94]. See *e.g.* [CHM97, §1.1.1] and the references given there for a more detailled account on Lang conjecture and its variants.

The aim of this concluding section is to discuss the connection between conjecture 1.2, Lang conjecture and the (ℓ-primary) torsion conjectures for abelian varieties.

Conjecture 4.2. (Torsion conjectures)
Arithmetic form: *Let k be a finitely generated field of characteristic 0. Then, for every integer $g \geq 1$ there exists an integer $N := N(k, g)$ such that for every g-dimensional abelian variety $\mathfrak{a}$ over k one has:*

$$\mathfrak{a}(k)_{tors} \subset \mathfrak{a}[N].$$

Geometric form: *Let k be an algebraically closed field of characteristic 0. Then, for every function field K/k and integer $g \geq 1$, there exists an integer $N := N(K/k, g)$ such that for every g-dimensional abelian variety $\mathfrak{a}$ over k which contains no non-trivial k-isotrivial abelian subvariety, one has:*

$$\mathfrak{a}(K)_{tors} \subset \mathfrak{a}[N].$$

One can weaken the torsion conjectures by considering only the ℓ-primary torsion submodule

$$\mathfrak{a}(k)[\ell^\infty] := \bigcup_{n \geq 1} \mathfrak{a}(k)[\ell^n] \subset \mathfrak{a}(k)_{tors}$$

(and requiring that the integers $N := N(k, g, \ell)$ and $N := N(K/k, g, \ell)$ depend also on ℓ). We will refer to these weak variants of the torsion conjectures as the ℓ-primary form of the torsion conjectures. Similarly, we will refer to the weak variant of conjecture 1.2 for the $S_{A,1}(\ell^n)$, $n \geq 1$ as the ℓ-primary form of conjecture 1.2.

The only known case of the (ℓ-primary) torsion conjectures is for $g = 1$ (See [M69] for the ℓ-primary arithmetic form and works of B. Mazur, S. Kammienny, L. Merel and others for the whole conjecture [M77], [Me96]; the geometric form simply follows from the fact that the genus of the modular curves $Y_1(n)$ goes to $+\infty$ with n). See *e.g.* [Si01] for an introduction to the folklore of torsion conjectures for abelian varieties.

4.1. **Reformulation of the arithmetic torsion conjecture in terms of abstract modular schemes.**

Lemma 4.3. *Let k be a finitely generated field of characteristic 0. The following three statements are equivalent:*

1. *The arithmetic torsion conjecture;*
2. *For every $S \in \mathcal{B}(k)$, abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety and n large enough, one has:*
$$S_{A,1}(n)(k) = \emptyset;$$
3. *For every $S \in \mathcal{B}(k)$, abelian scheme $A \to S$, and n large enough, one has:*
$$S_{A,1}(n)(k) = \emptyset.$$

Proof. (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are straightforward. For (3) $\Rightarrow$ (1), we first observe that the arithmetic form of conjecture 4.2 is equivalent to the arithmetic form of conjecture 4.2 for principally polarized abelian varieties. This follows from 'Zarhin's trick' ([Mi86, Rem. 16.12]), according to which any g-dimensional abelian variety over k embeds into a $8g$-dimensional principally polarized abelian variety over k, namely $(A \times A^\vee)^4$. Next, for any integer $g \geq 1$, one can a scheme $S_g \in \mathcal{B}(\mathbb{Q})$ with generic points η and an abelian scheme $A_g \to S_g$ such that any g-dimensional principally polarized abelian variety is a fiber of $A_g \to S_g$ (representability of Hilbert schemes - see [CT10b, §2.1.3], where the scheme built there is actually known to be smooth and irreducible). The conclusion follows from the fact that $A_{g,i\overline{\eta}_i}$ is then automatically simple (since $\mathrm{End}(A_{g\overline{\eta}_i})$ embeds into any of the endomorphism rings $\mathrm{End}(A_{g,i\overline{s}})$, $s \in S_{g,i}$) hence contains no non-trivial weakly $\overline{\mathbb{Q}}$-isotrivial abelian subvariety. □

Lemma 4.4. *Lang conjecture and the ℓ-primary form of conjecture 1.2 imply the ℓ-primary arithmetic torsion conjecture.*

Proof. Assume first that k is a number field. From lemma 4.3, it is enough to prove that Lang conjecture and the ℓ-primary form of conjecture 1.2 imply the ℓ-primary form of assertion (2) of lemma 4.3. So, let $S \in \mathcal{B}^{\delta}(k)$ and $A \to S$ an abelian scheme such that $A_{\overline{\eta}}$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety. Observe that if $\delta = 0$, the statement of lemma 4.4 holds unconditionally (this is just Mordell-Weil theorem). So, assume that $\delta > 0$. We are to show that:

$$S_{A,1}(\ell^n)(k) = \emptyset,$$

for n large enough.

Otherwise, we would have $S_{A,1}(\ell^n)(k) \neq \emptyset$ for all $n \geq 0$. Let T_n denote the reduced closed subscheme of $S_{A,1}(\ell^n)$ associated with the Zariski closure of $S_{A,1}(\ell^n)(k)$ and let δ_n denote the maximal dimension of the irreducible components of T_n. One has $\delta_{n+1} \leq \delta_n \leq \delta$ for $n \geq 0$. So, up to replacing S by $S_{A,1}(\ell^n)$ for some n large enough, we may assume that $\delta_n = \delta_0 \leq \delta$ for all $n \geq 0$. Write $\mathrm{Irr}_{\delta_0}(T_n)$ for the non-empty finite set of δ_0-dimensional irreducible components of T_n. Since the transition morphisms $S_{A,1}(\ell^{n+1}) \to S_{A,1}(\ell^n)$ are finite, they induce a projective system of non-empty finite sets:

$$(\mathrm{Irr}_{\delta_0}(T_{n+1}) \to \mathrm{Irr}_{\delta_0}(T_n))_{n\geq 0}$$

hence:

$$\varprojlim \mathrm{Irr}_{\delta_0}(T_n) \neq \emptyset.$$

So one can consider a projective subsystem of δ_0-dimensional irreducible components $(U_{n+1} \to U_n)_{n\geq 0}$ of $(T_{n+1} \to T_n)_{n\geq 0}$. If $\delta_0 = 0$, this contradicts Mordell-Weil theorem. If $\delta_0 > 0$ then, since U_n is irreducible, U_n is contained in S_{v_n} for some $v_n \in A_\eta[\ell^n]^\times$. Up to replacing U_n by a dense open subscheme, we may furthermore assume that U_n is normal and, base changing $A \to S$ *via* $U_0 \to S$, one gets a dominant morphism $U_n \to U_{0v_n}$. In particular, $U_{0v_n}(k)$ is Zariski-dense in U_{0v_n} since, by definition, $U_n(k)$ is Zariski dense in U_n. But from the ℓ-primary form of conjecture 1.2 for δ_0-dimensional schemes applied to $A \times_S U_0$, we may assume that U_{0v_n} is of general type for n large enough, which contradicts conjecture 4.1 for δ_0-dimensional schemes. Hence $S_{A,1}(\ell^n)(k) = \emptyset$ for n large enough as announced.

The conclusion for an arbitrary finitely generated field of characteristic 0 now follows from:

Claim: *The arithmetic form of conjecture 4.2 is equivalent to the arithmetic form of conjecture 4.2 for number fields.*

Proof of the claim The argument is rather standard. Let K is a finitely generated field of characteristic 0 and $\mathfrak{a}$ an abelian variety over K. Let k denote the algebraic closure of $\mathbb{Q}$ in K. Then there exists a smooth irreducible scheme S over k with generic point η and such that:

- $S(k) \neq \emptyset$;
- $k(\eta) = K$;
- $\mathfrak{a} \to \mathrm{spec}(K)$ extends to an abelian scheme $A \to S$.

So, the action of Γ_K on $\mathfrak{a}[\ell^\infty]$ factors through $\Gamma_K \twoheadrightarrow \pi_1(S)$. Now, given $s \in S(k)$, one can again consider the specialization isomorphism:

$$sp_s : \mathfrak{a}(\overline{K})_{tors} = (A_{\overline{\eta}}(k(\overline{\eta})))_{tors} \tilde{\to} (A_{\overline{s}}(\overline{k}))_{tors},$$

which, we recall, is Galois-equivariant in the sense that for all $\tau \in \Gamma_k$, $v \in (A_{\overline{\eta}}(k(\overline{\eta})))_{tors}$, one has:

$$sp_s(\sigma_s(\tau) \cdot v) = \tau \cdot sp_s(v).$$

In particular, if $\mathfrak{a}$ has a K-rational point of order exactly ℓ^n then A_s has a k-rational point of order exactly ℓ^n as well. □

The proof of lemma 4.4 shows, more precisely, that when k is a number field conjecture 4.1 and the ℓ-primary form of conjecture 1.2 for all $S \in \mathcal{B}^{\delta'}(k)$, $\delta' \leq \delta$ imply the ℓ-primary form of assertion (2) of lemma 4.3 for all $S \in \mathcal{B}^{\delta'}(k)$, $\delta' \leq \delta$. One may ask whether they also imply the ℓ-primary form of assertion (3) of lemma 4.3 for all $S \in \mathcal{B}^{\delta'}(k)$, $\delta' \leq \delta$. This is true for $\delta \leq 2$ but unclear for $\delta \geq 3$.

Lemma 4.5. *Assume that k is a number field. Then, the ℓ-primary form of conjecture 1.2 and conjecture 4.1 for all $S \in \mathcal{B}^{\delta}(k)$, $\delta \leq 2$ imply the ℓ-primary form of assertion (3) of lemma 4.3 for all $S \in \mathcal{B}^{\delta}(k)$, $\delta \leq 2$.*

Proof. Under the hypotheses of lemma 4.5, given $S \in \mathcal{B}^{\delta}(k)$ with $\delta \leq 2$ and an abelian scheme $A \to S$ we are to prove that there exists an integer $N := N(A, \ell)$ such that $S_{A,1}(\ell^n)(k) = \emptyset$, $n \geq N$. When $\delta \leq 1$, the following stronger assertion holds:

Lemma 4.6. ([CT09b, Cor. 4.3.1]) *Let k be a finitely generated field of characteristic* 0. *For any $S \in \mathcal{B}^1(k)$, abelian scheme $A \to S$, integer $b \geq 1$ and prime ℓ there exists an integer $n := n(A, \ell, b)$ such that:*

$$A_s(k(s))[\ell^\infty] \subset A_s[\ell^n],\ s \in S^{\leq b}.$$

So, we are only to deal with the case $\delta = 2$, which we assume from now on.

Given a function field K/k and an abelian variety $\mathfrak{a}$ over K, let $\mathfrak{a}_0$ denote the largest $\overline{k}$-isotrivial abelian subvariety of $\mathfrak{a}$ (see [CT08, §2.1]). By Poincaré's complete reducibility theorem, A_η is isogenous to $(A_\eta)_0 \times \mathfrak{b}$ for some abelian subvariety $\mathfrak{b} \subset A_\eta$ such that $\mathfrak{b}_0 = 0$. Again, by Poincaré's complete reducibility theorem, $\mathfrak{b}$ is isogenous to a product:

$$\mathfrak{b}_1 \times \cdots \times \mathfrak{b}_n \times \mathfrak{b}^0,$$

where $\mathfrak{b}^0$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety and, for each $i = 1, \ldots, n$, $\mathfrak{b}_i$ is weakly $\overline{k}$-isotrivial but not $\overline{k}$-isotrivial that is, there exists a curve $T_i \in \mathcal{B}(k)$, a dominant morphism $S \to T_i$, a finite morphism $T'_i \to T_i$ and an abelian scheme $B'_i \to T'_i$ (whose geometric generic fibre contains no non-trivial $\overline{k}$-isotrivial abelian subvariety) such that, if η'_i denotes the generic point of $S'_i := S \times_{T_i} T'_i$, then $(B'_i \times_{T'_i} S'_i)_{\eta'_i} = \mathfrak{b}_i \times_{k(\eta)} k(\eta'_i)$. Let $\nu \geq 0$ denote the maximal integer such that ℓ^ν divides the degree of the kernel of the isogeny:

$$A_\eta \to (A_\eta)_0 \times \mathfrak{b}_1 \times \cdots \times \mathfrak{b}_n \times \mathfrak{b}^0.$$

Up to replacing $N(A, \ell)$ by $N(A, \ell) + \nu$ in the statement of lemma 4.5, we may assume that $\nu = 0$ hence that:

$$A_\eta[\ell^\infty] = (A_\eta)_0[\ell^\infty] \oplus \mathfrak{b}_1[\ell^\infty] \oplus \cdots \oplus \mathfrak{b}_n[\ell^\infty] \oplus \mathfrak{b}^0[\ell^\infty].$$

For X one of the abelian varieties $(A_\eta)_0,\ \mathfrak{b}_1, \cdots, \mathfrak{b}_n,\ \mathfrak{b}^0$, set:

$$S_X(\ell^n) := \bigsqcup_{v \in X[\ell^n] \smallsetminus \ell X[\ell^n]} S_v.$$

Then the projection $A_\eta[\ell^\infty] \twoheadrightarrow X[\ell^\infty]$ induces by functoriality an étale cover:

$$S_{A,1}(\ell^n) \to S_X(\ell^n).$$

Thus it is enough to show that $S_{A,1}(\ell^n)(k) = \emptyset$ for $\ell \gg 0$ in each of the following cases:

(1) $A_{\overline{\eta}}$ is $\overline{k}$-isotrivial;
(2) $A_{\overline{\eta}}$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety;
(3) There exists a curve $T \in \mathcal{B}(k)$, a dominant morphism $S \to T$, a finite morphism $T' \to T$ and an abelian scheme $B' \to T'$ such that, if η' denotes the generic point of $S' := S \times_T T'$, then $(B' \times_{T'} S')_{\eta'} = A_\eta \times_{k(\eta)} k(\eta')$.

Case (1) follows from [CT10a, Prop. 3.18] and case (2) is lemma 4.4 for $\delta = 2$.

Case (3) follows from lemma 4.6. Indeed, observe first that one can replace freely S by a non-empty open suscheme U. Indeed, then, the closed reduced subscheme associated with $S \backslash U$ is a curve and, up to remove finitely many points - say $s_1, \ldots, s_r$ - we may assume that it is normal. In particular, it is the disjoint union of its irreducible (=connected) components. Each of these connected components C is a curve in $\mathcal{B}(k)$ so considering $A \times_S C \to C$ it follows from lemma 4.6 that there is no points in $S_{A,1}(\ell^n)(k)$ above $C(k)$ for $n \gg 0$. Also, it follows from Mordell-Weil theorem that there is no points in $S_{A,1}(\ell^n)(k)$ above $s_1, \dots, s_r$ for $n \gg 0$.

In particular, we may assume that $T' \to T$ is an étale cover hence that $S' \to S$ is an étale cover as well. By assumption, there exists an open subscheme $U \subset S'$ such that $A \times_S U = B' \times_{T'} U$. Up to replacing S by the image of U in S, which is again open since $S' \to S$ is flat, we may assume that $U = S'$. Set $b := \deg(T' \to T)$. Then $S'_{A\times_S S',1}(\ell^n)$ is the normalization of $T'_{B,1}(\ell^n) \times_{T'} S'$, which is normal (being an étale cover of the normal scheme S') hence:

$$S'_{A\times_S S',1}(\ell^n) = T'_{B,1}(\ell^n) \times_{T'} S'.$$

Now, given $v \in A_\eta[\ell^n]$ of order ℓ^n with image v' in $A_\eta \times_{k(\eta)} k(\eta')$, let k_v and $k_{v'}$ denote the field of definition of S_v and $S'_{v'}$ respectively. Again, $S'_{v'} = S_v \times_S S'$ and, in particular, we have $[k'_{v'} : k_v] \leq b$. If $[k_v : k] > 1$ then $S_v(k) = \emptyset$. If $k = k_v$ and $S_v(k) \neq \emptyset$ then any point $s \in S_v(k)$ lifts to a point $s' \in S'_{v'}$ such that $[k(s') : k_{v'}] \leq b$ hence $[k(s') : k] \leq b^2$. The image t' of s' in $T'_{B',1}(\ell^n)$ then satisfies $[k(t') : k] \leq b^2$. But, with the notation of lemma 4.6, this is only possible for $n \leq n(B, \ell, b^2)$. □

As a result, theorem 1.3 implies the following uniform boundedness statement for the ℓ-primary torsion in families of abelian varieties parametrized by surfaces.

Corollary 4.7. *Assume that conjecture 4.1 holds for surfaces. Let k be a number field, $S \in \mathcal{B}^2(k)$ a surface which is not rational and $A \to S$ an abelian scheme. Then, for every prime ℓ there exists an integer $N := N(A, \ell) \geq 1$ such that, for every $s \in S(k)$, one has:*

$$A_s(k)[\ell^\infty] \subset A_s[\ell^N].$$

4.2. **A remark about the geometric form of conjecture 4.2.** We begin with the following general observation.

Lemma 4.8. *Let $S \in \mathcal{B}^2(k)$ which is not of general type. Then there exists a curve $B \in \mathcal{B}^1(k)$ with generic point ζ_B, a flat family $C \to B$ of genus ≤ 2 curves and a dominant morphism $C_{\zeta_B} \to S$.*

Proof. It is enough to prove the assertion of lemma 4.8 for $S \in \mathcal{P}^2(k)$. From the Enriques-Kodaira classification for surfaces, the only non-obvious case of lemma 4.8 are for $S \in \mathcal{B}^2(k)$ with $\kappa_S = 0$. For an abelian surface S, this follows from the fact that S is always isogenous to a principally polarized abelian surface $S^\#$ which, itself, is either a product of two elliptic curves or the jacobian of a genus 2 curve C (hence birational to $C \times_k C$). For a K3 surface S, this follows from the fact that S always contains a curve C of genus 1 [MoMu83] and that, up to replacing C by its normalization, one gets a morphism $C \to S$ from an elliptic curve to S which is birational onto its image. But such morphisms can always be deformed into a flat family satisfying the properties of lemma 4.8 (*e.g.* [H03, Prop. 10.6]). For an Enriques surfaces S, this follows again from the fact that the universal cover of S is a K3 surface (more generally, it is known that Enriques surfaces always admit elliptic fibrations [Ho78]). □

Among the classical conjectures alluded to in this paper, the geometric form of conjecture 4.2 is probably the most accessible one and it seems that diophantine geometers seriously believe in it. By a straightforward Weil restriction argument (see [CT10a, Footnote 1]), the geometric form of conjecture 4.2 is equivalent to the following apparently stronger uniform boundedness statement.

Conjecture 4.9. *Let k be an algebraically closed field of characteristic 0. Then, for any integers γ, $g \geq 1$, there exists an integer $N := N(k, \gamma, g)$ such that for any smooth, proper connected curve C over k of k-gonality $\leq \gamma$ and g-dimensional abelian variety $\mathfrak{a}$ over $k(C)$ containing no non-trivial k-isotrivial abelian subvariety, one has:*

$$\mathfrak{a}(k(C))_{tors} \subset \mathfrak{a}[N].$$

So, from lemma 4.8 the geometric form of conjecture 4.2 implies the following uniform variant of conjecture 1.2 (2) for surfaces. *Let k be an algebraically closed field of characteristic* 0*. Then, for any $S \in \mathcal{B}^2(k)$ and integer $g \geq 1$ there exists an integer $N := N(g) \geq 1$ such that for any abelian scheme $A \to S$ with* $\dim(A_\eta) = g$ *and such that $A_{\overline{\eta}}$ contains no non-trivial weakly k-isotrivial abelian subvariety, all the connected components of $S_{A,1}(n)$ are of general type for $n \geq N$.*

For full level n-structures, one has the following non-conjectural (but weak, in the sense that k-gonality is replaced by genus) analogue of conjecture 4.9.

Theorem 4.10. ([HwT06, Thm. 1.3]) *For any integers $g \geq 1$, $a \geq 0$ there exists an integer $N := N(g,a)$ such that for any curve $S \in \mathcal{B}^1(\mathbb{C})$ and abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ is a g-dimensional principally polarized abelian variety over $\mathbb{C}(\overline{\eta})$ containing no non-trivial $\mathbb{C}$-isotrivial abelian subvariety, the genus of $S_A[n]$ is $\geq a$ for $n \geq N$.*

Again by Zarhin's trick, one can remove the assumption that $A_{\overline{\eta}}$ is principally polarized in the statement of theorem 4.10 and, by the standard descent argument, the statement of theorem 4.10 remains true for algebraically closed field of characteristic 0. As a result, from lemma 4.8 and theorem 4.10, one deduces the following uniform form of conjecture 1.2 (2) for $S_A[n]$, when $S \in \mathcal{B}^2(k)$.

Corollary 4.11. *Let k be an algebraically closed field of characteristic* 0*. Then, for any integer $g \geq 1$ there exists an integer $N := N(g) \geq 1$ such that for any $S \in \mathcal{B}^2(k)$ and abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ is a g-dimensional abelian variety over $k(\overline{\eta})$ containing no non-trivial weakly k-isotrivial abelian subvariety, $S_A[n]$ is of general type for $n \geq N$.*

The above observations motivate the following question, which generalizes lemma 4.8.

Question 1: *Let k be an algebraically closed field of characteristic* 0*. Given an integer $\delta \geq 1$ does there exists an integer $a(\delta) \geq 0$ such that for any $S \in \mathcal{B}^\delta(k)$ of Kodaira dimension ≤ 0 there exists $B \in \mathcal{B}^{\delta-1}(k)$ with generic point ζ_B, a flat family of genus $\leq a(\delta)$ curves $C \to B$ and a dominant morphism $C_{\zeta_B} \to S$?*

The condition that $\dim(B) = \delta - 1$ in question 1 ensures that, given an abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ contains no non-trivial weakly k-isotrivial abelian subvariety then $A \times_S C_{\zeta_B} \to C_{\zeta(b)}$ contains no non-trivial $k(\overline{\zeta}_B)$-isotrivial subvariety.

A positive answer to question 1 would imply:

- Let k be an algebraically closed field of characteristic 0. Then, for any integer $g \geq 1$ there exists an integer $N := N(g) \geq 1$ such that for any $S \in \mathcal{B}(k)$ and abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ is a g-dimensional abelian variety over $k(\overline{\eta})$ containing no non-trivial weakly k-isotrivial abelian subvariety, $S_A[n]$ is of general type for $n \geq N$.

- Assume the geometric form of conjecture 4.2 and let k be an algebraically closed field of characteristic 0. Then, for any $S \in \mathcal{B}(k)$ and integer $g \geq 1$ there exists an integer $N := N(g) \geq 1$ such that for any abelian scheme $A \to S$ with $\dim(A_\eta) = g$ and such that $A_{\overline{\eta}}$ contains no non-trivial weakly k-isotrivial abelian subvariety, all the connected components of $S_{A,1}(n)$ are of general type for $n \geq N$.

Actually, for this purpose, it would be enough to answer positively question 2 below, which seems more accessible. This follows from:

Lemma 4.12. *Let k be an algebraically closed field of characteristic* 0*. Then, for any $S \in \mathcal{B}(k)$ and abelian scheme $A \to S$ such that $A_{\overline{\eta}}$ is weakly k-isotrivial, the set of curves C contained in S such that the geometric generic fibre of $A \times_S C \to C$ is k-isotrivial is not Zariski dense in S.*

Proof. First, $A_{\overline{\eta}}$ is weakly k-isotrivial if and only if $\mathfrak{a} := (A_{\overline{\eta}} \times_{k(\overline{\eta})} A_{\overline{\eta}}^{\vee})^4$ is. From Zarhin's trick, $\mathfrak{a}$ is principally polarized and since the Néron-Severi group of an abelian variety is invariant under algebraically closed field extension, it follows that $\mathfrak{a}$ is weakly k-isotrivial if and only if for any principal polarization $\lambda : \mathfrak{a} \tilde{\rightarrow} \mathfrak{a}^{\vee}$, the pair $(\mathfrak{a}, \lambda)$ is. So, choose any principal polarization $\lambda : \mathfrak{a} \tilde{\rightarrow} \mathfrak{a}^{\vee}$ on $\mathfrak{a}$. This defines a morphism:

$$f : \mathrm{spec}(k(\eta)) \rightarrow A_{g,1},$$

where $g = \dim(A_{\eta})$ and $A_{g,1}$ denotes the coarse moduli scheme of principally polarized g-dimensional abelian varieties. Let x denote the image of this morphism and $Z \hookrightarrow A_{g,1}$ the reduced subscheme associated with the Zariski closure of x in $A_{g,1}$. Saying that $(\mathfrak{a}, \lambda)$ is not weakly k-isotrivial is equivalent to saying that the induced morphism $f : \mathrm{spec}(k(\eta)) \rightarrow Z$ is generically finite. Also, being of finite type, it extends to a generically finite morphism:

$$f^0 : U \rightarrow Z,$$

where $U \subset S$ is a non-empty open subscheme. Up to shrinking U, we may assume that $f^0 : U \rightarrow Z$ is quasi-finite. As a result, any curve C contained in S such that the geometric generic fibre of $A \times_S C \rightarrow C$ is k-isotrivial is necessarily contained in $S \setminus U$. □

Question 2: *Let k be an algebraically closed field of characteristic 0. Given an integer $\delta \geq 1$ does there exists an integer $a(\delta) \geq 0$ such that any $S \in \mathcal{B}^{\delta}(k)$ of Kodaira dimension ≤ 0 is the Zariski closure of the curves of genus $\leq a(\delta)$ it contains?*

This question is answered positively for abelian varieties in [CT10b].

References

[B78] A. Beauville, *Surfaces algébriques complexes*, Astérisque **54**, S.M.F., 1978.

[Bl06] J. Blanc, *Finite abelian subgroups of the Cremona group of the plane*, C.R. Acad. Sci. Paris Ser. I **344**, p. 21-26, 2006.

[CT08] A. Cadoret and A. Tamagawa, *Uniform boundedness of p-primary torsion on abelian schemes*, preprint 2008.

[CT09a] A. Cadoret and A. Tamagawa, *A uniform open image theorem for ℓ-adic representations I*, preprint 2009.

[CT09b] A. Cadoret and A. Tamagawa, *A uniform open image theorem for ℓ-adic representations II*, preprint 2009.

[CT10a] A. Cadoret and A. Tamagawa, *On a weak variant of the geometric torsion conjecture*, preprint 2010.

[CT10b] A. Cadoret and A. Tamagawa, *On a weak variant of the geometric torsion conjecture II*, preprint 2010.

[CT10b] A. Cadoret and A. Tamagawa, *A note on torsion conjecture*, preprint 2010.

[CHM97] L. Caporaso, J. Harris and B. Mazur, *Uniformity of Rational Points*, Journal of A.M.S., **10**, p. 1-35, 1997.

[D71] P. Deligne, *théorie de Hodge II*, Publ. Math. I.H.E.S. **40**, 1971.

[E81] H. Esnault, *Classification des variétés de dimension* 3 *et plus*, Séminaire Bourbaki 1980-1981, **568**, p. 111-131, 1980-1981.

[FW92] G. Faltings, G. Wüstholz (eds.), *Rational Points*, Aspects of Mathematics, E6, Friedr. Vieweg & Sohn, 1992.

[F94] G. Faltings, *The general case of S. Lang's Conjecture*, Barsotti Symposium in Algebraic Geometry, Perspectives in Mathematics , p. 175-182, Academic Press, Inc. 1994.

[Fr94] G. Frey, *Curves with infinitely many points of finite degree*, Israel Journal of Math. **85**, p. 79-85, 1994.

[SGA1] A. Grothendieck, *Revêtements étales et groupe fondamental - S.G.A.1*, L.N.M. **224**, Springer-Verlag, 1971.

[EGAIV2] A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique IV 2 - E.G.A.IV 2: Etude locale des schémas et des morphismes de schémas, seconde partie*, Publ. Math. I.H.E.S. **24**, 1965.

[H03] B. Hassett, *Potential density of rational points on algebraic varieties* in *Higher Dimensional Varieties and Rational Points*, K.J. Boroczky, J. Kollar, T. Szamuely eds., Bolyai Society Mathematical Studies, Vol. **12**, Springer Verlag, 2003

[Hi64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. **79**, I: p. 109-203, II: p. 205-326, 1964.

[Ho78] E. Horikawa, *On the periods of Enriques surfaces I*, Mathematische Annalen **234**, p. 73-88, 1978.

[HwT06] J.-M. Hwang and W.-K. To, *Uniform boundedness of level structures on abelian varieties over complex function fields*, Mathematische Annalen **335** (2), p. 363-377, 2006.

[LNe59] S. Lang and A. Néron, *Rational points of abelian varieties over function fields*, American Journal of Mathematics **81**, p. 95-118, 1959.

[M69] Ju. I. Manin, *The p-torsion of elliptic curves is uniformly bounded* (Russian), Izv. Akad. Nauk SSSR. Ser. Mat. **33**, p. 459-465, 1969.

[M77] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Etudes Sci. Publ. Math. **47**, p. 33-186, 1977.

[Me96] L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124**, p. 437-449, 1996.

[Mi86] J. Milne, *Abelian varieties*, in *Arithmetic Geometry*, G.Cornell and J.H. Silverman eds., Springer Verlag, 1986.

[MoMu83] S. Mori and S. Mukai *The uniruledness of the moduli space of curves of genus 11*, in Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. **1016**, Springer, Berlin, 1983.

[N62] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto **2**, 1962.

[N63] M. Nagata, *A generalization of the imbedding problem*, J. Math. Kyoto **3**, 1963.

[RZ00] L. Ribes and P. Zalesskii, *Profinite groups*, E.M.G. **40**, 2000.

[S09] J.-P. Serre, *Le groupe de Cremona et ses sous-groupes finis*, Séminaire Bourbaki, **1000**, 2008-2009.

[Si01] A. Silvernerg, *Open questions in arithmetic algebraic geometry* in *Arithmetic algebraic geometry (Park City, UT, 1999)*, IAS/Park City Math. Ser., **9**, Amer. Math. Soc., Providence, RI, p.83-142, 2001.

[X96] G. Xiao, *Galois covers between $K3$ surfaces*, Ann. Institut Fourier **46** no1, p. 73-88, 1996.

anna.cadoret@math.u-bordeaux1.fr

Institut de Mathématiques de Bordeaux - Université Bordeaux 1,

351 Cours de la Libération,

F33405 TALENCE cedex, FRANCE.