RELAXATION APPROXIMATION OF SOME INITIAL-BOUNDARY VALUE PROBLEM FOR P-SYSTEMS *

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Abstract. We consider the Suliciu model which is a relaxation approximation of the p-system. In the case of the Dirichlet boundary condition we prove that the local smooth solution of the p-system is the zero limit of the Suliciu model solutions.

Key words. Zero relaxation limit, p-system, Suliciu model, boundary conditions.

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1. Introduction

We study a relaxation approximation of the following p-system

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \\ \partial_t u_2 - \partial_x p(u_1) = 0. \end{cases} \tag{1.1}$$

For the viscoelastic case, Suliciu introduces in [19] the following approximation

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \\ \partial_t u_2 - \partial_x v = 0, \\ \\ \partial_t v - \mu \partial_x u_2 = \dfrac{1}{\varepsilon}(p(u_1) - v), \end{cases} \tag{1.2}$$

where ε and μ are positive.
The aim of this paper is to prove convergence results for the initial-boundary value problem when the relaxation coefficient ε tends to zero.
Under the classical assumption

$$\forall \xi \in \mathbb{R}, p'(\xi) > 0, \tag{1.3}$$

the p-system is strictly hyperbolic with eigenvalues

$$\lambda_1(u_1) = -\sqrt{p'(u_1)} < \lambda_2(u_1) = \sqrt{p'(u_1)}. \tag{1.4}$$

The semi-linear approximation system (1.2) is strictly hyperbolic with 3 constant eigenvalues

$$\mu_1 = -\sqrt{\mu} < \mu_2 = 0 < \mu_3 = \sqrt{\mu}. \tag{1.5}$$

In all the paper we assume that μ is chosen great enough so that the subcharacteristic-type condition holds

$$\mu > p'(u_1) \tag{1.6}$$

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for all the values of u_1 under consideration.
Formally, when ε tends to zero, the behaviour of the solution $w^\varepsilon=(u^\varepsilon,v^\varepsilon)=((u_1^\varepsilon,u_2^\varepsilon),v^\varepsilon)$ for the relaxation system (1.2) is the following: $p(u_1^\varepsilon)-v^\varepsilon$ tends to zero, so that u^ε tends to a solution $u=(u_1,u_2)$ of the p-system (1.1).

Recent papers are devoted to the zero relaxation limit in the case of the Cauchy problem. In [22] Wen-An Yong establishes a general framework to study the strong convergence for the smooth solutions. This convergence result is obtained describing the boundary layer which appears at $t=0$. We can apply Yong's tools for the Suliciu approximation

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu\partial_x u_2^\varepsilon = \dfrac{1}{\varepsilon}(p(u_1^\varepsilon)-v^\varepsilon), \end{cases} \tag{1.7}$$

for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$, with the smooth initial data:

$$w^\varepsilon(0,x)=w_0(x), x\in\mathbb{R}. \tag{1.8}$$

We give more details about this question in the annex at the end of this paper.
Since the lifespan for a smooth solution u of the Cauchy problem for the p-system is generally finite (see [12]), the strong convergence of the solution u^ε to u can only be obtained locally in time. Nevertheless, under the assumption

$$\forall\,\xi\in\mathbb{R}, p'(\xi)\le\Gamma<\mu, \tag{1.9}$$

if w_0 is smooth, the solution for the semi-linear Cauchy problem (1.7)-(1.8) is global and smooth. In this case, the question is: what about the global convergence ?
Under further additional assumptions (in particular $p'(\xi)\ge\gamma>0$) the weak convergence to a global weak solution of the p-system is obtained by Tzavaras in [21] using the compactness methods of [17].
Other convergence results in some particular cases can be found in [8] and [10].
For other connected papers see also [13, 16, 20]...

In this paper we study the zero relaxation limit for the initial-boundary value problem. To our knowledge general convergence results are not available for hyperbolic relaxation systems in domains with boundary in the literature. A special well investigated problem is the semi-linear relaxation approximation to the boundary value problem for a scalar quasilinear equation, see [11, 15, 9, 14], and [5, 1] for related numerical considerations.
A first example of convergence result for a particular p-system (1.1) is obtained in [4]. In that paper the p-system is the one-dimensionnal Kerr model, so p is the inverse function of $\xi\mapsto(1+\xi^2)\xi$. The relaxation approximation is given by the Kerr-Debye model which is the following quasilinear hyperbolic system

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x\left((1+v^\varepsilon)^{-1}u_1^\varepsilon\right) = 0, \\ \partial_t v^\varepsilon = \dfrac{1}{\varepsilon}\left((1+v^\varepsilon)^{-2}(u_1^\varepsilon)^2 - v^\varepsilon\right). \end{cases}$$

For these two models we consider the ingoing wave boundary condition. In the case of the smooth solutions we obtained a local strong convergence result. The main tool of the proof is the use of the entropic variables as proposed in [7]. In these variables, the system is symmetrized and the equilibrium manifold is linearized.

Here we study the zero relaxation limit for the Suliciu approximation

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \dfrac{1}{\varepsilon}(p(u_1^\varepsilon) - v^\varepsilon), \end{cases} \tag{1.10}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$, with the null initial data

$$w^\varepsilon(0,x) = 0, x \in \mathbb{R}^+, \tag{1.11}$$

and with the Dirichlet boundary condition

$$u_2^\varepsilon(t,0) = \varphi(t), t \in \mathbb{R}^+. \tag{1.12}$$

For the null initial data to be in equilibrium we assume that $p(0) = 0$. We prove the strong convergence of u^ε to the smooth solution of the initial-boundary value problem for the p-system

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \partial_t u_2 - \partial_x p(u_1) = 0, \end{cases} \tag{1.13}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$, with the initial-boundary conditions

$$u(0,x) = 0, x \in \mathbb{R}^+, \tag{1.14}$$

$$u_2(t,0) = \varphi(t), t \in \mathbb{R}^+. \tag{1.15}$$

2. Main Results

Let us specify the assumptions on the source term φ in the boundary condition (1.12) or (1.15). In order to simplify we chose φ smooth enough on $\mathbb{R}$ and such that supp $\varphi \subset [0,b]$, with $b > 0$. In this case the boundary conditions and the null initial data (1.11) and (1.14) match each other so both initial-boundary value problem (1.10)-(1.11)-(1.12) and (1.13)-(1.14)-(1.15) admit local smooth solutions.

First we consider the solutions for the second problem (1.13)-(1.14)-(1.15) and using the methods of [12] we establish that the lifespan T^* is generally finite with formation of shock waves.

Theorem 2.1. *Assume the property (1.3). Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ with supp $\varphi \subset [0,b]$, $b > 0$, $\varphi \neq 0$. Let g the function defined by*

$$g(\xi) = \int_0^\xi \sqrt{p'(s)}\, ds.$$

We assume that

$$p'' \text{ does not vanish on the interval } g^{-1}(-\varphi(\mathbb{R})). \tag{2.1}$$

Then the local smooth solution of (1.13)-(1.14)-(1.15) exhibits a shock wave at the time $T^* < +\infty$ *and we have*

$$\|u\|_{L^\infty([0,T^*]\times\mathbb{R}^+)} \leq C\|\varphi\|_{L^\infty(\mathbb{R})}. \tag{2.2}$$

We now investigate the smooth solutions of the initial-boundary value problem (1.10)-(1.11)-(1.12) for a fixed $\varepsilon > 0$. The system is semi-linear strictly hyperbolic and the boundary $\{x=0\}$ is characteristic. It is easy to prove that the local smooth solution w exists and, if the lifespan T^*_ε is finite, we have

$$\|w\|_{L^\infty([0,T^*_\varepsilon]\times\mathbb{R}^+)} = +\infty \tag{2.3}$$

(for general semi-linear hyperbolic systems, see [18]).
If we assume that p is globally lipschitz we establish that the smooth solutions are global.
Theorem 2.2. *Assume the properties (1.3) and (1.9). Let* $\varphi \in H^3(\mathbb{R})$ *with supp* $\varphi \subset \mathbb{R}^+$. *Then the solution of (1.10)-(1.11)-(1.12) is global and*

$$w \in \mathcal{C}^0(\mathbb{R}^+; H^1(\mathbb{R})),\, \partial_t w \in \mathcal{C}^0(\mathbb{R}^+; L^2(\mathbb{R})). \tag{2.4}$$

Finally, let us describe the convergence result.
Theorem 2.3. *We suppose (1.3). Let* $\varphi \in H^3(\mathbb{R})$ *with supp* $\varphi \subset \mathbb{R}^+$. *We consider a smooth solution* $u = (u_1^0, u_2^0)$ *of (1.13)-(1.14)-(1.15) defined on* $[0, T^*[$. *We suppose that*

$$\mu > \sup_{(t,x)\in[0,T^*[\times\mathbb{R}^+} p'(u_1^0(t,x)). \tag{2.5}$$

Let $T < T^*$. *For* ε *small enough, the relaxation problem (1.10)-(1.11)-(1.12) admits a solution* $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ *defined on* $[0,T]$ *such that*

$$u^\varepsilon = u^0 + \varepsilon u^1_\varepsilon,$$

and there exists a constant K *such that*

$$\|u^1_\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^+))} + \|\partial_t u^1_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \leq K. \tag{2.6}$$

In this result we can remark that no boundary layer appears in the time variable because the null initial data belongs to the equilibrium manifold $\mathcal{V} = \{v = p(u_1)\}$. For the space variable, we have the same boundary condition for both systems, so no space boundary layer appears.
To prove Theorem 2.3 we don't use the method in [4]: as observed in [7], with the entropic variables, we lose the semi-linear character of the system (1.10). We prefer write the following expansion of w^ε

$$w^\varepsilon = w^0 + \varepsilon w^1_\varepsilon = ((u_1^0, u_2^0), p(u_1^0)) + \varepsilon w^1_\varepsilon$$

so that the rest term w^1_ε satisfies a semi-linear hyperbolic system. In order to estimate w^1_ε, we use the conservative-dissipative variables introduced in [2]. With these variables the system is symmetrized and its semi-linear character is preserved. Furthermore by this method we obtain a more precise result : for ε small enough the lifespan T^*_ε is greater that the lifespan T^* of the limit system solution and the convergence is proved on all compact subset of $[0, T^*[$.

3. Proof of Theorem 2.1

We use the methods proposed by Majda in [12] for the Cauchy problem. We denote by l and r the left and right Riemann invariants of the system (1.1):

$$\begin{cases} l = \dfrac{1}{2}(u_2 + g(u_1)), \\ r = \dfrac{1}{2}(u_2 - g(u_1)). \end{cases}$$

These variables define a diffeomorphism which inverse is given by

$$\begin{cases} u_1 = g^{-1}(l-r), \\ u_2 = l + r. \end{cases}$$

These invariants (l,r) satisfy the diagonal system

$$\begin{cases} \partial_t l - \nu(l-r)\partial_x l = 0, \\ \partial_t r + \nu(l-r)\partial_x r = 0, \\ l(0,x) = r(0,x) = 0, x > 0, \\ (l+r)(t,0) = \varphi(t), t > 0, \end{cases} \tag{3.1}$$

where $\nu(l-r) = \sqrt{p'(g^{-1}(l-r))}$. The smooth solution of (3.1) is $(0,r)$ where r is the solution of the scalar equation

$$\begin{cases} \partial_t r + \nu(-r)\partial_x r = 0, \\ r(0,x) = 0, x > 0, \\ r(t,0) = \varphi(t), t > 0. \end{cases} \tag{3.2}$$

Under the assumptions (1.3) and (2.1) we will prove that the lifespan T^* of the solution of the problem (3.2) is finite and that this solution exhibits shock waves in T^*.
For solving (3.2) we can use the method of characteristics. The function r is constant on the characteristic curves which are the straight lines $t = T + \dfrac{1}{\nu(-\varphi(T))}x$, $T \in \mathbb{R}$.
Denoting $\alpha(s) = \dfrac{1}{\nu(-s)}$ we obtain then that

$$r(T,0) = \varphi(T) = r(T + \alpha(\varphi(T))x, x).$$

Let us introduce the mapping

$$(T,X) \mapsto \Phi(T,X) = (t,x) = (T + \alpha(\varphi(T))X, X).$$

This map is a diffeomorphism for $X < \bar{X}$ with

$$\bar{X} = \left[\max_{T \in [0,b]} -\frac{d}{dT}\alpha(\varphi(T)) \right]^{-1}.$$

Under assumption (2.1) we have $0<\bar{X}<+\infty$ and we have

$$\|r\|_{L^\infty(\mathbb{R}^+\times[0,\bar{X}[)}\leq\|\varphi\|_{L^\infty(\mathbb{R})}.$$

The characteristic curves through (0,0) and $(b,0)$ cut the straight line $\{x=\bar{X}\}$ at times
$T_1=\sqrt{p'(0)}^{-1}\bar{X}$ and $T_2=b+\sqrt{p'(0)}^{-1}\bar{X}$ so $T^*\in[T_1,T_2]$.

4. Proof of Theorem 2.2

In this section $\varepsilon>0$ and $\mu>0$ are fixed. We rewrite system (1.10)

$$\partial_t w+A\partial_x w=h(w)$$

where

$$A=\begin{pmatrix}0&-1&0\\0&0&-1\\0&-\mu&0\end{pmatrix}\text{ and }h(w)=\begin{pmatrix}0\\0\\\frac{1}{\varepsilon}(p(u_1)-v)\end{pmatrix}$$

and by (1.3) and (1.9) p is globally lipschitz. As zero is an eigenvalue of the matrix A, the boundary $\{x=0\}$ is characteristic, so for completeness we give the proof of the global existence. Using (2.3) it is sufficient to prove that the solution w is bounded on any domain $[0,T]\times\mathbb{R}^+$. In a first step we lift the boundary condition (1.12). We set $\omega(t,x)=\varphi(t)\eta(x)$ where η is a smooth function compactly supported with $\eta(0)=1$. We replace u_2 by $u_2-\omega$ and we obtain the following initial-boundary value problem

$$\begin{cases}\partial_t w+A\partial_x w=h(w)+\begin{pmatrix}\partial_x\omega\\-\partial_t\omega\\\mu\partial_x\omega\end{pmatrix},\\ w(0,x)=0,x\in\mathbb{R}^+,\\ u_2(t,0)=0,t\in\mathbb{R}^+.\end{cases}\tag{4.1}$$

We diagonalize the matrix A by the matrix P: $w=PW$ with

$$P=\begin{pmatrix}1&1&1\\\sqrt{\mu}&0&-\sqrt{\mu}\\\mu&0&\mu\end{pmatrix}.$$

We obtain

$$\begin{cases}\partial_t W+\begin{pmatrix}-\sqrt{\mu}&0&0\\0&0&0\\0&0&\sqrt{\mu}\end{pmatrix}\partial_x W=H(W)+\Phi,\\ W(0,x)=0,x\in\mathbb{R}^+,\\ W_1(t,0)-W_3(t,0)=0,t\in\mathbb{R}^+.\end{cases}\tag{4.2}$$

We have $H(W)=P^{-1}h(PW)$ so H is globally lipschitz

$$\exists K>0,|\partial_W H|\leq K.\tag{4.3}$$

In addition, Φ is given by

$$\Phi = P^{-1}\begin{pmatrix} \partial_x \omega \\ -\partial_t \omega \\ \mu \partial_x \omega \end{pmatrix}.$$

We denote by T^* the lifespan of the solution W for system (4.2) and we assume that $T^* < +\infty$. We will prove that $\|W\|_{L^\infty([0,T^*]\times\mathbb{R}^+)} < +\infty$ so that by (2.3) we obtain a contradiction.

L^2 estimate

We take the inner product of the first equation in (4.2) by W and we obtain

$$\frac{1}{2}\frac{d}{dt}\|W\|^2_{L^2(\mathbb{R}^+)} + \int_{\mathbb{R}^+} \sqrt{\mu}(-W_1\partial_x W_1 + W_3 \partial_x W_3)dx = \int_{\mathbb{R}^+} H(W)W\,dx + \int_{\mathbb{R}^+} \Phi W\,dx.$$

Using the third equation in (4.2) and (4.3) we obtain

$$\frac{1}{2}\frac{d}{dt}\|W\|^2_{L^2(\mathbb{R}^+)} \leq C(1+\|W\|^2_{L^2(\mathbb{R}^+)}). \tag{4.4}$$

H^1 estimate

We derivate system (4.2) with respect to t and with similar computations we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t W\|^2_{L^2(\mathbb{R}^+)} \leq C(1+\|\partial_t W\|^2_{L^2(\mathbb{R}^+)}). \tag{4.5}$$

By Gronwall lemma we obtain from (4.4) and (4.5) that

$$\|W\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} + \|\partial_t W\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*). \tag{4.6}$$

So using the first equation in (4.2) we have

$$\|\partial_x W_1\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} + \|\partial_x W_3\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*), \tag{4.7}$$

In addition we have

$$\partial_t \partial_x W_2 - \partial_{W_2} H_2(W)\partial_x W_2 = \mathcal{H}(t,x),$$

where

$$\mathcal{H} = \partial_{W_1} H_2(W)\partial_x W_1 + \partial_{W_3} H_2(W)\partial_x W_3 + \partial_x \Phi_2.$$

By (4.3) and (4.7) we have

$$\|\mathcal{H}\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*),$$

and since

$$\partial_x W_2(t,x) = \int_0^t \left(\exp \int_s^t \partial_{W_2} H_2(W(\tau,x))d\tau\right)\mathcal{H}(s,x)ds,$$

we conclude that

$$\|\partial_x W_2\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*).$$

By Sobolev injections we can apply the continuation principle and we conclude the proof of Theorem 2.2.

5. Proof of Theorem 2.3

We denote by T^* the lifespan of the smooth solution $u^0=(u_1^0,u_2^0)$ of system (1.13)-(1.14)-(1.15). Since the boundary data φ belongs to $H^3(\mathbb{R})$ we have

$$\partial_t^i u^0 \in \mathcal{C}^0([0,T^*[;H^{3-i}(\mathbb{R}^+)),\ i=0,1,2,3. \tag{5.1}$$

We define the profile w^0 by

$$w^0=(u^0,v^0)=((u_1^0,u_2^0),p(u_1^0)). \tag{5.2}$$

We denote

$$\gamma(t,x)=p'(u_1^0(t,x)),t<T^*,x>0, \tag{5.3}$$

$$\Gamma=\sup_{(t,x)\in[0,T^*[\times\mathbb{R}^+}\gamma(t,x), \tag{5.4}$$

and by (2.2), $\Gamma<+\infty$. We fix μ such that

$$\mu>\Gamma. \tag{5.5}$$

We will construct the solution w^ε of the relaxation problem (1.10)-(1.11)-(1.12) writing

$$w^\varepsilon=w^0+\varepsilon\begin{pmatrix}0\\0\\v^1\end{pmatrix}+\varepsilon r, \tag{5.6}$$

where

$$v^1=-\partial_t v^0+\mu\partial_x u_2^0, \tag{5.7}$$

so that r satisfies the following system

$$\begin{cases}\partial_t r_1-\partial_x r_2=0,\\[2mm] \partial_t r_2-\partial_x r_3=\partial_x v^1,\\[2mm] \partial_t r_3-\mu\partial_x r_2=\dfrac{1}{\varepsilon}(p'(u_1^0)r_1-r_3)+F(t,x,\varepsilon r_1)(r_1)^2-\partial_t v^1,\end{cases} \tag{5.8}$$

for $(t,x)\in[0,T^*[\times\mathbb{R}^+$, with the initial-boundary conditions

$$\begin{cases}r(0,x)=0,x\in\mathbb{R}^+,\\[2mm] r_2(t,0)=0,0\le t<T^*.\end{cases} \tag{5.9}$$

The function F is defined by

$$F(t,x,\xi)=\int_0^1(1-s)p''(u_1^0(t,x)+s\xi)ds. \tag{5.10}$$

First step: we want to construct a suitable symmetrization for system (5.8). We denote by A and B the matrices

$$A=\begin{pmatrix}0&-1&0\\0&0&-1\\0&-\mu&0\end{pmatrix},\ B=\begin{pmatrix}0&0&0\\0&0&0\\\gamma(t,x)&0&-1\end{pmatrix}.$$

With this object, we will use the conservative-dissipative form introduced in [2]. We first need a symmetric positive definite matrix A_0 such that AA_0 is a symmetric matrix, and such that

$$BA_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-d\end{pmatrix}\text{ with } d>0.$$

Following [7], such a matrix can be constructed using the entropic variables. For the special case of the Suliciu model we have

$$A_0(t,x)=\begin{pmatrix}(\gamma(t,x))^{-1}&0&1\\0&1&0\\1&0&\mu\end{pmatrix}=\begin{pmatrix}A_{0,11}&A_{0,12}\\ \\A_{0,21}&A_{0,22}\end{pmatrix}.$$

We obtain

$$AA_0=\begin{pmatrix}0&-1&0\\-1&0&-\mu\\0&-\mu&0\end{pmatrix},\ BA_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\gamma-\mu\end{pmatrix},$$

and we remark that with (5.5), we have $\mu-\gamma\geq\mu-\Gamma>0$. Finally we can apply Proposition 2.7 in [2]: the conservative-dissipative variables ρ is defined by $\rho=P(t,x)r$ with

$$P(t,x)=\begin{pmatrix}(A_{0,11})^{-\frac12}&0\\ \\((A_0^{-1})_{22})^{-\frac12}(A_0^{-1})_{21}&((A_0^{-1})_{22})^{\frac12}\end{pmatrix}=\begin{pmatrix}\gamma^{\frac12}&0&0\\0&1&0\\-\gamma(\mu-\gamma)^{-\frac12}&0&(\mu-\gamma)^{-\frac12}\end{pmatrix}.$$

In these variables, system (5.8) is equivalent to

$$\partial_t\rho+A_1\partial_x\rho+L\rho=-\frac{1}{\varepsilon}\begin{pmatrix}0\\0\\\rho_3\end{pmatrix}+\begin{pmatrix}0\\0\\F_1(t,x,\varepsilon\rho_1)\rho_1^2\end{pmatrix}+H,\tag{5.11}$$

for $(t,x)\in[0,T^*[\times\mathbb{R}^+$, with the initial-boundary conditions

$$\rho(0,x)=0 \text{ for } x\in\mathbb{R}^+ \text{ and } \rho_2(t,0)=0 \text{ for } t\in[0,T^*[.\tag{5.12}$$

The matrix $A_1=PAP^{-1}$ is symmetric

$$A_1(t,x)=\begin{pmatrix}0&-\gamma^{\frac12}&0\\-\gamma^{\frac12}&0&-(\mu-\gamma)^{\frac12}\\0&-(\mu-\gamma)^{\frac12}&0\end{pmatrix}.$$

The matrix L is given by $L(t,x)=P\partial_tP^{-1}+PA\partial_xP^{-1}$. In addition, F_1 and H are given by

$$F_1(t,x,\xi)=\gamma^{-1}(\mu-\gamma)^{-\frac12}F(t,x,\gamma^{-\frac12}\xi),\tag{5.13}$$

$$H(t,x)=\begin{pmatrix} 0 \\ \partial_x v^1 \\ -(\mu-\gamma)^{-\frac{1}{2}}\partial_t v^1 \end{pmatrix}.$$

From (5.1) we have

$$\partial_t^i \gamma \in \mathcal{C}^0([0,T^*[;H^{3-i}(\mathbb{R}^+)), i=0,1,2,3, \tag{5.14}$$

and using (2.2) there exists $\alpha>0$ such that

$$\gamma(t,x)\geq \alpha \text{ for } (t,x)\in[0,T^*[\times\mathbb{R}^+. \tag{5.15}$$

Using (5.14), (5.15) and (5.5) we have

$$A_1,\partial_t A_1,\partial_x A_1\in\mathcal{C}^0([0,T^*[;L^\infty(\mathbb{R}^+)), \tag{5.16}$$

$$L,\partial_t L,\partial_x L\in\mathcal{C}^0([0,T^*[;L^\infty(\mathbb{R}^+)). \tag{5.17}$$

Using (5.1) and (5.7) we have

$$\partial_t^i H\in\mathcal{C}^0([0,T^*[;H^{1-i}(\mathbb{R}^+)), i=0,1. \tag{5.18}$$

We recall that by (5.10) and (5.13) we have

$$F_1(t,x,\xi)=\gamma^{-1}(t,x)(\mu-\gamma(t,x))^{-\frac{1}{2}}\int_0^1(1-s)p''(u_1^0(t,x)+s\gamma^{-\frac{1}{2}}(t,x)\xi)ds,$$

so, by (5.14), (5.15) and (5.5) we have

$$F_1,\partial_t F_1,\partial_x F_1,\partial_\xi F_1\in\mathcal{C}^0([0,T^*[;L^\infty(\mathbb{R}^+\times[-1,1])). \tag{5.19}$$

Now we fix $T<T^*$ and we introduce T_ε defined by

$$T_\varepsilon=\sup\left\{t\leq T, \|\rho\|_{L^\infty([0,t]\times\mathbb{R}^+)}\leq\frac{1}{\varepsilon}\right\}. \tag{5.20}$$

We will prove that, for ε small enough, $T_\varepsilon=T$ and that there exists K such that for all ε small enough,

$$\|\rho\|_{L^\infty([0,T];H^1(\mathbb{R}^+))}+\|\partial_t\rho\|_{L^\infty([0,T];L^2(\mathbb{R}^+))}\leq K. \tag{5.21}$$

First, by variational methods, we obtain L^2-estimates on ρ and $\partial_t\rho$. To obtain L^2-estimates on $\partial_x\rho$ we use the equations taking into account that the boundary $\{x=0\}$ is characteristic.

Second step: variational estimates

We take the inner product of system (5.11) by ρ and we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2_{L^2(\mathbb{R}^+)}+\int_{\mathbb{R}^+}A_1\partial_x\rho\cdot\rho dx+\int_{\mathbb{R}^+}L\rho\cdot\rho dx+\frac{1}{\varepsilon}\int_{\mathbb{R}^+}\rho_3^2dx=\int_{\mathbb{R}^+}F_1(t,x,\varepsilon\rho_1)\rho_1^2\rho_3 +\int_{\mathbb{R}^+}H\cdot\rho dx.$$

Using (5.12) we obtain that

$$\int_{\mathbb{R}+} A_1 \partial_x \rho \cdot \rho dx = -\frac{1}{2} \int_{\mathbb{R}^+} (\partial_x A_1) \rho \cdot \rho dx.$$

With the estimates (5.16),.., (5.19) and since $\varepsilon|\rho| \leq 1$ on $[0, T_\varepsilon] \times \mathbb{R}^+$, there exists a constant $C > 0$ such that, for $t \leq T_\varepsilon$,

$$\frac{1}{2} \frac{d}{dt} \|\rho\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx \leq C(1 + \|\rho\|^2_{L^2(\mathbb{R}^+)} + \|\rho_1\|_{L^\infty(\mathbb{R}^+)} \|\rho_1\|_{L^2(\mathbb{R}^+)} \|\rho_3\|_{L^2(\mathbb{R}^+)}).$$

Therefore we obtain that for $t \leq T_\varepsilon$,

$$\frac{d}{dt} \|\rho\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx \leq C(1 + \|\rho\|^2_{L^2(\mathbb{R}^+)} + \varepsilon \|\rho_1\|^2_{L^\infty(\mathbb{R}^+)} \|\rho_1\|^2_{L^2(\mathbb{R}^+)}). \tag{5.22}$$

We can derivate (5.11)-(5.12) with respect to t

$$\begin{aligned} \partial_t \partial_t \rho + A_1 \partial_x \partial_t \rho + L \partial_t \rho + \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ 0 \\ \partial_t \rho_3 \end{pmatrix} &= -\partial_t A_1 \partial_x \rho - \partial_t L \rho + \begin{pmatrix} 0 \\ 0 \\ \partial_t F_1(t, x, \varepsilon\rho_1) \rho_1^2 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ \varepsilon \partial_\xi F_1(t, x, \varepsilon\rho_1) \partial_t \rho_1 \rho_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2F_1(t, x, \varepsilon\rho_1) \rho_1 \partial_t \rho_1 \end{pmatrix} + \partial_t H. \end{aligned}$$

With the same arguments as before we obtain that there exists $C > 0$ such that for $\leq T_\varepsilon$,

$$\begin{aligned} \frac{d}{dt} \|\partial_t \rho\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} (\partial_t \rho_3)^2 dx \leq C(1 + \|\rho\|^2_{L^2(\mathbb{R}^+)} + \|\partial_t \rho\|^2_{L^2(\mathbb{R}^+)} + \|\partial_x \rho\|^2_{L^2(\mathbb{R}^+)}) \\ + C\varepsilon \|\rho_1\|^2_{L^\infty(\mathbb{R}^+)} (\|\rho_1\|^2_{L^2(\mathbb{R}^+)} + \|\partial_t \rho_1\|^2_{L^2(\mathbb{R}^+)})). \end{aligned} \tag{5.23}$$

We define ψ by

$$\psi(t) = \left(\|\rho(t)\|^2_{L^2(\mathbb{R}^+)} + \|\partial_t \rho(t)\|^2_{L^2(\mathbb{R}^+)} \right)^{\frac{1}{2}}, \tag{5.24}$$

so we obtain by (5.22) and (5.23) the L^2-estimate: there exists $C > 0$ such that for $t \leq T_\varepsilon$,

$$\begin{aligned} \frac{d}{dt} (\psi(t))^2 + \frac{1}{\varepsilon} (\|\rho_3\|^2_{L^2(\mathbb{R}^+)} + \|\partial_t \rho_3\|^2_{L^2(\mathbb{R}^+)}) \leq C(1 + (\psi(t))^2 \\ + \varepsilon \|\rho_1\|^2_{L^\infty(\mathbb{R}^+)} (\psi(t))^2 + \|\partial_x \rho\|^2_{L^2(\mathbb{R}^+)}). \end{aligned} \tag{5.25}$$

Third step

We now estimate $\partial_x \rho$ using the equations

$$\begin{cases} \partial_t \rho_1 - \gamma^{\frac{1}{2}} \partial_x \rho_2 + (L\rho)_1 = 0, \\ \partial_t \rho_2 - \gamma^{\frac{1}{2}} \partial_x \rho_1 - (\mu - \gamma)^{\frac{1}{2}} \partial_x \rho_3 + (L\rho)_2 = H_2, \\ \partial_t \rho_3 - (\mu - \gamma)^{\frac{1}{2}} \partial_x \rho_2 + (L\rho)_3 + \frac{1}{\varepsilon} \rho_3 = F_1(t, x, \varepsilon\rho_1) \rho_1^2 + H_3. \end{cases} \tag{5.26}$$

From the first equation in (5.26), and with (5.15) and (5.17) we have for $t \in [0, T_\varepsilon]$

$$\|\partial_x \rho_2\|_{L^2(\mathbb{R}^+)} \leq C\psi. \tag{5.27}$$

Let us introduce $\tilde{\rho}_1=\rho_1+\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\rho_3$. From the second equation in (5.26) we have

$$\partial_t\rho_2-\gamma^{\frac{1}{2}}\partial_x\tilde{\rho}_1+\gamma^{\frac{1}{2}}\partial_x(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})\rho_3+(L\rho)_2=H_2,$$

so, by (5.15), (5.14), (5.17) and (5.18) we obtain that

$$\|\partial_x\tilde{\rho}_1\|_{L^2(\mathbb{R}^+)}\le C(1+\psi). \tag{5.28}$$

We cannot estimate $\partial_x\rho_1$ or $\partial_x\rho_3$ by the same method because the boundary $\{x=0\}$ is characteristic. We rewrite the third equation in (5.26)

$$\partial_t\rho_3+\frac{1}{\varepsilon}\rho_3=\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}(\partial_t\rho_1+(L\rho)_1)-(L\rho)_3+F_1(t,x,\varepsilon\rho_1)\rho_1^2+H_3.$$

So eliminating ρ_1 we obtain

$$\begin{aligned}\mu\gamma^{-1}\partial_t\rho_3+\tfrac{1}{\varepsilon}\rho_3=\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}[\partial_t\tilde{\rho}_1-\partial_t(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})\rho_3]+M_1(t,x)\tilde{\rho}_1+M_2(t,x)\rho_2\\+M_3(t,x)\rho_3+H_3+F_1(t,x,\varepsilon\rho_1)\rho_1^2,\end{aligned} \tag{5.29}$$

with $\rho_1=\tilde{\rho}_1-\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\rho_3$.We derivate (5.29) with respect to x and we obtain the equation satisfied by $\partial_x\rho_3$

$$\partial_t\partial_x\rho_3+\tau(t,x)\partial_x\rho_3=\sum_{i=1}^{6}T_i, \tag{5.30}$$

with

$$\begin{aligned}\tau=\ &\mu^{-1}\gamma\left(\frac{1}{\varepsilon}+\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\partial_t(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})+\varepsilon\partial_\xi F_1(t,x,\varepsilon\rho_1)\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\rho_1^2\right.\\&\left.+2F_1(t,x,\varepsilon\rho_1)\rho_1\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}-M_3(t,x)\right),\end{aligned}$$

$$T_1=\mu^{-1}\gamma^{\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\partial_t\partial_x\tilde{\rho}_1,$$

$$\begin{aligned}T_2=\mu^{-1}\gamma&\Big(\partial_x(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})\partial_t\tilde{\rho}_1-\partial_x(\gamma^{-1}\mu)\partial_t\rho_3\\&-\partial_x(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}\partial_t(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}}))\rho_3\\&+(\partial_xM_1)\tilde{\rho}_1+(\partial_xM_2)\rho_2+(\partial_xM_3)\rho_3\Big),\end{aligned}$$

$$T_3=\mu^{-1}\gamma\partial_xH_3,$$

$$T_4=\mu^{-1}\gamma(M_1\partial_x\tilde{\rho}_1+M_2\partial_x\rho_2),$$

$$\begin{aligned}T_5=\mu^{-1}\gamma&\Big(\partial_xF_1(t,x,\varepsilon\rho_1)\rho_1^2-\varepsilon\partial_\xi F_1(t,x,\varepsilon\rho_1)\partial_x(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})\rho_1^2\rho_3\\&-2F_1(t,x,\varepsilon\rho_1)\partial_x(\gamma^{-\frac{1}{2}}(\mu-\gamma)^{\frac{1}{2}})\rho_1\rho_3\Big),\end{aligned}$$

$$T_6=\mu^{-1}\gamma\left(\varepsilon\partial_\xi F_1(t,x,\varepsilon\rho_1)\rho_1^2\partial_x\tilde{\rho}_1+2F_1(t,x,\varepsilon\rho_1)\rho_1\partial_x\tilde{\rho}_1\right).$$

For $t\in[0,T_\varepsilon]$, using (5.5), (5.14) (5.15) and (5.19) we obtain that

$$\left|\tau(t,x)-\frac{\mu^{-1}\gamma}{\varepsilon}\right|\le C+C_0\|\rho_1\|_{L^\infty(\mathbb{R}^+)}.$$

We define $T_\varepsilon^1 \leq T_\varepsilon$ by

$$T_\varepsilon^1 = \max\left\{ t \leq T_\varepsilon, \|\rho_1\|_{L^\infty([0,t]\times\mathbb{R}^+)} \leq \frac{1}{2C_0\varepsilon} \right\}, \tag{5.31}$$

so there exists $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$\forall\, t \leq T_\varepsilon^1, \forall\, x > 0, \frac{\tau_1}{\varepsilon} \leq \tau(t,x) \leq \frac{\tau_2}{\varepsilon}. \tag{5.32}$$

We solve Equation (5.30) by Duhamel formula

$$\partial_x \rho_3 = \sum_{i=1}^{6} \mathcal{T}_i, \tag{5.33}$$

with

$$\mathcal{T}_i(t,x) = \int_0^t \exp(-\int_s^t \tau(\sigma,x)d\sigma) T_i(s,x)ds.$$

We define Ψ by

$$\Psi(t) = \sup_{[0,t]} \psi(s), \tag{5.34}$$

where ψ is given by (5.24). Integrating by parts in $\mathcal{T}_1$ we obtain

$$\begin{aligned}\mathcal{T}_1(t,x) = &-\int_0^t \mu^{-1}\gamma^{\frac12}(\mu-\gamma)^{\frac12}\tau(s,x)\exp(-\int_s^t \tau(\sigma,x)d\sigma)\partial_x\tilde\rho_1(s,x)ds \\ &-\int_0^t \exp(-\int_s^t \tau(\sigma,x)d\sigma)\partial_s(\mu^{-1}\gamma^{\frac12}(\mu-\gamma)^{\frac12})(s,x)\partial_x\tilde\rho_1(s,x)ds \\ &+\mu^{-1}\gamma^{\frac12}(\mu-\gamma)^{\frac12}\partial_x\tilde\rho_1(t,x).\end{aligned}$$

Using (5.32), (5.5), (5.14), (5.15) and (5.28) we have

$$\|\mathcal{T}_1(t,\cdot)\|_{L^2(\mathbb{R}^+)} \leq \int_0^t \exp(-\frac{\tau_1}{\varepsilon}(t-s))C(\psi(s)+1)(1+\frac{\tau_2}{\varepsilon})ds + C(\psi(t)+1),$$

and we obtain that

$$\forall\, t \leq T_\varepsilon^1, \|\mathcal{T}_1\|_{L^2(\mathbb{R}^+)} \leq C(1+\Psi(t)). \tag{5.35}$$

Using (5.5) (5.14) (5.15) (5.24) (5.34) and also (5.18) for T_3 and (5.27) and (5.28) for T_4, we obtain

$$\forall\, t \leq T_\varepsilon^1, \|\mathcal{T}_2\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_3\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_4\|_{L^2(\mathbb{R}^+)} \leq C\varepsilon(1+\Psi(t)). \tag{5.36}$$

For the nonlinear terms T_5 and T_6 we use in addition (5.19) (5.20) and we obtain

$$\forall\, t \leq T_\varepsilon^1, \|\mathcal{T}_5\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_6\|_{L^2(\mathbb{R}^+)} \leq C(1+\Psi(t)). \tag{5.37}$$

Therefore we obtain the following estimation for $\partial_x\rho$ using (5.27) (5.28) (5.33) (5.35) (5.36) (5.37)

$$\forall\, t \leq T_\varepsilon^1, \|\partial_x\rho\|_{L^2(\mathbb{R}^+)} \leq C(1+\Psi(t)), \tag{5.38}$$

so we have

$$\forall t \leq T_\varepsilon^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1+\Psi(t)). \tag{5.39}$$

Fourth step

By a comparison method we estimate Ψ. For $t \leq T_\varepsilon^1$, integrating (5.25) from 0 to t, using (5.38) and (5.39) we obtain that

$$(\Psi(t))^2 \leq C_2 \int_0^t (1+(\Psi(s))^2+\varepsilon(\Psi(s))^4) ds. \tag{5.40}$$

We introduce the differential equation

$$y_\varepsilon' = C_2(1+y_\varepsilon+\varepsilon y_\varepsilon^2),\, y_\varepsilon(0)=0. \tag{5.41}$$

There exists $\varepsilon_0 > 0$ such that, for $\varepsilon \leq \varepsilon_0$, the lifespan of y_ε is greater than T. So we have

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T, y_\varepsilon(t) \leq y_{\varepsilon_0}(t) \leq y_{\varepsilon_0}(T) = C_3.$$

By comparison principle we deduce from (5.40) that

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T_\varepsilon^1, (\Psi(t))^2 \leq C_3,$$

and from (5.39),

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T_\varepsilon^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1+\sqrt{C_3}).$$

Let $\varepsilon_1 > 0$ such that $\varepsilon_1 \leq \varepsilon_0$ such that

$$\forall \varepsilon \leq \varepsilon_1, C_1(1+\sqrt{C_3}) \leq \frac{1}{2C_0\varepsilon}.$$

So, by (5.20) and (5.31), we have for $\varepsilon \leq \varepsilon_1$, $T_e^1 = T_\varepsilon = T$ and we conclude the proof by the estimate

$$\exists K > 0, \forall \varepsilon \leq \varepsilon_1, \|\rho\|_{L^\infty([0,T];H^1(\mathbb{R}^+))} + \|\partial_t \rho\|_{L^\infty([0,T];L^2(\mathbb{R}^+))} \leq K.$$

6. Annex

Using the method in W.A. Yong [22] we show the convergence result for the Cauchy problem

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \dfrac{1}{\varepsilon}(p(u_1^\varepsilon) - v^\varepsilon), \end{cases} \tag{6.1}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ with the smooth initial data

$$w^\varepsilon(0,x) = w_0(x) = (u_0(x), v_0(x)) \text{ for } x \in \mathbb{R}. \tag{6.2}$$

Let us introduce u^0 the smooth solution of the Cauchy problem

$$\begin{cases} \partial_t u_1^0 - \partial_x u_2^0 = 0, \\ \partial_t u_2^0 - \partial_x p(u_1^0) = 0, \end{cases} \tag{6.3}$$

with the initial data

$$u^0(0,x) = u_0(x). \tag{6.4}$$

As in Tzavaras [21] we assume that there exists $\gamma > 0$ and $\Gamma > 0$ such that

$$\forall \xi \in \mathbb{R}, \gamma \leq p'(\xi) \leq \Gamma < \mu, \tag{6.5}$$

so the problem (6.1)-(6.2) admits a global solution $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ such that

$$w^\varepsilon \in \mathcal{C}^0(\mathbb{R}^+; H^s(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^{s-1}(\mathbb{R})).$$

We will prove the following convergence theorem.

THEOREM 6.1. *Under assumption (6.5), if $w_0 \in H^s(\mathbb{R})$ with $s \geq 2$, then there exists $T_1 > 0$ such that when ε tends to zero, u^ε tends to u^0 in $L^\infty([0,T_1]; H^s(\mathbb{R}))$.*

REMARK 6.1. *It would be possible to relax hypothesis (6.5) as in Theorem 2.3; in this case, the lifespan of w^ε is uniformly greater that T_1.*

REMARK 6.2. *In fact it appears a boundary layer in time which affects only the third component of w^ε.*

Sketch of the proof

First step: the stability assumption in [22] are satisfied. As in [21] and [7], we consider the strictly convex entropy function for the system (6.1)

$$\mathcal{E}(u_1, u_2, v) = \frac{1}{2} u_2^2 + u_1 v - \frac{\mu}{2} u_1^2 - \int_0^{v - \mu u_1} h^{-1}(y) dy,$$

where $h(\xi) = p(\xi) - \mu\xi$ which is strictly decreasing by (6.5). So $A_0(w) = \mathcal{E}''(w)$ is a symmetrizer for the system. Denoting $a = (h^{-1})'(v - \mu u_1)$ we obtain

$$A_0(w) = \begin{pmatrix} -\mu - \mu^2 a & 0 & 1 + \mu a \\ 0 & 1 & 0 \\ 1 + \mu a & 0 & -a \end{pmatrix},$$

and the system (6.1) is equivalent to the quasilinear symmetric system

$$A_0(w)\partial_t w + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \partial_x w = \frac{1}{\varepsilon}(p(u_1) - v) \begin{pmatrix} 1 + \mu a \\ 0 \\ -a \end{pmatrix}. \tag{6.6}$$

We denote

$$Q(w) = \begin{pmatrix} 0 \\ 0 \\ p(u_1) - v \end{pmatrix} \text{ and } P(w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p'(u_1) & 0 & 1 \end{pmatrix},$$

and we obtain

$$P(w)Q'(w)P^{-1}(w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.7}$$

On the equilibrium manifold $\mathcal{V}=\{v=p(u_1)\}$, we have

$$A_0(w)Q'(w)+Q'(w)A_0(w)=\frac{2}{p'(u_1)-\mu}\begin{pmatrix}(p'(u_1))^2 & 0 & -p'(u_1)\\ 0 & 0 & 0\\ -p'(u_1) & 0 & 1\end{pmatrix}. \tag{6.8}$$

Using (6.6), (6.7) and (6.8) we obtain the stability conditions in [22].

Second step: we use Theorems 6.1 and 6.2 in [22]. We introduce the interior profile $w^0=((u_1^0,u_2),p(u_1^0))$ and the boundary layer term $I^0=\tilde{I}^0-w^0(0,x)$ where $\tilde{I}^0$ is the solution of

$$\frac{d\tilde{I}_0}{d\tau}=Q(\tilde{I}_0),\tilde{I}(\tau=0)=w_0(x).$$

We have $I_1^0=I_2^0=0$ and

$$I_3^0(\tau,x)=(v_0(x)-p(u_1,0))e^{-\tau},$$

and we obtain

$$w^\varepsilon(t,x)=w^0(t,x)+I^0(\frac{t}{\varepsilon},x)+\mathcal{O}(\varepsilon),$$

so we conclude the proof of Theorem 6.1.
Remark 6.3. *If w_0 belongs to the equilibrium manifold then the order zero boundary layer term vanishes.*
Remark 6.4. *In fact using more precisely [22] and the appendix of [3] we can prove that T_1 can be arbitrarily close to the lifespan of u^0 as in Theorem 2.3.*
Remark 6.5. *In this annex the matrix P introduced in [22] plays an analogous role as the matrix P in section 5.*

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