Contrast re-enhancement of Total-Variation regularization jointly with the Douglas-Rachford iterations

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Abstract

Restoration of a piece-wise constant signal can be performed using anisotropic Total-Variation (TV) regularization. Anisotropic TV may capture well discontinuities but suffers from a systematic loss of contrast. This contrast can be re-enhanced in a post-processing step known as least-square refitting. We propose here to jointly estimate the refitting during the Douglas-Rachford iterations used to produce the original TV result.

Problem statement

Consider the **anisotropic TV** regularization defined, for $\lambda > 0$, as [Rudin et al., 1992]

$$u^{\mathrm{TV}} \in \underset{u \in \mathbb{R}^N}{\operatorname{argmin}} \ \tfrac{1}{2}\|\Phi u - f\|^2 + \lambda \|\nabla u\|_1,$$

which aims at recovering u_0 from its linear noisy observation

$$f = \Phi u_0 + w$$

where we consider

- $u_0 \in \mathbb{R}^N$ the representation of a 2D signal,
- $f \in \mathbb{R}^P$ the linear noisy observation of u_0,
- $w \in \mathbb{R}^P$ a zero-mean noise component,
- $\Phi \in \mathbb{R}^{P \times N}$ a linear operator accounting for a loss of information,
- $\nabla u \in \mathbb{R}^{2N}$ the discrete gradient vector field of u,
- $\|\nabla u\|_1 = \sum_i |(\nabla u)_i|$ a sparsity promoting term.

Anisotropic TV is known to

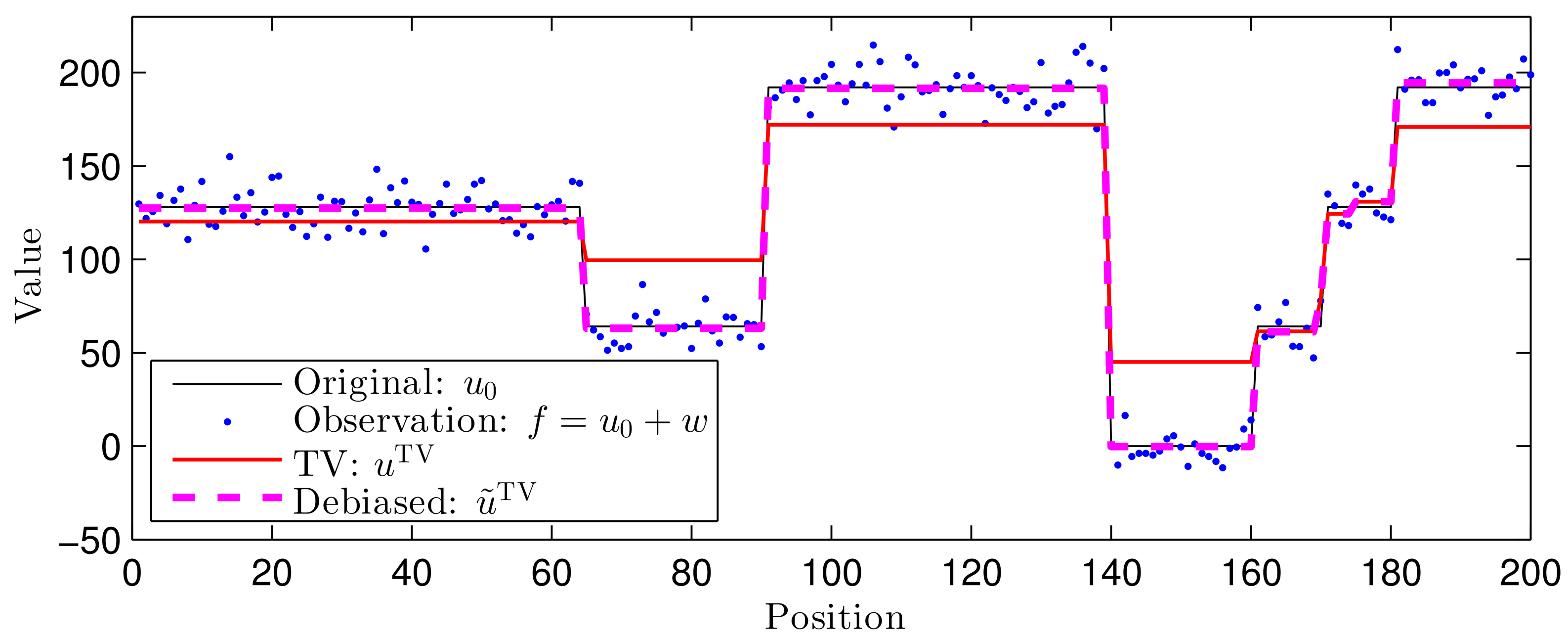
- Recover piece-wise constant signals,
- Recover the discontinuities of u_0 in some cases,
- Suffer from a **systematic loss of contrast** [Strong and Chan, 2003].

Least-Square refitting problem

Least-square refitting re-enhances the amplitudes and preserves the discontinuities as

$$\tilde{u}^{\mathrm{TV}} \in \underset{u\,;\,\mathrm{supp}(\nabla u) \subset \mathrm{supp}(\nabla u^{\mathrm{TV}})}{\operatorname{argmin}} \|\Phi u - f\|^2$$

where, for $x \in \mathbb{R}^{2N}$, $\mathrm{supp}(x) = \{i \in [1, 2N] \,;\, \|x_i\| \neq 0\}$ denotes the support of x.



Post-refitting, see, e.g., [Efron et al., 2004]

- Estimates $\mathrm{supp}(\nabla u^{\mathrm{TV}})$ and computes $\tilde{u}^{\mathrm{TV}}$, e.g., with a conjugate gradient,
- However, u^{TV} is usually obtained thanks to a converging sequence u^k,
- Unfortunately, $u^k \approx u^{\mathrm{TV}} \not\Rightarrow \mathrm{supp}(\nabla u^k) \approx \mathrm{supp}(\nabla u^{\mathrm{TV}})$,
- Erroneous support identifications can lead to **strong numerical instabilities**.

The Douglas-Rachford sequence

- Consider the splitting TV reformulation given by

$$u^{\mathrm{TV}} \in \underset{u \in \mathbb{R}^N}{\operatorname{argmin}} \ \min_{z \in \mathbb{R}^{N \times 2}} \tfrac{1}{2}\|\Phi u - f\|^2 + \lambda \|z\|_1 + \iota_{\{z,u\,;\,z = \nabla u\}}(z, u)$$

where ι_S is the indicator function of a set S.

- The associated Douglas-Rachford sequence u^k given, for $\tau > 0$, as

$$\begin{cases} \mu^{k+1} = (\mathrm{Id} + \Delta)^{-1}(2u^k - \mu^k - \mathrm{div}(2z^k - \zeta^k))/2 + \mu^k/2, \\ \zeta^{k+1} = \nabla \mu^{k+1}, \\ u^{k+1} = \mu^{k+1} + \tau \Phi^t (\mathrm{Id} + \tau \Phi \Phi^t)^{-1}(f - \Phi \mu^{k+1}), \\ z^{k+1} = \Psi_{\zeta^{k+1}}(\zeta^{k+1}, \lambda) \end{cases}$$

where $\Psi_\zeta(\zeta, \lambda)_i = \begin{cases} 0 & \text{if } |\zeta_i| \leqslant \tau\lambda, \\ \zeta_i - \tau\lambda \,\mathrm{sign}\,\zeta_i & \text{otherwise,} \end{cases}$

converges towards a solution u^{TV} [Combettes and Pesquet, 2007].

Can we build a stable sequence $\tilde{u}^k$ jointly with u^k converging towards $\tilde{u}^{\mathrm{TV}}$?

Theorem (Proposed joint-refitting)

Let $\alpha > 0$ be the minimum non zero value of $|(\nabla u)_i|$, $i \in [1, 2N]$.
The sequence, $\tilde{u}^k$ given, for $0 < \beta < \alpha\lambda$, as

$$\begin{cases} \tilde{\mu}^{k+1} = (\mathrm{Id} + \Delta)^{-1}(2\tilde{u}^k - \tilde{\mu}^k - \mathrm{div}(2\tilde{z}^k - \tilde{\zeta}^k))/2 + \tilde{\mu}^k/2, \\ \tilde{\zeta}^{k+1} = \nabla \tilde{\mu}^{k+1}, \\ \tilde{u}^{k+1} = \tilde{\mu}^{k+1} + \tau \Phi^t (\mathrm{Id} + \tau \Phi \Phi^t)^{-1}(f - \Phi \tilde{\mu}^{k+1}), \\ \tilde{z}^{k+1} = \Pi_{\zeta^{k+1}}(\tilde{\zeta}^{k+1}, \lambda) \end{cases}$$

where $\Pi_\zeta(\tilde{\zeta}, \lambda)_i = \begin{cases} 0 & \text{if } |\zeta_i| \leqslant \tau\lambda + \beta, \\ \tilde{\zeta}_i & \text{otherwise,} \end{cases}$

converges towards a solution $\tilde{u}^{\mathrm{TV}}$.

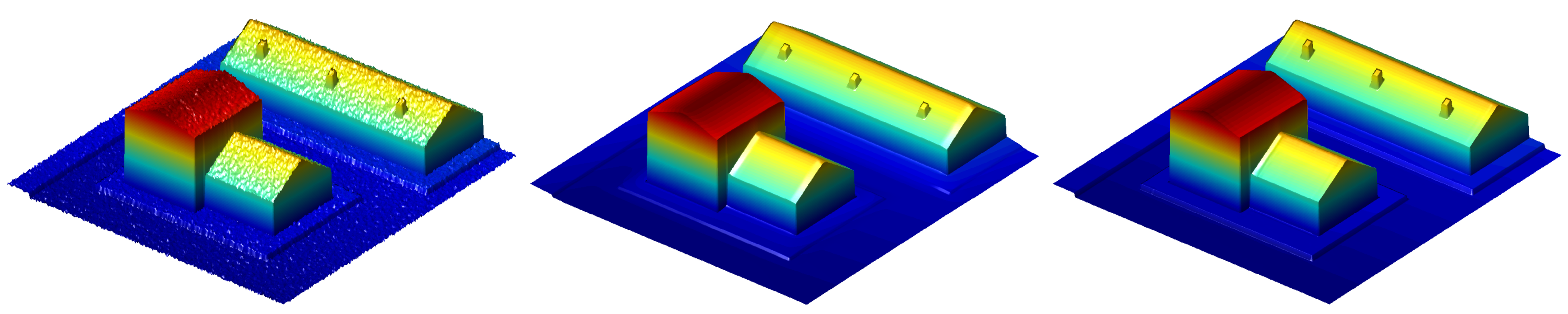
Our joint-refitting does not require any support identification[1].

[1] *We have a similar result for the Chambolle-Pock sequence in [Deledalle et al., 2015].*

Conclusion and perspectives

Computed during the Douglas-Rachford iterations, our refitting strategy

- is free of post-processing steps, such as support identification,
- is moreover easy to implement,
- can be used likewise for other ℓ_1 penalties, e.g., with TGV [Bredies et al., 2010].



(a) Noisy obs. f (b) Anisotropic TGV[2] (c) Joint-refitting

Perspectives

- Extensions for isotropic TV, block sparsity and non ℓ_1-based estimators,
- Explore alternative definitions of contrast re-enhancement,
- Establish links with debiasing [Deledalle et al., 2015].

Experimental results

(d) Underlying image u_0 (e) Observed image f (f) Anisotropic TV u^{TV} (g) Post-refitting (h) Joint-refitting

Experimental setting

- u_0 chosen from a 8bits image,
- f damaged version of u_0 by a Gaussian blur of 2px and white noise $\sigma = 20$,
- λ chosen to enforce large homogeneous regions faithful to the edges of u_0,
- β chosen as the smallest positive value up to machine precision.

Numerical observations

- TV reduces the contrast,
- Refitting recovers the original amplitudes and keep unchanged the discontinuities,
- Post-refitting creates suspicious oscillations due to wrong support identification,
- Joint-refitting re-enhances the contrast without introducing new artifacts.

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