On generalized Hermite constants.

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We give a brief survey on some generalizations of the Hermite constant of quadratic forms. Our main concern is the theory of Humbert forms and quaternionic Humbert forms. In that cases, we proved an analogue of Voronoï's theorem about extreme forms. We also give a brief account of Watanabe's theory of generalized Hermite constants.

Introduction

The density of a regular sphere packing, the centers of which are the points of a Euclidean lattice Λ, is proportional to the Hermite invariant $\gamma(\Lambda)$ of the underlying lattice. The lattices achieving a local maximum of this density function, the so-called extreme lattices, are characterized by a celebrated theorem of Voronoï, in terms of *perfection* and *eutaxy*.

Over the recent years, this theory has received several generalizations, due to the work of C. Bavard [2], A.-M. Bergé, J. Martinet [3], J. Opgenorth [14], and the author ([5,6]) among others .

Another active direction of research is the adelic geometry of numbers initiated by T. Watanabe [19], in which a very general notion of Hermite constant over the group of adèles $G(\mathbb{A})$ of an algebraic group G is defined. The classical Hermite constant corresponds to the general linear group GL_n over $\mathbb{Q}$. The theory of Humbert forms studied in [6] might be viewed as a particular case of Watanabe's theory, corresponding to the general linear group GL_n over a number field k. In a work in progress with T.Watanabe [7], we investigate in detail the case of GL_n over a quaternion algebra.

In this paper, we give a brief survey of these subjects, focusing on Humbert forms and quaternionic Humbert forms.

1. Hermite constant

To start with, we recall some classical definitions and results about the Hermite invariant of quadratic forms. Let $S_n(\mathbb{R})$ be the space of nxn real symmetric matrices, or equivalently, the space of n-ary real quadratic forms. Inside this space, we consider the cone $P_n := S_n(\mathbb{R})_{>0}$ of positive definite quadratic forms. If A is in P_n, the set $\{A[X], X \in \mathbb{Z}^n \setminus \{0\}\}$ admits a minimum (as usual, the notation $A[X]$ stands for XAX'), so one can set

$$m(A) = \min_{X \in \mathbb{Z}^n \setminus \{0\}} A[X]. \tag{1}$$

The *Hermite number* of A is then the following homogenized version of $m(A)$

$$\gamma(A) = \frac{m(A)}{(\det A)^{\frac{1}{n}}}. \tag{2}$$

Finally, we let $S(A)$ be the (finite) set of minimal vectors of A *i.e.*

$$S(A) = \{X \in \mathbb{Z}^n : A[X] = m(A)\} \tag{3}$$

The *Hermite constant* in dimension n is then defined as the supremum of $\gamma(A)$ over all $A \in P_n$. Its actual value is known only in dimensions 1 to 8. Recall that the search for quadratic forms with high Hermite number is motivated by the following geometric interpretation : one writes $A = PP'$, with $P \in \mathrm{GL}_n(\mathbb{R})$ and consider the lattice $L = \mathbb{Z}^n P$ in $\mathbb{R}^n$. Then it is readily checked that $\gamma(A)^{n/2}$ is proportional to the density of the sphere packing associated to L.

In 1873, Korkine and Zolotareff initiated a method to compute γ_n by looking for the local maxima of the function $A \mapsto \gamma(A)$ ([11],[12]). A form achieving such a local maximum is *extreme*. Some thirty years later, Voronoï ([18]) completed their work by establishing the following fundamental theorem

Theorem 1 (Voronoï 1908)**.** *A is* extreme *if and only if it is both* perfect *and* eutactic*.*

This is of course meaningless as long as we have not defined perfection and eutaxy. Since this will be done in full generality in the next section, we will content ourselves here with the definition of perfection: a positive definite quadratic form A is perfect if it is completly determined by the set of equations $A[X] = m(A), X \in S(A)$. In other words if $B[X] = m(A)$ for all $X \in S(A)$, then $B = A$. As a noticeable consequence, one gets that $\gamma_n = \max_{A \in S_n(\mathbb{Q})_{>0}} \gamma(A)$, and that $\gamma_n^n \in \mathbb{Q}$.

To conclude this introductory section, we give another geometric interpretation of Hermite constant in the case of binary forms, which will be of some help in the next section. In that case indeed, there is a well-known 1-1 correpondance between the Poincaré upper half-plane $\mathfrak{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ and the cone P_2, up to scaling:

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} & \longleftrightarrow & \mathbb{R}_{>0}\backslash P_2/\mathrm{SL}_2(\mathbb{Z}) \\ z = x + iy & \longleftrightarrow & \begin{pmatrix} 1 & x \\ x & x^2 + y^2 \end{pmatrix} \end{array}$$

Thus, a form in P_2 with high γ on the right-hand side corresponds to a point in $\mathfrak{H}$ with small imaginary part. This is illustrated by the picture below:

2. Humbert forms.

There have been several attempts to develop a "geometry of numbers in a relative context", *i.e.* replacing the field of rationals $\mathbb{Q}$ by an arbitrary number field (see for instance [16]). In the 40's P. Humbert wrote two long papers about the reduction theory of quadratic forms over number fields ([8,9]). In fact, the objects under consideration in this reduction theory are not exactly quadratic forms, but rather tuples of such forms, which we will call Humbert forms in the sequel, according to the terminology introduced by Icaza ([10]). These are defined as follows: first, let k be a number field, with $[k : \mathbb{Q}] = d$. We denote by $\{v_1, \dots, v_r\}$ the set of real *places*, corresponding to the real embeddings of k, and by $\{v_{r+1}, \dots, v_{r+s}\}$ the set of complex *places*, corresponding to the couples of pairwise conjugated complex embeddings of k, so that $d = r + 2s$. In keeping with the classical definitions recalled in the first section, we have to define a space of forms $S_{n,k}$ and a cone $P_{n,k}$ of positive elements inside it. This is done as follows:

$$\begin{array}{l} S_{n,k} := S_n(\mathbb{R})^r \times H_n(\mathbb{C})^s \\ \quad\cup \\ P_{n,k} := S_n(\mathbb{R})^r_{>0} \times H_n(\mathbb{C})^s_{>0}. \end{array}$$

Definition 1. A *Humbert form* of rank n over k is a $(r+s)$-tuple $\mathcal{A} = (A_1, \ldots, A_r, A_{r+1}, \ldots, A_{r+s})$ in $P_{n,k}$.

The value of a Humbert form $\mathcal{A}$ at $u \in k^n$ is defined as follows: to $u \in k^n$ we associate its conjugates $u^{(1)}, \ldots, u^{(r+s)}$ and we set

$$\begin{aligned} \mathcal{A}[u] &:= \prod_{i=1}^{r+s} A_i[u^{(i)}]^{d_i} \text{ with } d_i = \begin{cases} 1 \text{ if } v_i \text{ real} \\ 2 \text{ if } v_i \text{ complex} \end{cases} \\ &= \prod_{i=1}^{r} A_i[u^{(i)}] \prod_{i=r+1}^{r+s} A_i[u^{(i)}]^2. \end{aligned}$$

Let $\mathcal{O}_k$ be the ring of integers of k. From now on, we assume, for simplicity, that the class number h_k of k is 1, *i.e.* $\mathcal{O}_k$ is a PID. Restricting to vectors with entries in $\mathcal{O}_k$, we set the following definition

Definition 2. The *minimum* of a Humbert form $\mathcal{A} \in P_{n,k}$ is defined as

$$m(\mathcal{A}) = \min\{\mathcal{A}[u] \ : \ u \in \mathcal{O}_k^n \setminus \{0\}\}.$$

The *Hermite-Humbert number* of $\mathcal{A}$ is

$$\gamma(\mathcal{A}) = \frac{m(\mathcal{A})}{(\det \mathcal{A})^{1/n}},$$

where $\det \mathcal{A} = \prod_i \det A_i^{d_i} = \prod_{i=1}^r \det A_i \prod_{i=r+1}^{r+s} (\det A_i)^2$

Remark 1.
(1) If $k = \mathbb{Q}$, then $P_{n,k} = P_n$, and the Hermite-Humbert function γ coincides with the classical Hermite function.
(2) When $h_k > 1$, one has to consider simultaneously h_k distinct Hermite numbers for a given Humbert form $\mathcal{A}$, indexed by the ideal classes $\{[\mathfrak{a}_1], \cdots, [\mathfrak{a}_{h_k}]\}$, and defined by replacing the lattice $\mathcal{O}_k^n$ by the lattices $\mathcal{O}_k^{n-1} \oplus \mathfrak{a}_i$, $i = 1, \cdots, h_k$ in the above definition.

Vectors in $\mathcal{O}_k^n$ achieving the minimum are called *minimal.* There might be infinitely many, since if u is minimal, then ξu is minimal for any ξ in U_k, the group of units of $\mathcal{O}_k$. It turns out that modulo multiplication by units, the set of minimal vectors is indeed finite. We set

$$S(\mathcal{A}) := \{u \in \mathcal{O}_k^n \text{ minimal for } \mathcal{A}\}/U_k$$

In ([10]), Icaza established the following theorem

Theorem 2 (Icaza, 1997)**.** $\gamma_{n,k} := \sup_{\mathcal{A} \in P_{n,k}} \{\gamma(\mathcal{A})\} < \infty$ *and is attained.*

Following Korkine and Zolotareff's terminology, Humbert forms at which γ achieves a local maximum are called *extreme*. These extreme forms can be characterised by a Voronoï type theorem. This requires a few more definitions. To a vector $u \in \mathcal{O}_k^n$ we associate the tuple

$$u'u_{\mathcal{A}} := \left(d_1 \frac{\overline{u^{(1)}}' u^{(1)}}{A_1[u^{(1)}]}, \ldots, d_{r+s} \frac{\overline{u^{(r+s)}}' u^{(r+s)}}{A_{r+s}[u^{(r+s)}]} \right) \in S_{n,k}.$$

Definition 3. A Humbert form $\mathcal{A} = (A_1, \ldots, A_{r+s})$ is *perfect* if

$$\dim_{\mathbb{R}} \sum_{u \in S(\mathcal{A})} \mathbb{R} u'u_{\mathcal{A}} = r\frac{n(n+1)}{2} + sn^2 - (r+s-1).$$

Definition 4. A Humbert form $\mathcal{A} = (A_1, \ldots, A_{r+s})$ is *eutactic* if its adjoint form $\overline{\mathcal{A}^{-1}} = (\overline{A_1^{-1}}, \ldots, \overline{A_{r+s}^{-1}})$ lies in the open convex hull of the $(r+s)$-tuples $u'u_{\mathcal{A}},\ u \in S(\mathcal{A})$. In other words, $\mathcal{A}$ is eutactic if there exist $\rho_u > 0$, $u \in S(\mathcal{A})$ such that:

$$\overline{A_i^{-1}} = \sum_{u \in S(\mathcal{A})} \rho_u d_i \frac{\overline{u^{(i)}}' u^{(i)}}{A_i[u^{(i)}]} \text{ for } i = 1, \ldots, r+s.$$

Of course, when $k = \mathbb{Q}$, these definitions coincide with the classical notion of perfect and eutactic form mentionned in the previous section. In [6], we obtained a characterization of extreme Humbert forms:

Theorem 3 (C, 2001)**.** *A Humbert form $\mathcal{A}$ is extreme if and only if it is both eutactic and perfect.*

Although the statement of this theorem is quite similar to Voronoï's theorem, its proof is slightly more complicated. A general tool to get this kind of results, is Bavard's condition (C) (see [2]).

As a consequence, we get the following properties:

Theorem 4 (C, 2001)**.**
(1) *(Algebraicity) Any perfect Humbert form $\mathcal{A}$ of dimension n is equivalent, modulo rescalation, to a Humbert form $\mathcal{B}$ with entries in a finite extension L of k.*
(2) *(Finiteness) The set of perfect n-ary Humbert forms over a number field k, up to scaling and equivalence under $GL_n(\mathcal{O}_k)$, is finite.*

Unfortunately, there is no general Voronoï's algorithm to determine all the perfect forms in a given dimension. Nevertheless, in dimension 2, in a joint work with M.I. Icaza, R. Baeza and M. O'Ryan (University of Talca, Chile), we were able to compute some Hermite-Humbert constants over real quadratic fields with small discriminant:

Theorem 5 (B-C-I-O, 2001)**.** *The value of $\gamma_{2,k}$ for the real quadratic fields $k = \mathbb{Q}(\sqrt{d})$, $d = 2, 3, 5$ are as follows:*

d	*2*	*3*	*5*
$\gamma_{2,\mathbb{Q}(\sqrt{d})}$	$\frac{4}{2\sqrt{6}-3}$	4	$4/\sqrt{5}$

The previous theorem may be interpretated in terms of Hilbert modular surfaces (see [1] for details). Indeed, if k is a totally real number field with $[k:\mathbb{Q}]=r$, there is a natural action of $\mathrm{SL}_2(\mathcal{O}_k)$ on $\mathfrak{H}^r$, induced by the diagonal embedding of $\mathrm{SL}_2(\mathcal{O}_k)$ in $\mathrm{SL}_2(\mathbb{R})^r$ and the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$. The quotient $\mathrm{SL}_2(\mathcal{O}_k)\backslash\mathfrak{H}^r$ is by definition a Hilbert modular surface. As in the first section, we get a 1-1 correspondance

$$\mathrm{SL}_2(\mathcal{O}_k)\backslash\mathfrak{H}^r \longleftrightarrow \mathbb{R}^r_{>0}\backslash P_{2,k}/\mathrm{SL}_2(\mathcal{O}_k)$$

If we assume again, for simplicity, that $h_k=1$, then the Hermite-Humbert number of a form $\mathcal{A}\in P_{2,k}$ can be interpreted as the distance to infinity of the corresponding point in $\mathrm{SL}_2(\mathcal{O}_k)\backslash\mathfrak{H}^r$:

$$\begin{aligned} \text{“distance to } \infty\text{”} &\longleftrightarrow \text{“}\gamma(\mathcal{A})\text{”} \\ \text{“local lowest point”} &\longleftrightarrow \text{“extreme Humbert form”} \end{aligned}$$

The problem of finding lowest points on Hilbert modular surface had been investigated by H. Cohn in the 60's (see [4]). For the real quadratic fields $\mathbb{Q}(\sqrt{d})$, $d=2,3,5$, he had made some conjectures which, once translated in the appropriate language, appear as consequences of our theorem.

Remark 2. If $h_k>1$, then there are h_k “cusps” and h_k Humbert invariants, corresponding to the distance to these various cusps.

3. Generalized Hermite constants (after T. Watanabe).

The Hermite type invariants mentionned so far, may be viewed as particular instances of a general theory developed recently by Watanabe (see [19],[13]). The general setting is as follows: we take k a number field, and G a connected reductive algebraic group over k. We let $\mathfrak{V}=\mathfrak{V}_f\cup\mathfrak{V}_\infty$ be the set of places of k, and $\mathbb{A}$ the adele ring of k . One has to fix $\rho: G\longrightarrow GL(V)$ a strongly k-rational absolutely irreducible representation of G on a k-vector space V (see [19],[13]for details). Let D be the highest weight space of ρ, with stabilizer P (parabolic subgroup). Then $X=G/P$ is a smooth projective variety embedded in $\mathbb{P}(V)$ via ρ. On each localisation $V_v=V\otimes_k k_v$, $v\in\mathfrak{V}$, we fix a norm $||\cdot||_v$. Then, one has to choose a suitable maximal compact subgroup K in $G(\mathbb{A})$. For $x\in GL(V(\mathbb{A}))V(k)$, we set $||x||_{\mathbb{A}}:=\prod_{v\in\mathfrak{V}}||x_v||_v$. One of the conditions that K must satisfy is that $||.||_{\mathbb{A}}$ is K-invariant. Finally, $||.||_{\mathbb{A}}$ is normalized by the condition $||x_0||_{\mathbb{A}}=1$ for $x\in D(k)\setminus\{0\}$ (this might be achieved by a suitable choice of the local norms $||\cdot||_v$).

For each $g\in G(\mathbb{A})^1:=\{g\in G(\mathbb{A}):\forall\chi\in X_k(G)\quad|\chi(g)|_{\mathbb{A}}=1\}$, define $H_g(x):=||\rho(g\gamma)x_0||_{\mathbb{A}}^{1/[k:\mathbb{Q}]}$, where $x=\rho(\gamma)x_0$. Then

Theorem 6 (Watanabe, 2000)**.**

$$\begin{aligned} K\setminus G(\mathbb{A})^1/G(k) &\longrightarrow \mathbb{R}_+ \\ g &\mapsto \min_{x\in X(k)} H_g(x) \end{aligned}$$

is a bounded continuous function. The generalized Hermite constant associated to $(\rho, ||.||_{\mathbb{A}})$ *is*

$$\mu(\rho, ||.||_{\mathbb{A}}) := \max_{g \in G(\mathbb{A})^1} \min_{x \in X(k)} H_g(x)^2.$$

Example 1. $G = GL_n$

(1) $k = \mathbb{Q}$,
  (a) if ρ is the natural representation in $\mathbb{Q}^n$, then $\mu(\rho, ||.||_{\mathbb{A}}) = \gamma_n$, the classical Hermite constant.
  (b) if ρ_d is the natural *exterior* representation in $\bigwedge^d \mathbb{Q}^n$, then $\mu(\rho, ||.||_{\mathbb{A}}) = \gamma_{n,d}$ the Rankin constant (see [15],[5]).

(2) $k =$ number field,
  (a) if ρ is the natural representation in k^n, then $\mu(\rho, ||.||_{\mathbb{A}}) = \gamma_{n,k}$ the Hermite-Humbert constant.
  (b) if ρ_d is the natural *exterior* representation in $\bigwedge^d k^n$, then $\mu(\rho, ||.||_{\mathbb{A}}) = \gamma_{n,d}$ the Rankin-Thunder constant (see [17]).

An opened question is wether there exists a general Voronoï type theorem to characterize the local maxima of $\mu(\rho, ||.||_{\mathbb{A}})$. In the next section, we present a new example where this holds true.

4. Quaternionic Humbert forms (work in progress with T. Watanabe).

Let k be a number field, and D a quaternion field over k. Let $\mathfrak{V} = \mathfrak{V}_f \cup \mathfrak{V}_\infty$ be the set of places of k. where $\mathfrak{V}_f$ and $\mathfrak{V}_\infty$ stand for the set of finite and infinite places respectively. The set of infinite places splits into real and complex places, denoted $\mathfrak{V}_{\infty,1}$ and $\mathfrak{V}_{\infty,2}$ respectively, and $\mathfrak{V}_{\infty,1}$ splits into ramified ($\mathfrak{V}'_{\infty,1}$) and split ($\mathfrak{V}''_{\infty,1}$) places. The cardinality of $\mathfrak{V}_{\infty,1}$ and $\mathfrak{V}_{\infty,2}$ are denoted r and s respectively. In other words, setting $D_v = D \otimes k_v$ for $v \in \mathfrak{V}$, one has $D_v = \begin{cases} \mathbb{H} \text{ if } v \in \mathfrak{V}'_{\infty,1} \\ M_2(\mathbb{R}) \text{ if } v \in \mathfrak{V}''_{\infty,1} \\ M_2(\mathbb{C}) \text{ if } v \in \mathfrak{V}_{\infty,2} \end{cases}$

We apply the previous theory to the affine algebraic k-group G defined by $G(R) = GL_n(D \otimes_k R)$ for any k-algebra R, and ρ the natural representation in D^n. In order to get a Voronoï type theorem for the corresponding $\mu(\rho, ||.||_{\mathbb{A}})$, we first translate the definitions from the adelic to global setting. This is done *via* the introduction of *quaternionic Humbert forms*

Definition 5. A n-ary quaternionic Humbert form over D is a $(r+s)$-tuple $S = (S_v)_{v \in \mathfrak{V}_\infty}$, where:
- if $v \in \mathfrak{V}'_{\infty,1}$, S_v is an n-ary positive definite hermitian form on $\mathbb{H}^n$.
- if $v \in \mathfrak{V}''_{\infty,1}$ (resp. $v \in \mathfrak{V}_{\infty,2}$), S_v is a $2n$-ary positive definite symmetric (resp. hermitian) form on $\mathbb{R}^{2n}$ (resp. $\mathbb{C}^{2n}$).

The set of such n-ary quaternionic Humbert forms is denoted $P_{n,D}$. To a given $S \in P_{n,D}$, one can assign a value $S[u]$ at $u \in D^n$ (the precise definition is not straightforward, so we refer to [7] for the details).

Let $\mathfrak{O}$ be a maximal order of D. Assume, to simplify, that $h_D = 1$, and set

$$m(\mathcal{S}) = \min\{\mathcal{S}[u] : u \in \mathfrak{O}^n \setminus \{0\}\}$$
$$\gamma(\mathcal{S}) = \frac{m(\mathcal{S})}{(\det \mathcal{S})^{1/n}}$$
$$\text{and } \gamma_{n,D} := \sup_{S \in P_{n,D}} \{\gamma(S)\}.$$

Then

Theorem 7 (C-W, 2003). $\mu(\rho, ||.||_{\mathbb{A}}) = \gamma_{n,D}$.

Moreover, we get a Voronoï type theorem, with suitable notions of perfection and eutaxy that are too complicated to be detailed here (see [7]).

Example 2. We consider the three euclidean quaternion fields over $\mathbb{Q}$, namely, with the standard notation,

$$D_2 = (-1,-1)_{\mathbb{Q}}, D_3 = (-1,-3)_{\mathbb{Q}} \text{ and } D_5 = (-2,-5)_{\mathbb{Q}}$$

(recall that $(a,b)_k$ stands for the quaternion algebra over k generated by i and j with $i^2 = a$, $j^2 = b$ and $ij = -ji$). Then

Proposition 1 (C-W, 2003). *For $m = 2, 3, 5$, one has*
(i) $\gamma_n(D_m) \leq m^{n(n-1)/2}$,
(ii) $\gamma_2(D_m) = m$.

REFERENCES

(1) R. Baeza, R. Coulangeon, M.I. Icaza and M. O'Ryan (2001), *Hermite's constant for quadratic number fields*, Experimental Mathematics **10**, no. 4, 543–551.

(2) C. Bavard (1997), *Systole et invariant d'Hermite.* J. Reine Angew. Math., **482**, 93–120.

(3) A.-M. Bergé and J. Martinet (1991), *Réseaux extrêmes pour un groupe d'automorphismes.* Journées Arithmétiques, 1989 (Luminy, 1989). Astérisque, **No. 198-200**, 41–66

(4) H., Cohn (1965), *A numerical survey of the floors of various Hilbert fundamental domains*, Math. Comp., **19**, 594-605.

(5) R. Coulangeon, (1996) *Réseaux k-extrêmes.*, Proc. London Math. Soc. (3), **73** , no. 3, 555–574.

(6) R. Coulangeon (2001), *Voronoï theory over algebraic number fields*, Monographies de l'Enseignement Mathématique, **no 37**, 147-162.

(7) R. Coulangeon and T. Watanabe, T. *Hermite constant and Voronoï theory over a quaternion skew field*, submitted

(8) P. Humbert (1940), *Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini*, Comment. Math. Helv., **12**, 263–306.

(9) P. Humbert (1949), *Réduction de formes quadratiques dans un corps algébrique fini*, Comment. Math. Helv., **23**, 50–63.

(10) M.I. Icaza (1997), *Hermite constant and extreme forms for algebraic number fields*, J. London Math. Soc. (2), **55**, (1997), 11–22.

(11) A. Korkine and G. Zolotareff (1873), *Sur les formes quadratiques*, Math. Ann, **6**, 366–389.

(12) A. Korkine and G. Zolotareff (1877), *Sur les formes quadratiques positives*, Math. Ann, **11**, 242–292.

(13) M. Masanori and T. Watanabe (2001), *Adèle geometry of numbers*, Class field theory—its centenary and prospect (Tokyo, 1998), Math. Soc. Japan, Tokyo (2001), 509–536.

(14) J. Opgenorth (2001), *Dual cones and the Voronoï algorithm.* Experiment. Math., **10**, no. 4, 599–608.

(15) R. A. Rankin, *On positive definite quadratic forms*, J. London Math. Soc., **28** (1953), 309–314.

(16) K. Rogers and H. P. F. Swinnerton-Dyer (1958), *The geometry of numbers over algebraic number fields*, Trans. Amer. Math. Soc., **88**, 227–242

(17) J. L. Thunder (1998), *Higher-dimensional analogs of Hermite's constant*, Michigan Math. J., **45**, no. 2, 301–314.

(18) G. Voronoï (1908), *Nouvelles applications des paramètres continus à la théorie des formes quadratiques : 1 Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine angew. Math, **133**, 97–178.

(19) T. Watanabe (2000), *On an analog of Hermite's constant.*, J. Lie Theory, **10**, no. 1, 33–52.

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