SIAM J. FINANCIAL MATH.
Vol. 2, pp. 587–626

On the Robustness of the Snell Envelope*

Pierre Del Moral†, Peng Hu†, Nadia Oudjane‡, and Bruno Rémillard§

Abstract. We analyze the robustness properties of the Snell envelope backward evolution equation for the discrete time optimal stopping problem. We consider a series of approximation schemes, including cut–off-type approximations, Euler discretization schemes, interpolation models, quantization tree models, and the stochastic mesh method of Broadie and Glasserman. In each situation, we provide nonasymptotic convergence estimates, including $\mathbb{L}_p$-mean error bounds and exponential concentration inequalities. We deduce these estimates from a single and general robustness property of Snell envelope semigroups. In particular, this analysis allows us to recover existing convergence results for the quantization tree method and to improve significantly the rates of convergence obtained for the stochastic mesh estimator of Broadie and Glasserman. In the second part of the article, we propose a new approach based on a genealogical tree approximation model of the reference Markov process in terms of a neutral-type genetic model. In contrast to Broadie–Glasserman Monte Carlo models, the computational cost of this new stochastic approximation is linear in the number of particles. Some simulation results are provided and confirm the interest of this new algorithm.

Key words. Snell envelope, optimal stopping, American option pricing, genealogical trees, interacting particle model

AMS subject classifications. Primary, 60G40; Secondary, 91G60

DOI. 10.1137/100798016

1. Introduction.

The evaluation of optimal stopping time of random processes, based on a given optimality criterion, is one of the major problems in stochastic control and optimal stopping theory, particularly in financial mathematics with American options pricing and hedging. The present paper is restricted to the case of the discrete time optimal stopping problem corresponding in finance to the case of Bermudan options.

It is well known that the price of Bermudan options giving the opportunity to exercise a payoff f_k at discrete dates $k = 0, \ldots, n$ can be calculated by a backward dynamic programming formula. This recursion consists in comparing at each time step k the immediate payoff f_k and the expectation of the future gain (or the so-called continuation value), which precisely involves the Markov transition M_{k+1} of the underlying assets process (X_k).

The first objective of this paper is to provide a simple framework to analyze in unison most of the numerical schemes currently used in practice to approximate the Snell envelope, which are precisely based on the approximation of the dynamic programming recursion. The

*Received by the editors June 9, 2010; accepted for publication (in revised form) May 25, 2011; published electronically August 25, 2011.
http://www.siam.org/journals/sifin/2/79801.html

†Centre INRIA Bordeaux et Sud-Ouest & Institut de Mathématiques de Bordeaux, Université de Bordeaux I, 351 cours de la Libération, 33405 Talence cedex, France (Pierre.Del-Moral@inria.fr, Peng.Hu@inria.fr).

‡EDF R&D Clamart, Clamart, France (nadia.oudjane@edf.fr), and Université Paris 13 and FiME (Laboratoire de Finance des Marchs de l'Energie (Dauphine, CREST, EDF R&D)), Paris, France.

§HEC Montréal, Montréal, Canada (bruno.remillard@hec.ca).

idea is to analyze the related approximation error in terms of robustness properties of the Snell envelope with respect to the pair parameters (f_k, M_k). Hence, we include in our analysis approximation schemes which are defined in terms of some *approximate* pairs of functions and transitions $(\widehat{f}_k, \widehat{M}_k)_{k\geq 0}$. After stating the robustness lemma (Lemma 2.1) in section 2, we deduce from it nonasymptotic convergence theorems, including $\mathbb{L}_p$-mean error bounds and related exponential inequalities for the deviations of Monte Carlo–type approximation models.

In section 3, that approach allows us to derive nonasymptotic error bounds for deterministic approximation schemes such as *cut-off techniques, Euler-type discrete time approximations, quantization tree models, interpolation-type approximations* and then recover or improve some existing results or in some cases provide new bounds. We emphasize that this non asymptotic robustness analysis also allows us to combine in a natural way several approximation models. For instance, under appropriate tightness conditions, cut-off techniques can be used to reduce the numerical analysis of the Snell envelope to compact state spaces and bounded functions $\widehat{f}_n$. In the same line of ideas, in designing any type of Monte Carlo approximation model, we can suppose that the transitions of the chain X_n are known, based on a preliminary analysis of Euler-type approximation models.

In section 4, we focus on two kinds of Monte Carlo importance sampling approximation schemes. The first one is the stochastic mesh method introduced by Broadie and Glasserman in their seminal paper [5] (see also [24] for some recent refinements). The principal idea of that methodology is to operate a change of measures to replace conditional expectations by simple expectations involving Markov transition densities with respect to some reference measures. The number of sampled points with respect to the reference measures η_n required by this model can be constant in every exercise date. This technique avoids the explosion issue of the naive Monte Carlo method. As with any full Monte Carlo–type technique, the main advantage of their approach is that it applies to high-dimensional Bermudan options with a finite but possibly large number of exercise dates. In [5], the authors provided a set of conditions under which the Monte Carlo importance scheme converges as the computational effort increases. However, the computing time grows quadratically with the number of sampled points in the stochastic mesh. In this context, in section 4.2, we provide new nonasymptotic estimates, including $\mathbb{L}_p$-mean error bounds and exponential concentration inequalities. Our analysis allows us to derive (4.7), improving significantly existing convergence results (see [5] or [1]).

The second type of Monte Carlo importance sampling scheme discussed in this article is another version of the Broadie–Glasserman model, called *average density* in their original article. The main advantage of this strategy comes from the fact that the sampling distribution η_n can be chosen as the distribution of the random states X_n of the reference Markov chain, even if the Radon–Nikodym derivatives $R_n(x,y) = \frac{dM_n(x,\cdot)}{d\eta_n}(y)$ are not known explicitly. Here, we assume only that the Markov transitions $M_n(x,\cdot)$ are absolutely continuous with respect to some measures λ_n. We can then approximate these functions with empirical measures. In this situation, we can recover an approximation similar to the original stochastic mesh method, except that the Radon–Nikodym derivatives $R_{k+1}(\xi_k^i, \xi_{k+1}^j)$ are replaced by approximations. The stochastic analysis of this particle model is provided in the second part of section 4.2 and follows essentially the same line of arguments as that of the Broadie–Glasserman model.

In the final part of the article, section 5, we present a new Monte Carlo approach based on the genealogical tree evolution model associated with a neutral genetic model with mutations given by the Markov transitions M_n. The main advantage of this new strategy comes from the fact that the computational effort of the algorithm is now linear in the number of sampled points. We recall that a neutral genetic model is a Markov chain with a selection/mutation transition. During the mutation phase, the particles explore the state space independently according to the Markov transitions while the selection step induces interactions between the various particles. This type of model is frequently used in biology and genetic algorithms literature (see, for instance, [16] and the references therein).

An important observation concerns the genealogical tree structure of the genetic particle model that we consider. The main advantage of this path particle model comes from the fact that the occupation measure of the ancestral tree model converges in some sense to the distribution of the path of the reference Markov chain. It is also well known that the Snell envelope associated with a Markov chain evolving on some finite state space is easily computed using the tree structure of the chain evolution. Therefore, replacing the reference distribution $\mathbb{P}_n$ by its N-approximation $\mathbb{P}_n^N$, we define an N-approximated Markov model whose evolutions are described by the genealogical tree model defined above. We can then construct the approximation $\widehat{u}_k$ as the Snell envelope associated with this N-approximated Markov chain. Several estimates of convergence are provided in section 5. Finally, some numerical simulations are performed, illustrating the interest of our new algorithm.

2. Preliminary.

In a discrete time setting, the problem is related to the pricing of Bermuda options and is defined in terms of a given real-valued stochastic process $(Z_k)_{0\leq k\leq n}$, adapted to some increasing filtration $\mathcal{F} = (\mathcal{F}_k)_{0\leq k\leq n}$ that represents the available information at any time $0 \leq k \leq n$. For any $k \in \{0, \dots, n\}$, let $\mathcal{T}_k$ be the set of all stopping times τ taking values in $\{k, \dots, n\}$. The Snell envelope of $(Z_k)_{0\leq k\leq n}$ is the stochastic process $(U_k)_{0\leq k\leq n}$ defined for any $0 \leq k < n$ by the following backward equation:

$$U_k = Z_k \vee \mathbb{E}(U_{k+1}|\mathcal{F}_k),$$

with the terminal condition $U_n = Z_n$, where $a \vee b = \max(a, b)$. The main property of this stochastic process is that

$$\begin{aligned} U_k &= \sup_{\tau\in\mathcal{T}_k} \mathbb{E}(Z_\tau|\mathcal{F}_k) = \mathbb{E}(Z_{\tau_k^*}|\mathcal{F}_k) \\ \text{with} \quad \tau_k^* &= \min\{k \leq l \leq n \ : \ U_l = Z_l\} \in \mathcal{T}_k. \end{aligned} \tag{2.1}$$

At this level of generality, in the absence of any additional information on the filtration $\mathcal{F}$ or on the terminal random variable Z_n, no numerical computation of the Snell envelope is available. To get one step further, we assume that $(\mathcal{F}_n)_{n\geq 0}$ is the natural filtration associated with some Markov chain $(X_n)_{n\geq 0}$ taking values in some sequence of measurable state spaces $(E_n, \mathcal{E}_n)_{n\geq 0}$. Let $\eta_0 = \text{Law}(X_0)$ be the initial distribution on E_0, and define by $M_n(x_{n-1}, dx_n)$ the elementary Markov transition of the chain from E_{n-1} into E_n. We also assume that $Z_n = f_n(X_n)$ for some collection of nonnegative measurable functions f_n on E_n. In this situation, the computation of the Snell envelope amounts to solving the following backward functional equation:

$$u_k = \mathcal{H}_{k+1}(u_{k+1}) = f_k \vee M_{k+1}(u_{k+1}), \tag{2.2}$$

for any $0 \leq k < n$, with the terminal value $u_n = f_n$. In the above displayed formula, $M_{k+1}(u_{k+1})$ stands for the measurable function on E_k defined for any $x_k \in E_k$ by the conditional expectation formula

$$\begin{aligned} M_{k+1}(u_{k+1})(x_k) &= \int_{E_{k+1}} M_{k+1}(x_k, dx_{k+1})\ u_{k+1}(x_{k+1}) \\ &= \mathbb{E}\left(u_{k+1}(X_{k+1})|X_k = x_k\right). \end{aligned}$$

Let $\mathcal{H}_{k,l} = \mathcal{H}_{k+1} \circ \mathcal{H}_{k+1,l}$, with $k \leq l \leq n$, be the nonlinear semigroups associated with the backward equation (2.2). We use the convention $\mathcal{H}_{k,k} = \mathrm{Id}$, the identity operator, so that $u_k = \mathcal{H}_{k,l}(u_l)$, for any $k \leq l \leq n$. Given a sequence of bounded integral operators M_k from some state space E_{k-1} into another E_k, let us denote by $M_{k,l}$ the composition operator such that $M_{k,l} := M_{k+1}M_{k+2}\ldots M_l$, for any $k \leq l$, with the convention $M_{k,k} = \mathrm{Id}$, the identity operator. With this notation, one can check that a necessary and sufficient condition for the existence of the Snell envelope $(u_k)_{0\leq k\leq n}$ is that $M_{k,l}f_l(x) < \infty$ for any $1 \leq k \leq l \leq n$ and any state $x \in E_k$. To check this claim, we simply notice that

$$f_k \leq u_k \leq f_k + M_{k+1}u_{k+1} \quad \forall\, 1 \leq k \leq n \implies f_k \leq u_k \leq \sum_{k\leq l\leq n} M_{k,l}f_l\,,\ \forall\, 1 \leq k \leq n. \tag{2.3}$$

From the readily proved Lipschitz property $|\mathcal{H}_k(u) - \mathcal{H}_k(v)| \leq M_{k+1}\left(|u - v|\right)$, for any functions u, v on E_k, we also have that

$$|\mathcal{H}_{k,l}(u) - \mathcal{H}_{k,l}(v)| \leq M_{k,l}\left(|u - v|\right) \tag{2.4}$$

for any functions u, v on E_l and any $k \leq l \leq n$.

Even if it looks simple, the numerical solving of the recursion (2.2) often requires extensive computations. The central problem is to compute the conditional expectation $M_{k+1}(u_{k+1})$ on the whole state space E_k at every time step $0 \leq k < n$. For Markov chain models taking values in some finite state spaces (with a reasonably large cardinality), the above expectations can be easily computed by a simple backward inspection of the whole realization tree that lists all possible outcomes and every transition of the chain. In more general situations, we need to resort to some approximation strategy. Most of the numerical approximation schemes amount to replacing the pair of functions and Markov transitions $(f_k, M_k)_{0\leq k\leq n}$ by some approximation model $(\widehat{f}_k, \widehat{M}_k)_{0\leq k\leq n}$ on some possibly reduced measurable subsets $\widehat{E}_k \subset E_k$. Let $\widehat{u}_k$ be the Snell envelope on $\widehat{E}_k$ associated with the functions $\widehat{f}_k$ and the sequence of integral operators $\widehat{M}_k$ from $\widehat{E}_{k-1}$ into $\widehat{E}_k$. As in (2.2), the computation of the Snell envelope $\widehat{u}_k$ amounts to solving the following backward functional equation:

$$\widehat{u}_k = \widehat{\mathcal{H}}_{k+1}(\widehat{u}_{k+1}) = \widehat{f}_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1}). \tag{2.5}$$

Let $\widehat{\mathcal{H}}_{k,l} = \widehat{\mathcal{H}}_{k+1} \circ \widehat{\mathcal{H}}_{k+1,l}$, with $k \leq l \leq n$, be the nonlinear semigroups associated with the backward equations (2.5), so that $\widehat{u}_k = \widehat{\mathcal{H}}_{k,l}(\widehat{u}_l)$, for any $k \leq l \leq n$. Using the elementary inequality $|(a \vee a') - (b \vee b')| \leq |a - b| + |a' - b'|$, which is valid for any $a, a', b, b' \in \mathbb{R}$, for any $0 \leq k < n$ and for any functions u on E_{k+1} one readily obtains the local approximation inequality

$$\left|\mathcal{H}_{k+1}(u) - \widehat{\mathcal{H}}_{k+1}(u)\right| \leq |f_k - \widehat{f}_k| + |(M_{k+1} - \widehat{M}_{k+1})(u)|. \tag{2.6}$$

To transfer these local estimates to the semigroups $\mathcal{H}_{k,l}$ and $\widehat{\mathcal{H}}_{k,l}$ we use the same perturbation analysis as in [9, 13, 23, 28] in the context of nonlinear filtering semigroups and particle approximation models. The difference between the approximate and the exact Snell envelope can be written as a telescoping sum

$$u_k - \widehat{u}_k = \sum_{l=k}^{n} \left[\widehat{\mathcal{H}}_{k,l}(\mathcal{H}_{l+1}(u_{l+1})) - \widehat{\mathcal{H}}_{k,l}(\widehat{\mathcal{H}}_{l+1}(u_{l+1}))\right],$$

setting for simplicity $\mathcal{H}_{n+1}(u_{n+1}) = u_n$ and $\widehat{\mathcal{H}}_{n+1}(u_{n+1}) = \widehat{u}_n$, for $l = n$. Combining the Lipschitz property (2.4) of the semigroup $\widehat{\mathcal{H}}_{k,l}$ with the local estimate (2.6), one finally gets the following robustness lemma, which is a natural and fundamental tool for the analysis of the Snell envelope approximations.

Lemma 2.1. *For any $0 \le k < n$, on the state space $\widehat{E}_k$, we have that*

$$|u_k - \widehat{u}_k| \le \sum_{l=k}^{n} \widehat{M}_{k,l}|f_l - \widehat{f}_l| + \sum_{l=k}^{n-1} \widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})u_{l+1}|.$$

The perturbation analysis of nonlinear semigroups described above and the resulting robustness lemma are not really new. As mentioned previously, it is a rather standard tool in approximation theory and numerical probability. More precisely, these Lipschitz-type estimates are often used by induction or as an intermediate technical step in the proof of a convergence theorem of some particular approximation scheme.

In the context of optimal stopping problems, similar induction arguments are developed to prove the convergence of some specific approximation models, for instance, in the papers of Egloff [18], Gobet, Lemor, and Warin [21], or Pagès [25]. However, to the best of our knowledge, the general and abstract semigroup formulation given above and its direct application to different approximation models seem to represent the first result of this type for that class of models.

Besides the fact that the convergence of many Snell approximation schemes results from a single robustness property, Lemma 2.1 can be used sequentially and without further work to obtain nonasymptotic estimates for models combining several levels of approximations. In the same vein, and whenever it is possible, Lemma 2.1 can also be used as a technical tool to reduce the analysis of Snell approximation models on compact state spaces or even on finite but possibly large quantization trees or Monte Carlo–type grids.

We end this section with an exponential inequality that can be readily deduced from the $\mathbb{L}_p$-mean error bounds presented in this article. For a more thorough discussion on the connection between Khintchine style $\mathbb{L}_p$-mean error bounds and concentration inequalities, we refer the reader to [10, 11, 12] and the more recent article on the concentration properties of mean field–type particle models [15].

Lemma 2.2. *Suppose the estimates have the following form:*

$$\sqrt{N} \sup_{x \in E_k} \mathbb{E}\left(|u_k(x) - \widehat{u}_k(x)|^p\right)^{\frac{1}{p}} \le a(p) b_k(n),$$

where $b_k(n)$ are some finite constants whose values do not depend on the parameter p and $a(p)$

is a collection of constants such that for all nonnegative integers r,

$$a(2r)^{2r} = (2r)_r \; 2^{-r} \quad \textit{and} \quad a(2r+1)^{2r+1} = \frac{(2r+1)_{r+1}}{\sqrt{r+1/2}} \; 2^{-(r+1/2)}, \tag{2.7}$$

with the notation $(q)_p = q!/(q-p)!$, *for any* $1 \le p \le q$. *Then we deduce the following exponential concentration inequality:*

$$\sup_{x \in E_k} \mathbb{P}\left(|u_k(x_k) - \widehat{u}_k(x_k)| > \frac{b_k(n)}{\sqrt{N}} + \epsilon \right) \le \exp\left(-N\epsilon^2/(2b_k(n)^2)\right). \tag{2.8}$$

Proof. This result is a direct consequence of the fact that for any nonnegative random variable U, if there exists a bounded positive real b such that

$$\forall\, r \ge 1, \qquad \mathbb{E}\,(U^r)^{\frac{1}{r}} \le a(r)b,$$

where $a(r)$ is defined by (2.7), then

$$\mathbb{P}\,(U \ge b+\epsilon) \le \exp\left(-\epsilon^2/(2b^2)\right).$$

To check this implication, we first notice that

$$\mathbb{P}\,(U \ge b+\epsilon) \le \inf_{t \ge 0} \{e^{-t(b+\epsilon)} \mathbb{E}[e^{tU}]\}.$$

Then, developing the exponential and using the moments boundedness assumption, one obtains that for all $t \ge 0$,

$$\mathbb{E}\left(e^{tU}\right) \le \exp\left(\frac{(bt)^2}{2} + bt\right).$$

As a result,

$$\mathbb{P}\,(U \ge b+\epsilon) \le \exp\left(-\sup_{t\ge 0}\left(\epsilon t - \frac{(bt)^2}{2}\right)\right). \qquad \blacksquare$$

3. Some deterministic approximation models. In this section, we analyze the robustness of the Snell envelope with respect to some deterministic approximation schemes that are parts of many algorithms proposed to approximate the Snell envelope. Hence, the nonasymptotic error bounds provided in this section can be applied and combined to derive convergence rates for such algorithms. We recover or improve previous results and in some cases state new error bounds.

3.1. Cut–off-type models. It is often useful, when computing the Snell envelope, to approximate the state space by a compact set. Indeed, Glasserman and Yu [20] showed that for standard (unbounded) models (like Black–Scholes), the Monte Carlo estimation requires samples of exponential size in the number of variables of the value function, whereas the bounded state space assumption enables one to estimate the Snell envelope from samples of polynomial size in the number of variables. For instance, in [19], the authors proposed a new algorithm that first requires a cut-off step which consists in replacing the price process by another process killed at first exit from a given bounded set. However, no bound is provided for the error induced by this cut-off approximation. In this section, we formalize a general cut-off model and provide some bounds on the error induced on the Snell envelope.

We suppose that for each n, E_n is a topological space with σ-fields $\mathcal{E}_n$ that contains the Borel σ-field on E_n. Our next objective is to find conditions under which we can reduce the backward functional equation (2.2) to a sequence of compact sets $\widehat{E}_n$.

To that end, we further assume that the initial measure η_0 and the Markov transition M_n of the chain X_n satisfy the following tightness property: For every sequence of positive numbers $\epsilon_n \in [0,1[$, there exists a collection of compact subsets $\widehat{E}_n \subset E_n$ such that

$$(\mathcal{T}) \qquad \eta_0(\widehat{E}_0^c) \leq \epsilon_0 \quad \text{and} \quad \forall n \geq 0, \quad \sup_{x_n \in \widehat{E}_n} M_{n+1}(x_n, \widehat{E}_{n+1}^c) \leq \epsilon_{n+1}.$$

For instance, this condition is clearly met for regular Gaussian-type transitions on the Euclidean space, for some collection of increasing compact balls.

In this situation, a natural cut-off consists in considering the Markov transitions $\widehat{M}_k$ restricted to the compact sets $\widehat{E}_k$:

$$\forall x \in \widehat{E}_{k-1}, \qquad \widehat{M}_k(x, dy) := \frac{M_k(x, dy)\ 1_{\widehat{E}_k}}{M_k(1_{\widehat{E}_k})(x)}.$$

These transitions are well defined as soon as $M_k(x, \widehat{E}_k) > 0$ for any $x \in \widehat{E}_{k-1}$. Using the decomposition

$$\begin{aligned}
[\widehat{M}_k - M_k](u_k) &= \widehat{M}_k(u_k) - M_k(1_{\widehat{E}_k} u_k) - M_k(1_{\widehat{E}_k^c} u_k) \\
&= \left(1 - \frac{1}{M_k(1_{\widehat{E}_k})}\right) M_k(u_k 1_{\widehat{E}_k}) - M_k(1_{\widehat{E}_k^c} u_k) \\
&= \frac{M_k(1_{\widehat{E}_k^c})}{M_k(1_{\widehat{E}_k})} M_k(u_k 1_{\widehat{E}_k}) - M_k(1_{\widehat{E}_k^c} u_k)
\end{aligned}$$

and then using Lemma 2.1 yields

$$\begin{aligned}
\|u_k - \widehat{u}_k\|_{\widehat{E}_k} &:= \sup_{x \in \widehat{E}_k} |u_k(x) - \widehat{u}_k(x)| \\
&\leq \sum_{l=k+1}^{n} \left[\left\| \frac{M_l(1_{\widehat{E}_l^c})}{M_l(1_{\widehat{E}_l})} \right\|_{\widehat{E}_{l-1}} \|M_l(u_l 1_{\widehat{E}_l})\|_{\widehat{E}_{l-1}} + \|M_l(u_l 1_{\widehat{E}_l^c})\|_{\widehat{E}_{l-1}} \right] \\
&\leq \sum_{l=k+1}^{n} \left[\frac{\epsilon_l}{1-\epsilon_l} \|M_l(u_l)\|_{\widehat{E}_{l-1}} + \|M_l(u_l^2)\|_{\widehat{E}_{l-1}}^{1/2} \epsilon_l^{1/2} \right].
\end{aligned}$$

We summarize the above discussion with the following result.

Theorem 3.1. *We assume that the tightness condition $(\mathcal{T})$ is met for every sequence of positive numbers $\epsilon_n \in [0,1[$ and for some collection of compact subsets $\widehat{E}_n \subset E_n$. In this situation, for any $0 \leq k \leq n$, we have that*

$$\|u_k - \widehat{u}_k\|_{\widehat{E}_k} \leq \sum_{l=k+1}^{n} \frac{\epsilon_l^{1/2}}{1-\epsilon_l^{1/2}} \|M_l(u_l^2)\|_{\widehat{E}_{l-1}}^{1/2}.$$

Note that

$$u_k \leq \sum_{l=k}^{n} M_{k,l}(f_l),$$

and therefore

$$\left\|M_k(u_k^2)\right\|_{\widehat{E}_{k-1}} \leq (n-k+1) \sum_{l=k}^{n} \left\|M_{k-1,l}(f_l)^2\right\|_{\widehat{E}_{k-1}}.$$

Consequently, one can find sets $(\widehat{E}_l)_{k<l\leq n}$ so that $\|u_k - \widehat{u}_k\|_{\widehat{E}_k}$ is as small as one wants as soon as $\left\|M_{k,l}(f_l)^2\right\|_{\widehat{E}_k} < \infty$ for any $0 \leq k < l \leq n$. A similar cut-off approach was introduced and analyzed in Bouchard and Touzi [6], but the cut-off was operated on some regression functions and not on the transition kernels.

3.2. Euler approximation models. In several application model areas, the discrete time Markov chain $(X_k)_{k\geq 0}$ is often given in terms of an $\mathbb{R}^d$-valued and continuous time process $(X_t)_{t\geq 0}$ given by a stochastic differential equation of the following form:

$$dX_t \;=\; a(X_t)dt + b(X_t)dW_t, \qquad \text{law}(X_0) \;=\; \eta_0, \tag{3.1}$$

where η_0 is a known distribution on $\mathbb{R}^d$, a, b are known functions, and W is a d-dimensional Wiener process. Except in some particular instances, the time homogeneous Markov transitions $M_k = M$ are usually unknown, and we need to resort to an Euler approximation scheme.

In this situation, any approximation of the Snell envelope which is based on simulations of the price process will be impacted by the error induced by the Euler scheme used in simulations. We propose here to provide bounds for that error. Notice that in this setting, the exercise dates are discrete and fixed, so that our results are not comparable with those from Dupuis and Wang [17], who analyzed the convergence of the discrete time optimal stopping problem to the continuous time optimal stopping problem when the frequency of exercise dates increases to infinity. Similarly, for numerical approximations of backward stochastic differential equations, [6] and [21] also analyzed the case where the number of exercise opportunities grows to infinity.

The discrete time approximation model with a fixed time step $1/m$ is defined by the following recursive formula:

$$\begin{aligned}
&\widehat{\xi}_0(x) = x,\\
&\widehat{\xi}_{\frac{(i+1)}{m}}(x) = \widehat{\xi}_{\frac{i}{m}}(x) + a\left(\widehat{\xi}_{\frac{i}{m}}(x)\right)\,\frac{1}{m} + b\left(\widehat{\xi}_{\frac{i}{m}}(x)\right)\,\frac{1}{\sqrt{m}}\,\epsilon_i,
\end{aligned}$$

where the ϵ_i's are independent and identically distributed (i.i.d.) centered and $\mathbb{R}^d$-valued Gaussian vectors with unit covariance matrix. The chain $(\widehat{\xi}_k)_{k\geq 0}$ is an homogeneous Markov with a transition kernel which we denote by $\widehat{M}$.

We further assume that the functions a and b are twice differentiable, with bounded partial derivatives of orders 1 and 2, and the matrix $(bb^*)(x)$ is uniformly nondegenerate.

In this situation, the integral operators M and $\widehat{M}$ admit densities, denoted by p and $\widehat{p}$. According to Bally and Talay [3, 4], we have that

$$[p \vee \widehat{p}] \;\leq\; c\,q \quad \text{and} \quad m\,|\widehat{p}-p| \;\leq\; c\,q, \tag{3.2}$$

with the Gaussian density $q(x,x') := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}|x-x'|^2}$, and a pair of constants (c,σ) depending only on the pair of functions (a,b). Let Q be the Markov integral operator on $\mathbb{R}^d$ with density q. We consider a sequence of functions $(f_k)_{0\leq k\leq n}$ on $\mathbb{R}^d$. Let $(u_k)_{0\leq k\leq n}$ and $(\widehat{u}_k)_{0\leq k\leq n}$ be the Snell envelope on $\mathbb{R}^d$ associated with the pair (M,f_k) and $(\widehat{M},f_k)$. Using Lemma 2.1, we readily obtain the following estimate:

$$|u_k - \widehat{u}_k| \leq \sum_{l=k}^{n-1} \widehat{M}^{l-k} |(M-\widehat{M})u_{l+1}| \leq \frac{c}{m} \sum_{l=k}^{n-1} \widehat{M}^{l-k} Q |u_{l+1}|.$$

Rather crude upper bounds that do not depend on the approximation kernels $\widehat{M}$ can be derived using the first inequality in (3.2):

$$|u_k - \widehat{u}_k| \leq \frac{1}{m} \sum_{l=1}^{n-k} c^l \; Q^l |u_{l+k}|.$$

Recalling that $u_{l+k} \leq \sum_{l+k\leq l'\leq n} M^{l'-(l+k)} f_{l'}$, we also have that

$$\begin{aligned} |u_k - \widehat{u}_k| &\leq \frac{1}{m} \sum_{l=1}^{n-k} c^l \; Q^l \sum_{l+k\leq l'\leq n} c^{l'-(l+k)} \; Q^{l'-(l+k)} f_{l'} \\ &\leq \frac{1}{m} \sum_{l=1}^{n-k} \sum_{l+k\leq l'\leq n} c^{l'-k} \; Q^{l'-k} f_{l'} = \frac{1}{m} \sum_{1\leq l\leq n-k} l \; c^l \; Q^l f_{k+l}. \end{aligned}$$

We summarize the above discussion with the following theorem.

Theorem 3.2. *Suppose the functions $(f_k)_{0\leq k\leq n}$ on $\mathbb{R}^d$ are chosen such that $Q^l f_{k+l}(x) < \infty$, for any $x \in \mathbb{R}^d$, and $1 \leq k+l \leq n$. Then, for any $0 \leq l \leq n$, we have the inequalities*

$$|u_k - \widehat{u}_k| \leq \frac{c}{m} \sum_{l=k}^{n-1} \widehat{M}^{l-k} Q |u_{l+1}| \leq \frac{1}{m} \sum_{1\leq l\leq n-k} l \; c^l \; Q^l f_{k+l}.$$

3.3. Interpolation-type models. Most algorithms proposed to approximate the Snell envelope provide discrete approximations $\hat{u}_k^i$ at some discrete (potentially random) points ξ_k^i of E_k. However, for several purposes, it can be interesting to consider approximations $\hat{u}_k$ of functions u_k on the whole space E_k. One motivation to do so is, for instance, to be able to define a new (low biased) estimator, $\bar{U}_k$, using a Monte Carlo approximation of (2.1), with a stopping rule $\hat{\tau}_k$ associated with the approximate Snell envelope $\hat{u}_k$, by replacing u_k by $\hat{u}_k$ in the characterization of the optimal stopping time τ_k^* (2.1), i.e.,

$$\bar{U}_k = \frac{1}{M} \sum_{i=1}^{M} f_{\hat{\tau}_k^i}(X^i_{\hat{\tau}_k^i}) \quad \text{with} \quad \hat{\tau}_k^i = \min\{k \leq l \leq n \;:\; \hat{u}_l(X_l^i) = f_l(X_l^i)\}, \tag{3.3}$$

where $X^i = (X_1^i, \ldots, X_n^i)$ are i.i.d. path according to the reference Markov chain dynamic.

In this section, we analyze nonasymptotic errors of some specific approximation schemes providing such interpolated estimators $\hat{u}_k$ of u_k on the whole state E_k. Let $\widehat{M}_{k+1} = \mathcal{I}_k \widetilde{M}_{k+1}$ be the composition of the Markov transition $\widetilde{M}_{k+1}$ from a finite set S_k into the whole state space E_{k+1}, with an auxiliary interpolation-type and Markov operator $\mathcal{I}_k$ from E_k into S_k, so that

$$\forall x_k \in S_k, \qquad \mathcal{I}_k(x_k, ds) = \delta_{x_k}(ds),$$

and such that the integrals

$$x \in E_k \mapsto \mathcal{I}_k(\varphi_k)(x) = \int_{S_k} \mathcal{I}_k(x, ds)\ \varphi_k(s)$$

of any function φ_k on S_k are easily computed starting from any point x_k in E_k. We further assume that the finite state spaces S_k are chosen so that

$$\|f - \mathcal{I}_k f\|_{E_k} \leq \epsilon_k(f, |S_k|) \to 0 \quad \text{as} \quad |S_k| \to \infty \tag{3.4}$$

for continuous functions f_k on E_k. An example of interpolation transition $\mathcal{I}_k$ is provided hereafter. Let $\widehat{M}_k = \mathcal{I}_{k-1} \widetilde{M}_k$ be the composition operator on the state spaces $\widehat{E}_k = E_k$.

The approximation models $\widetilde{M}_k$ are not necessarily deterministic. In [14], the authors examined the situation where

$$\forall s \in S_k, \qquad \widetilde{M}_k(s, dx) = \frac{1}{N_k} \sum_{1 \leq i \leq N_k} \delta_{X_k^i(s)}(dx),$$

where $X_k^i(s)$ stands for a collection of N_k independent random variables with common law $M_k(s, dx)$.

Theorem 3.3. *We suppose that the Markov transitions M_k are Feller, in the sense that $M_k(C(E_k)) \subset C(E_{k-1})$, where $C(E_k)$ stands for the space of continuous functions on the E_k. Let $(u_k)_{0 \leq k \leq n}$ and, respectively, $(\widehat{u}_k)_{0 \leq k \leq n}$ be the Snell envelopes associated with the functions $f_k = \widehat{f}_k$ and the Markov transitions M_k (and, respectively, $\widehat{M}_k = \mathcal{I}_{k-1} \widetilde{M}_k$) on the state spaces $\widehat{E}_k = E_k$. Then*

$$\|u_k - \widehat{u}_k\|_{E_k} \leq \sum_{l=k}^{n-1} \left[\epsilon_l \left(M_{l+1} u_{l+1}, |S_l| \right) + \|(M_{l+1} - \widetilde{M}_{l+1}) u_{l+1}\|_{S_l} \right].$$

The proof of the theorem is a direct consequence of Lemma 2.1 combined with the following decomposition:

$$\begin{aligned} &\|u_k - \widehat{u}_k\|_{E_k} \\ &\quad \leq \sum_{l=k}^{n-1} \left[\|(\mathrm{Id} - \mathcal{I}_l) M_{l+1}) u_{l+1}\|_{E_l} + \|\mathcal{I}_l (M_{l+1} - \widetilde{M}_{l+1}) u_{l+1}\|_{E_l} \right]. \end{aligned} \tag{3.5}$$

We illustrate these results in the typical situation where the spaces E_k are the convex hull generated by the finite sets S_k. First, we present the definition of the interpolation operators. Let $\mathcal{P} = \{\mathcal{P}^1, \ldots, \mathcal{P}^m\}$ be a partition of a convex and compact space E into simplexes with disjoint nonempty interiors, so that $E = \cup_{1\leq i\leq m}\mathcal{P}_i$. We denote by $\delta(\mathcal{P})$ the refinement degree of the partition $\mathcal{P}$:

$$\delta(\mathcal{P}) := \sup_{1\leq i\leq m}\sup_{x,y\in\mathcal{P}_i}\|x-y\|.$$

Let $S = \mathcal{V}(\mathcal{P})$ be the set of vertices of these simplexes. We denote by $\mathcal{I}$ the interpolation operator defined by $\mathcal{I}(f)(s) = f(s)$, if $s\in S$, and if x belongs to some simplex $\mathcal{P}^j$ with vertices $\{x_1^j,\dots,x_{d_j}^j\}$,

$$\mathcal{I}(f)\left(\sum_{1\leq i\leq d_j}\lambda_i\ x_j^i\right) = \sum_{1\leq i\leq d_j}\lambda_i\ f(x_i^j),$$

where the barycenters $(\lambda_i)_{1\leq i\leq d_j}$ are the unique solution of

$$x = \sum_{1\leq i\leq d_j}\lambda_i\ x_i^j \quad\text{with}\quad (\lambda_i)_{1\leq i\leq d_j}\in[0,1]^{d_j}\quad\text{and}\quad \sum_{1\leq i\leq d_j}\lambda_i = 1.$$

The Markovian interpretation is that starting from x, one chooses the "closest simplex" and then one chooses one of its vertices x_i with probability λ_i.

For any $\delta>0$, let $\omega(f,\delta)$ be the δ-modulus of continuity of a function $f\in C(E)$:

$$\omega(f,\delta) := \sup_{(x,y)\in E:\|x-y\|\leq\delta}|f(x)-f(y)|.$$

The following technical lemma provides a simple way to check condition (3.4) for interpolation kernels.

Lemma 3.4. *Then, for any $f,g\in C(E)$,*

$$\sup_{x\in E}|f(x)-\mathcal{I}g(x)| \leq \max_{x\in S}|f(x)-g(x)| + \omega(f,\delta(\mathcal{P})) + \omega(g,\delta(\mathcal{P})). \tag{3.6}$$

In particular, we have that

$$\sup_{x\in E}|f(x)-\mathcal{I}f(x)| \leq \omega(f,\delta(\mathcal{P})).$$

Proof. Suppose x belongs to some simplex $\mathcal{P}^j$ with vertices $\{x_1^j,\dots,x_{d_j}^j\}$, and let $(\lambda_i)_{1\leq i\leq d_j}$ be the barycenter parameters $x=\sum_{1\leq i\leq d_j}\lambda_i\ x_j^i$. Since we have $\mathcal{I}g(x_i^j) = g(x_i^j)$, and $\mathcal{I}g(x_i^j) = g(x_i^j)$ for any $i\in\{1,\dots,d_j\}$, it follows that

$$\begin{aligned}
|f(x)-\mathcal{I}g(x)| &\leq \sum_{i=1}^{d_j}\lambda_i|(f(x)-f(x_i^j)| + \sum_{i=1}^{d_j}\lambda_i|f(x_i^j)-\mathcal{I}g(x_i^j)|\\
&\quad+\sum_{i=1}^{d_j}\lambda_i|\mathcal{I}g(x_i^j)-g(x)|\\
&=\sum_{i=1}^{d_j}\lambda_i|(f(x)-f(x_i^j)| + \sum_{i=1}^{d_j}\lambda_i|f(x_i^j)-g(x_i^j)|\\
&\quad+\sum_{i=1}^{d_j}\lambda_i|g(x_i^j)-g(x)|.
\end{aligned}$$

This implies that

$$\sup_{x\in\mathcal{P}^j}|f(x)-\mathcal{I}g(x)|\leq\max_{x\in\mathcal{P}^j}|f(x)-g(x)|+\omega(f,\delta(\mathcal{P}^j))+\omega(g,\delta(\mathcal{P}^j)),$$

with

$$\omega(f,\delta(\mathcal{P}^j))=\sup_{\|x-y\|\leq\delta(\mathcal{P}^j)}|f(x)-f(y)|\quad\text{and}\quad\delta(\mathcal{P}^j):=\sup_{x,y\in\mathcal{P}^j}\|x-y\|.$$

The end of the proof is now clear. ■

Combining (3.5) and (3.6), we obtain the following result.

Proposition 3.5. *Let $\mathcal{P}_k=\{\mathcal{P}_k^1,\dots,\mathcal{P}_k^{m_k}\}$ be a partition of a convex and compact space E_k into simplexes with disjoint nonempty interiors, so that $E_k=\cup_{1\leq i\leq m_k}\mathcal{P}_i$. Let $S_k=\mathcal{V}(\mathcal{P}_k)$ be the set of vertices of these simplexes. Let $(\widehat{u}_k)_{0\leq k\leq n}$ be the Snell envelope associated with the functions $\widehat{f}_k=f_k$ and the Markov transitions $\widehat{M}_k=\mathcal{I}_{k-1}\widetilde{M}_k$ on the state spaces $E_k=\widehat{E}_k$:*

$$\|u_k-\widehat{u}_k\|_{E_k}\leq\sum_{l=k}^{n-1}\left[\omega(M_{l+1}u_{l+1},\delta(\mathcal{P}_l))+\|(M_{l+1}-\widetilde{M}_{l+1})u_{l+1}\|_{S_l}\right].$$

To illustrate the results of Theorem 3.3 and Proposition 3.5, we have derived the effective convergence rate induced by the interpolation in a specific example.

Following the previous section, let us consider the $\mathbb{R}^d$-valued Markov chain $(\hat{\xi}_k)_{0\leq k\leq n}$ defined as the Euler time discretization of the stochastic differential equation (3.1), with a time step $\Delta t=1$, i.e.,

$$\begin{aligned}\hat{\xi}_0&=x,\\ \hat{\xi}_{k+1}&=\hat{\xi}_k+a(\hat{\xi}_k)\Delta t+b(\hat{\xi}_k)\sqrt{\Delta t}\epsilon_k,\end{aligned}\tag{3.7}$$

where ϵ_k are i.i.d. centered Gaussian vectors on $\mathbb{R}^d$ with unit covariance matrix.

Let $\mathrm{Lip}(\mathbb{R}^d)$ be the set of all Lipschitz functions f on $\mathbb{R}^d$, and set

$$L(f)=\sup_{x,y\in\mathbb{R}^d,x\neq y}\frac{\|f(x)-f(y)\|}{\|x-y\|},\qquad f\in\mathrm{Lip}(\mathbb{R}^d).\tag{3.8}$$

We assume that $a:\mathbb{R}^d\to\mathbb{R}$ and $b:\mathbb{R}^d\to\mathcal{M}(d,d)$ are Lipschitz continuous functions. Then we can prove that the time homogeneous Markov transitions $M_k=M$ associated with the Markov chain $(\hat{\xi}_k)_{0\leq k\leq n}$ is such that for any Lipschitz continuous function f on $\mathbb{R}^d$,

$$|M(f)(x)-M(f)(y)|\leq(1+\alpha)L(f)\|x-y\|,\tag{3.9}$$

with $\alpha:=\alpha(L(a),L(b),\Delta t):=L(a)\Delta t+dL(b)\sqrt{\Delta t}\geq0$. Hence, we observe that $M_k(\mathrm{Lip}(\mathbb{R}^d))\subset\mathrm{Lip}(\mathbb{R}^d)$. We also observe that

$$\begin{gathered}\left(f_k\quad\text{and}\quad u_{k+1}\in\mathrm{Lip}(\mathbb{R}^d)\right)\\ \Downarrow\\ \left(u_k\in\mathrm{Lip}(\mathbb{R}^d)\quad\text{with}\quad L(u_k)\leq L(f_k)\vee L(M_{k+1}(u_{k+1}))\right).\end{gathered}\tag{3.10}$$

Moreover, assume that the payoff function $f_k = f$ for all $k = 0, \dots, n$. Using (3.9) together with (3.10) implies

$$L(u_k) \leq (1+\alpha)^{n-k} L(f).$$

Using again (3.9) yields

$$\omega\big(M_{l+1}u_{l+1}, \delta(\mathcal{P}_l)\big) \leq (1+\alpha)^{n-l} L(f)\delta(\mathcal{P}_l).$$

Finally, in the specific case of model (3.7), with payoff functions $f_k = f$ and some refinement degrees of the partitions $\delta(\mathcal{P}_k) \leq \delta$, we obtain the following bound for the convergence of our interpolation model:

$$\|u_k - \widehat{u}_k\|_{E_k} \leq \frac{(1+\alpha)^{n-k+1}}{\alpha} L(f)\delta + \sum_{l=k}^{n-1} \|(M_{l+1} - \widetilde{M}_{l+1})u_{l+1}\|_{S_l}.$$

3.4. Quantization tree models. Quantization tree models belong to the class of deterministic grid approximation methods. The basic idea consists in choosing finite space grids

$$\widehat{E}_k = \left\{x_k^1, \dots, x_k^{m_k}\right\} \subset E_k = \mathbb{R}^d$$

and some neighborhoods measurable partitions $(A_k^i)_{1 \leq k \leq m_k}$ of the whole space E_k such that the random state variable X_k is suitably approximated, as $m_k \to \infty$, by discrete random variables of the following form:

$$\widehat{X}_k := \sum_{1 \leq i \leq m_k} x_k^i \, 1_{A_k^i}(X_k) \simeq X_k.$$

The numerical efficiency of these quantization methods depends heavily on the choice of these grids. There exist various criteria to choose these objects judiciously, including minimal $\mathbb{L}_p$-quantization errors, that ensure that the corresponding Voronoi-type quantized variable $\widehat{X}_k$ minimizes the $\mathbb{L}_p$ distance to the real state variable X_k. For further details on this subject, we refer the interested reader to the pioneering article by Pagès [25], and the series of articles by Bally, Pagès, and Printemps [2], Pagès and Printems [27], Pagès, Pham, and Printems [26], Bucklew and Wise [8], and Graf and Luschgy [22], and the references therein. The second approximation step of these quantization models consists in defining the coupled distribution of any pair of variables $(\widehat{X}_{k-1}, \widehat{X}_k)$ by setting

$$\mathbb{P}\left(\widehat{X}_k = x_k^j,\ \widehat{X}_{k-1} = x_{k-1}^i\right) = \mathbb{P}\left(X_k \in A_k^j,\ X_{k-1} \in A_{k-1}^i\right),$$

for any $1 \leq i \leq m_{k-1}$, and $1 \leq j \leq m_k$. This allows one to interpret the quantized variables $(\widehat{X}_k)_{0 \leq k \leq n}$ as a Markov chain taking values in the states spaces $(\widehat{E}_k)_{0 \leq k \leq n}$ with Markov transitions

$$\widehat{M}_k(x_{k-1}^i, x_k^j) := \mathbb{P}\left(\widehat{X}_k = x_k^j \mid \widehat{X}_{k-1} = x_{k-1}^i\right) = \mathbb{P}\left(X_k \in A_k^j \mid X_k \in A_{k-1}^i\right).$$

Using the decompositions

$$M_k(f)(x_{k-1}^i) = \sum_{j=1}^{m_k} \int_{A_k^j} f(y)\ \mathbb{P}(X_k \in dy \mid X_{k-1} = x_{k-1}^i)$$
$$= \sum_{j=1}^{m_k} \int_{A_k^j} f(y)\ \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i)$$
$$+ \int \left[M(f)(x_{k-1}^i) - M(f)(x)\right] \mathbb{P}(X_{k-1} \in dx | X_{k-1} \in A_{k-1}^i)$$

and

$$\widehat{M}_k(f)(x_{k-1}^i) = \sum_{j=1}^{m_k} \int_{A_k^j} f(x_k^j)\ \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i),$$

we find that

$$[M_k - \widehat{M}_k](f)(x_{k-1}^i)$$
$$= \sum_{j=1}^{m_k} \int_{A_k^j} [f(y) - f(x_k^j)]\ \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i)$$
$$+ \int \left[M(f)(x_{k-1}^i) - M(f)(x)\right]\ \mathbb{P}(X_{k-1} \in dx \mid X_{k-1} \in A_{k-1}^i).$$

We further assume that $M_k(\mathrm{Lip}(\mathbb{R}^d)) \subset \mathrm{Lip}(\mathbb{R}^d)$. From previous considerations, we find that

$$|[M_k - \widehat{M}_k](f)(x_{k-1}^i)| \leq L(f)\ \mathbb{E}\left[|X_k - \widehat{X}_k|^p \mid \widehat{X}_{k-1} = x_{k-1}^i)\right]^{\frac{1}{p}}$$
$$+ L(M_k(f))\ \mathbb{E}(|X_{k-1} - \widehat{X}_{k-1}|^p \mid \widehat{X}_{k-1} = x_{k-1}^i)^{\frac{1}{p}}.$$

This clearly implies that

$$\widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})f|(x_k^i) \leq L(f)\ \left[\mathbb{E}(|X_{l+1} - \widehat{X}_{l+1}|^p \mid \widehat{X}_k = x_k^i)\right]^{\frac{1}{p}}$$
$$+ L(M_{l+1}(f))\ \mathbb{E}(|X_l - \widehat{X}_l|^p \mid \widehat{X}_k = x_k^i)^{\frac{1}{p}}.$$

Using (3.10), we also obtain that $L(u_k) \leq L(f_k) \vee L(M_{k+1}(u_{k+1}))$. Using Lemma 2.1, we readily arrive at the following proposition, similar to Theorem 2 in [2].

Proposition 3.6. *Assume that* $(f_k)_{0\leq k\leq n} \in \mathrm{Lip}(\mathbb{R}^d)^{n+1}$, *and* $M_k(\mathrm{Lip}(\mathbb{R}^d)) \subset \mathrm{Lip}(\mathbb{R}^d)$, *for any* $1 \leq k \leq n$. *In this case, we have* $(u_k)_{0\leq k\leq n} \in \mathrm{Lip}(\mathbb{R}^d)^{n+1}$, *and for any* $0 \leq k \leq n$, *we have the almost sure estimate*

$$|u_k - \widehat{u}_k|(\widehat{X}_k) \leq L(M_{k+1}(u_{k+1}))\ |X_k - \widehat{X}_k|$$
$$+ \sum_{l=k+1}^{n-1} (L(u_l) + L(M_{l+1}(u_{l+1})))\ \mathbb{E}(|X_l - \widehat{X}_l|^p \mid \widehat{X}_k)^{\frac{1}{p}}$$
$$+ L(f_n)\ \left[\mathbb{E}(|X_n - \widehat{X}_n|^p \mid \widehat{X}_k)\right]^{\frac{1}{p}}.$$

Proof. Using the decomposition

$$\widehat{u}_k(\widehat{X}_k) - u_k(X_k) = [\widehat{u}_k(\widehat{X}_k) - u_k(\widehat{X}_k)] + [u_k(\widehat{X}_k) - u_k(X_k)],$$

we have that

$$|u_k(\widehat{X}_k) - u_k(X_k)| \leq L(u_k)\ |\widehat{X}_k - X_k|.$$

Then the proof is completed by the following inequality:

$$\begin{aligned}|\widehat{u}_k(\widehat{\xi}_k) - u_k(X_k)| \leq L(f_n)\ &\left[\mathbb{E}(|X_n - \widehat{X}_n|^p \mid \widehat{X}_k)\right]^{\frac{1}{p}} \\ &+ \sum_{l=k}^{n-1}(L(u_l) + L(M_{l+1}(u_{l+1})))\ \mathbb{E}(|X_l - \widehat{X}_l|^p \mid \widehat{X}_k)^{\frac{1}{p}}. \qquad \blacksquare\end{aligned}$$

In contrast with [2], which focuses on optimizing deterministic grids, we remark that the independent applications of Lemma 2.1 in this model and in the previous examples illustrate the generality of our framework.

4. Monte Carlo importance sampling approximation schemes.

4.1. Path space models. The choice of nonhomogeneous state spaces E_n is not without consequences. In several applications, the underlying Markov model is a path space Markov chain

$$X_n = (X'_0, \ldots, X'_n) \in E_n = (E'_0 \times \cdots \times E'_n). \tag{4.1}$$

The elementary prime variables X'_n represent an elementary Markov chain with Markov transitions $M'_k(x_{k-1}, dx'_k)$ from E'_{k-1} into E'_k. In this situation, the historical process X_n can be seen as a Markov chain with transitions given for any $x_{k-1} = (x'_0, \ldots, x'_{k-1}) \in E_{k-1}$ and $y_k = (y'_0, \ldots, y'_k) \in E_k$ by the following formula:

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1})\ M'_k(y'_{k-1}, dy'_k).$$

This path space framework is, for instance, well suited when dealing with path dependent options such as Asian options.

Besides, this path space framework is also well suited for the analysis of the Snell envelope under different probability measures. To fix the ideas, we associate with the latter a canonical Markov chain $(\Omega, \mathcal{F}, (X'_n)_{n\geq 0}, \mathbb{P}'_{\eta'_0})$ with initial distribution η'_0 on E'_0, and with Markov transitions M'_n from E'_{n-1} into E'_n. We use the notation $\mathbb{E}'_{\eta'_0}$ to denote the expectations with respect to $\mathbb{P}'_{\eta'_0}$. We further assume that there exists a sequence of measures $(\eta_k)_{0\leq k\leq n}$ on the state spaces $(E'_k)_{0\leq k\leq n}$ such that

$$\eta'_0 \sim \eta_0 \quad \text{and} \quad M'_k(x'_{k-1}, .) \sim \eta_k, \tag{4.2}$$

for any $x'_{k-1} \in E'_{k-1}$, and $1 \leq k \leq n$. Let $(\Omega, \mathcal{F}, (X'_n)_{n\geq 0}, \mathbb{P}_{\eta_0})$ be the canonical space associated with a sequence of independent random variables X'_k with distribution η_k on the state space E'_k, with $k \geq 1$. Under the probability measure $\mathbb{P}_{\eta_0}$, the historical process $X_n = (X'_0, \ldots, X'_n)$ can be seen as a Markov chain with transitions

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1})\ \eta_k(dy'_k).$$

By construction, for any integrable function f'_k on E'_k, we have

$$\mathbb{E}_{\eta'_0}(f'_n(X'_n)) = \mathbb{E}_{\eta_0}\left(f_n(X_n)\right),$$

with the collection of functions f_k on E_k given for any $x_k = (x'_0, \ldots, x'_k) \in E_k$ by

$$f_k(x_k) = f'_k(x'_k) \times \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k) \quad \text{with} \quad \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k) = \frac{d\eta'_0}{d\eta_0}(x'_0) \prod_{1 \leq l \leq k} \frac{dM'_l(x'_{l-1}, .)}{d\eta_l}(x'_l). \tag{4.3}$$

Proposition 4.1. *The Snell envelopes u_k and u'_k associated with the pairs (f'_k, M'_k) and (f_k, M_k) are given, for any $0 \leq k < n$, by the backward recursions*

$$u'_k = f'_k \vee M'_{k+1}(u'_{k+1}) \quad \textit{and} \quad u_k = f_k \vee M_{k+1}(u_{k+1}) \quad \textit{with} \quad (u'_n, u_n) = (f'_n, f_n).$$

These functions are connected by the formula

$$\forall\, 0 \leq k \leq n, \quad \forall\, x_k = (x'_0, \ldots, x'_k) \in E_k, \qquad u_k(x_k) = u'_k(x'_k) \times \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k). \tag{4.4}$$

Proof. The first assertion is a simple consequence of the definition of the Snell envelope, and formula (4.4) is easily derived using the fact that

$$u'_k(x'_k) = f'_k(x'_k) \vee \left(\int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1})\ \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1})\ u'_{k+1}(x'_{k+1}) \right).$$

That completes the proof of the proposition. ∎

Under condition (4.2), the above proposition shows that the computation of the Snell envelope associated with a given pair of functions and Markov transitions (f'_k, M'_k) reduces to that of the path space models associated with a sequence of independent random variables with distributions η_n. More formally, the restriction $\mathbb{P}_{\eta_0,n}$ of reference measure $\mathbb{P}_{\eta_0}$ to the σ-field $\mathcal{F}_n$ generated by the canonical random sequence $(X'_k)_{0 \leq k \leq n}$ is given by the tensor product measure $\mathbb{P}_{\eta_0,n} = \otimes_{k=0}^n \eta_k$. Nevertheless, under these reference distributions, the numerical solving of the backward recursion stated in the above proposition still involves integrations with respect to the measures η_k. These equations can be solved if we replace these measures by some sequence of (possibly random) measures $\widehat{\eta}_k$ with finite support on some reduced measurable subset $\widehat{E}'_k \subset E'_k$, with a reasonably large and finite cardinality. We extend $\widehat{\eta}_k$ to the whole space E'_k by setting $\widehat{\eta}_k(E'_k - \widehat{E}'_k) = 0$.

Let $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$ be the distribution of a sequence of independent random variables $\widehat{\xi}'_k$ with distribution $\widehat{\eta}_k$ on the state space $\widehat{E}'_k$, with $k \geq 1$. Under the probability measure $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$, the historical process $X_n = (X'_0, \ldots, X'_n)$ can now be seen as a Markov chain taking values in the path spaces

$$\widehat{E}_k := \left(\widehat{E}'_0 \times \cdots \times \widehat{E}'_k\right),$$

with Markov transitions given for any $x_{k-1} = (x'_0, \ldots, x'_{k-1}) \in \widehat{E}_{k-1}$ and $y_k = (y'_0, \ldots, y'_k) \in \widehat{E}_k$ by the following formula:

$$\widehat{M}_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1})\ \widehat{\eta}_k(dy'_k).$$

Notice that the restriction $\widehat{\mathbb{P}}_{\widehat{\eta}'_0,n}$ of the approximated reference measure $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$ to the σ-field $\mathcal{F}_n$ generated by the canonical random sequence $(X'_k)_{0\le k\le n}$ is now given by the tensor product measure $\widehat{\mathbb{P}}_{\widehat{\eta}'_0,n} = \otimes_{k=0}^n \widehat{\eta}_k$.

Let $\widehat{u}_k$ be the Snell envelope on the path space $\widehat{E}_k$, associated with the pair $(\widehat{f}_k, \widehat{M}_k)$, with the sequence of functions $\widehat{f}_k = f_k$ given in (4.3). By construction, for any $0 \le k \le n$ and any path $x_k = (x'_0, \ldots, x'_k) \in \widehat{E}_k$, we have

$$\widehat{u}_k(x_k) = \widehat{u}'_k(x'_k) \times \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k),$$

with the collection of functions $(\widehat{u}'_k)_{0\le k\le n}$ on the state spaces $(E'_k)_{0\le k\le n}$ given by the backward recursions

$$\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left(\int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1})\ \widehat{u}'_{k+1}(x'_{k+1}) \right), \tag{4.5}$$

with the random integral operator $\widehat{M}'_k$ from E'_k into $\widehat{E}'_{k+1}$ defined by

$$\widehat{M}'_{k+1}(x'_k, dx'_{k+1}) = \widehat{\eta}_{k+1}(dx'_{k+1})\ R_{k+1}(x'_k, x'_{k+1}),$$

with the Radon–Nikodym derivatives $R_{k+1}(x'_k, x'_{k+1}) = \frac{dM'_{k+1}(x'_k,.)}{d\eta_{k+1}}(x'_{k+1})$.

4.2. Broadie–Glasserman models. We consider the path space models associated with the change of measures presented in section 4.1. We use the same notation. We further assume that $\widehat{\eta}_k = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_k^i}$ is the occupation measure associated with a sequence of independent random variables $\xi_k := (\xi_k^i)_{1\le i\le N}$ with common distribution η_k on $\widehat{E}'_k = E'_k$. This Monte Carlo–type model was introduced in 1997 by Broadie and Glasserman (see, for instance, [5] and the references therein). Let $\widehat{\mathbb{E}}$ be the expectation operator associated with this additional level of randomness, and set $\widehat{\mathbb{E}}_{\eta_0} := \widehat{\mathbb{E}} \otimes \mathbb{E}_{\mathbb{P}_{\eta_0}}$.

In this situation, we observe that

$$(M'_{k+1} - \widehat{M}'_{k+1})(x'_k, dx'_{k+1}) = \frac{1}{\sqrt{N}}\ \widehat{V}_{k+1}(dx'_{k+1})\ R_{k+1}(x'_k, x'_{k+1}),$$

with the random fields $\widehat{V}_{k+1} := \sqrt{N}\ [\eta_{k+1} - \widehat{\eta}_{k+1}]$. From these observations, we readily prove that the approximation operators $\widehat{M}'_{k+1}$ are unbiased, in the sense that

$$\forall 0 \le k \le l, \quad \forall x'_l \in E_l, \qquad \widehat{\mathbb{E}}_{\eta_0}\left(\widehat{M}'_{k,l}(f)(x'_l)\,|\mathcal{F}_k\right) = M'_{k,l}(f)(x'_l), \tag{4.6}$$

for any bounded function f on E_{l+1}. Furthermore, for any even integer $p \geq 1$, we have

$$\sqrt{N}\ \widehat{\mathbb{E}}_{\eta_0}\left(\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](f)(x'_l)\right|^p\right)^{\frac{1}{p}} \leq 2\ a(p)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^p\right]^{\frac{1}{p}}.$$

The above estimate is valid as soon as the right-hand side (r.h.s.) in the above inequality is well defined.

We are now in position to state and prove the following theorem.

Theorem 4.2. *For any integer $p \geq 1$, we denote by p' the smallest even integer greater than p. Then, for any time horizon $0 \leq k \leq n$ and any $x'_k \in E'_k$, we have that*

$$\sqrt{N}\widehat{\mathbb{E}}_{\eta_0}\left(|u'_k(x'_k) - \widehat{u}'_k(x'_k)|^p\right)^{\frac{1}{p}} \leq 2a(p)\sum_{k\leq l<n}\left\{\int M'_{k,l}(x'_k,dx'_l)\eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)u'_{l+1})^{p'}\right]\right\}^{\frac{1}{p'}}. \tag{4.7}$$

Note that, as stated in the introduction, this result implies exponential rate of convergence in probability. Hence, this allows us to improve noticeably existing convergence results stated in [5], where there was no rate of convergence, and in [1], where the rate of convergence in probability was polynomial.

Proof. For any even integers $p \geq 1$, any $0 \leq k \leq l$, any measurable function f on E_{l+1}, and any $x_k \in E'_k$, using the generalized Minkowski inequality we find that

$$\begin{aligned}\sqrt{N}\ \widehat{\mathbb{E}}_{\eta_0}&\left(\left|\widehat{M}'_{k,l}\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](f)\right|(x'_k)\right|^p |\mathcal{F}_l\right)^{\frac{1}{p}}\\ &\leq 2a(p)\int\ \widehat{M}'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^p\right]^{\frac{1}{p}}.\end{aligned}$$

By the zero-bias property (4.6), we conclude that

$$\begin{aligned}\sqrt{N}\ \widehat{\mathbb{E}}_{\eta_0}&\left(\left|\widehat{M}'_{k,l}\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](f)\right|(x'_k)\right|^p\right)^{\frac{1}{p}}\\ &\leq 2a(p)\left\{\int\ M'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^p\right]\right\}^{1/p}.\end{aligned}$$

For odd integers $p = 2q+1$, with $q \geq 0$, we use the fact that

$$\mathbb{E}(Y^{2q+1})^2 \leq \mathbb{E}(Y^{2q})\ \mathbb{E}(Y^{2(q+1)}) \quad \text{and} \quad \mathbb{E}(Y^{2q}) \leq \mathbb{E}(Y^{2(q+1)})^{\frac{q}{q+1}},$$

for any nonnegative random variable Y and

$$(2(q+1))_{q+1} = 2\ (2q+1)_{q+1} \quad \text{and} \quad (2q)_q = (2q+1)_{q+1}/(2q+1),$$

so that

$$a(2q)^{2q}a(2(q+1))^{2(q+1)} \leq 2^{-(2q+1)}(2q+1)^2_{q+1}/(q+1/2) = \left(a(2q+1)^{2q+1}\right)^2,$$

and

$$N\ \widehat{\mathbb{E}}_{\eta_0}\left(\left|\widehat{M}'_{k,l}\left|\left[M'_{l+1}-\widehat{M}'_{l+1}\right](f)\right|(x'_k)\right|^{2q+1}\right)^2$$
$$\leq\left(2^{(2q+1)}a(2q+1)^{2q+1}\right)^2\int M'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^{2(q+1)}\right]^{\frac{q}{q+1}}$$
$$\times\int M'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^{2(q+1)}\right].$$

Using the fact that $\mathbb{E}(Y^{\frac{q}{q+1}})\leq\mathbb{E}(Y)^{\frac{q}{q+1}}$, we prove that the r.h.s. term in the above display is upper bounded by

$$\left(2^{(2q+1)}a(2q+1)^{2q+1}\right)^2\left\{\int M'_{k,l}(x'_k,dx'_l)\eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^{2(q+1)}\right]\right\}^{2\left(1-\frac{1}{2(q+1)}\right)},$$

from which we conclude that

$$\sqrt{N}\ \widehat{\mathbb{E}}_{\eta_0}\left(\left|\widehat{M}'_{k,l}\left|\left[M'_{l+1}-\widehat{M}'_{l+1}\right](f)\right|(x'_k)\right|^{2q+1}\right)^{\frac{1}{2q+1}}$$
$$\leq 2a(2q+1)\ \left\{\int M'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)f)^{2(q+1)}\right]\right\}^{\frac{1}{2(q+1)}}.$$

That complete the proof of the theorem. ■

The $\mathbb{L}_p$-mean error estimates stated in Theorem 4.2 are expressed in terms of $\mathbb{L}_{p'}$ norms of Snell envelope functions and Radon–Nikodym derivatives. The terms in the r.h.s. of (4.7) have the following interpretation:

$$\int M'_{k,l}(x'_k,dx'_l)\ \eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)u_{l+1})^{p'}\right]$$
$$=\mathbb{E}\left[\left(R_{l+1}(X'_l,\xi^1_{l+1})u_{l+1}(\xi^1_{l+1})\right)^{p'}|X'_k=x'_k\right].$$

In the above display, $\mathbb{E}(\cdot)$ stands for the expectation with respect to some reference probability measure under which X'_l is a Markov chain with transitions M'_l, and ξ^1_{l+1} is an independent random variable with distribution η_{l+1}. Loosely speaking, the above quantities can be very large when the sampling distributions η_{l+1} are far from the distribution of the random states X'_{l+1} of the reference Markov chain at time $(l+1)$. Next, we provide an original strategy that allows us, for instance, to take $\eta_{l+1}=\text{law}(X'_{l+1})$ as the sampling distribution, even if R_{l+1} is not known (i.e., cannot be evaluated at any point of E_{l+1}). In what follows, we consider N independent copies $(\xi^i_0,\ldots,\xi^i_n)_{1\leq i\leq N}$ of the Markov chain $(X'_0,X'_1,\ldots,X'_n)$, from the origin $k=0$, up to the final time horizon $k=n$. Then, for all $k=0,\ldots,n$, we define the associated occupation measure $\widehat{\eta}_k=\frac{1}{N}\sum_{i=1}^N\delta_{\xi^i_k}$. For all $k=0,\ldots,n$, let $\mathcal{F}_k$ be the sigma field generated by the random sequence $(\xi_l)_{0\leq l\leq k}$.

We also assume that the Markov transitions $M'_n(x'_{n-1}, dx'_n)$ are absolutely continuous with respect to some measures $\lambda_n(dx'_n)$ on E'_n, and we have that

$$(H)_0 \qquad \forall (x'_{n-1}, x'_n) \in \left(E'_{n-1} \times E'_n\right), \quad H_n(x'_{n-1}, x'_n) = \frac{dM'_n(x'_{n-1}, .)}{d\lambda_n}(x'_n) > 0,$$

where H_n is supposed to be known up to a normalizing constant. In this situation, we have $\eta_{k+1} \ll \lambda_{k+1}$, with the Radon–Nikodym derivative given by

$$\eta_{k+1}(dx'_{k+1}) = \eta_k M'_{k+1}(dx'_{k+1}) = \eta_k \left(H_{k+1}(\cdot, x'_{k+1})\right) \ \lambda_{k+1}(dx'_{k+1}).$$

Also notice that the backward recursion of the Snell envelope u'_k can be rewritten as

$$\begin{aligned} u'_k(x'_k) &= f'_k(x'_k) \vee \left(\int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \ \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1}) \ u'_{k+1}(x'_{k+1}) \right) \\ &= f'_k(x'_k) \vee \left(\int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \ \frac{H_{k+1}(x'_k, x'_{k+1})}{\eta_k(H_{k+1}(\cdot, x'_{k+1}))} \ u'_{k+1}(x'_{k+1}) \right). \end{aligned}$$

Arguing as in (4.5), we define the approximated Snell envelope $(\widehat{u}'_k)_{0\leq k\leq n}$ on the state spaces $(E'_k)_{0\leq k\leq n}$ by setting

$$\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left(\int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) \ \widehat{u}'_{k+1}(x'_{k+1}) \right),$$

with the random integral operator $\widehat{M}'$ from E_k into $\widehat{E}_{k+1}$ defined by

$$\begin{aligned} \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) &= \widehat{\eta}_{k+1}(dx'_{k+1}) \ \frac{dM'_{k+1}(x'_k, .)}{d\widehat{\eta}_k M'_{k+1}}(x'_{k+1}) \\ &= \widehat{\eta}_{k+1}(dx'_{k+1}) \ \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\cdot, x'_{k+1}))}. \end{aligned}$$

By construction, these random approximation operators $\widehat{M}'_{k+1}$ satisfy the zero-bias property stated in (4.6), and we have that

$$(M'_{k+1} - \widehat{M}'_{k+1})(x'_k, dx'_{k+1}) = \frac{1}{\sqrt{N}} \ \widehat{V}_{k+1}(dx'_{k+1}) \ \widehat{R}_{k+1}(x'_k, x'_{k+1}),$$

with the random fields $\widehat{V}_{k+1}$ and the $\mathcal{F}_k$-measurable random functions $\widehat{R}_{k+1}$ defined by

$$\widehat{V}_{k+1} := \sqrt{N} \ [\widehat{\eta}_k M'_{k+1} - \widehat{\eta}_{k+1}] \quad \text{and} \quad \widehat{R}_{k+1}(x'_k, x'_{k+1}) := \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\cdot, x'_{k+1}))}.$$

Furthermore, for any even integer $p \geq 1$ and any measurable function f on E_l, we have that

$$\sqrt{N}\, \widehat{\mathbb{E}}_{\eta_0}\left(\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](f)(x'_l)\right|^p |\mathcal{F}_l\right)^{\frac{1}{p}} \leq 2\, a(p)\, \widehat{\eta}_l M'_{l+1}\left[(\widehat{R}_{l+1}(x'_l,\cdot)f)^p\right]^{\frac{1}{p}}.$$

The above estimate is valid as soon as the r.h.s. in the above inequality is well defined. For instance, assuming that

$$\begin{aligned}(H)_1 \qquad & \|M'_{l+1}(u^{2p}_{l+1})\| < \infty \\ \text{and} \quad & \sup_{x'_l, y'_l \in E'_l} \frac{H_{l+1}(x'_l, x'_{l+1})}{H_{l+1}(y'_l, x'_{l+1})} \leq h_{l+1}(x'_{l+1}) \text{ with } \|M'_{l+1}(h^{2p}_{l+1})\| < \infty,\end{aligned}$$

we find that

$$\begin{aligned}&\sqrt{N}\, \mathbb{E}\left(\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](u'_{l+1})(x'_l)\right|^p |\mathcal{F}_l\right)^{\frac{1}{p}} \\ &\qquad \leq 2\, a(p)\, \left(\|M'_{l+1}(h^{2p}_{l+1})\|\, \|M'_{l+1}((u'_{l+1})^{2p})\|\right)^{\frac{1}{2p}}.\end{aligned}$$

Rephrasing the proof of Theorem 4.2, we just proved the following result.

Theorem 4.3. *Under the conditions $(H)_0$ and $(H)_1$ stated above, for any even integer $p > 1$, any $0 \leq k \leq n$, and $x'_k \in E'_k$, we have that*

$$\begin{aligned}\sqrt{N}\, \mathbb{E}\left(\left|u'_k(x'_k) - \widehat{u}'_k(x'_k)\right|^p\right)^{\frac{1}{p}} \\ \leq 2a(p) \sum_{k \leq l < n} \left(\|M'_{l+1}(h^{2p}_{l+1})\|\, \|M'_{l+1}((u'_{l+1})^{2p})\|\right)^{\frac{1}{2p}}.\end{aligned} \tag{4.8}$$

In the end, recovering and extending results from [5], it is interesting to point out that both the Broadie–Glasserman estimator and this new Broadie–Glasserman-type adapted estimator have positive bias.

Proposition 4.4. *For any $0 \leq k \leq n$ and any $x'_k \in E'_k$,*

$$\mathbb{E}\left(\widehat{u}'_k(x'_k)\right) \geq u'_k(x'_k). \tag{4.9}$$

Proof. This inequality can be proved easily by a simple backward induction. The terminal condition $\widehat{u}'_n = u'_n$ implies directly the inequality on instant n. Assuming the inequality holds true in instant k, then Jensen's inequality implies that

$$\begin{aligned}\mathbb{E}\left(\widehat{u}'_k(x'_k)\right) &\geq f_k(x'_k) \vee \mathbb{E}\left(\widehat{M}'_{k+1}(\widehat{u}'_{k+1})(x'_k)\right) \\ &\geq f_k(x'_k) \vee M_{k+1}u'_{k+1}(x'_k) = u'_k(x'_k),\end{aligned}$$

completing the proof of the proposition. ∎

5. A genealogical tree based model.

5.1. Neutral genetic models. Using the notation of section 4.1, set

$$X_n = (X'_0, \dots, X'_n) \in E_n = (E'_0 \times \cdots \times E'_n).$$

Further assume that the state spaces E'_n are finite, and denote by η_k the distribution of the path-valued random variable X_k on E_k, with $0 \leq k \leq n$.

Further let M'_k be the Markov transition from X'_{k-1} to X'_k, and let M_k be the Markov transition from X_{k-1} to X_k. In section 4.1, we have seen that

$$M_k((x'_0, \dots, x'_{k-1}), d(y'_0, \dots, y'_k)) = \delta_{(x'_0,\dots,x'_{k-1})}(d(y'_0, \dots, y'_{k-1}))\; M'_k(y'_{k-1}, dy'_k).$$

In the further development, we fix the final time horizon n, and for any $0 \leq k \leq n$, we denote by π_k the kth coordinate mapping:

$$\pi_k \;:\; x_n = (x'_0, \dots, x'_n) \in E_n = (E'_0 \times \cdots \times E'_n) \mapsto \pi_k(x_n) = x'_k \in E'_k.$$

In this notation, for any $0 \leq k < n$, $x'_k \in E'_k$, and any function $f \in \mathcal{B}(E'_{k+1})$, we have that

$$\eta_n = \mathrm{Law}(X'_0, \dots, X'_n) \quad \text{and} \quad M'_{k+1}(f)(x) := \frac{\eta_n((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n((1_x \circ \pi_k))}. \tag{5.1}$$

By construction, it is also readily checked that the flow of measure $(\eta_k)_{0\leq k\leq n}$ also satisfies the following equation:

$$\eta_k := \Phi_k(\eta_{k-1}) \qquad \forall\, 1 \leq k \leq n, \tag{5.2}$$

with the linear mapping $\Phi_k(\eta_{k-1}) := \eta_{k-1} M_k$.

The genealogical tree based particle approximation associated with these recursions is defined in terms of a Markov chain $\xi_k^{(N)} = (\xi_k^{(i,N)})_{1\leq i\leq N_k}$ in the product state spaces $E_k^{N_k}$, where $N = (N_k)_{0\leq k\leq N}$ is a given collection of integers:

$$\mathbb{P}\left(\xi_k^{(N)} = (x_k^1, \dots, x_k^{N_k}) \mid \xi_{k-1}\right) = \prod_{1\leq i\leq N_k} \Phi_k\left(\frac{1}{N_{k-1}} \sum_{1\leq i\leq N_{k-1}} \delta_{\xi_{k-1}^i}\right)(x_k^i). \tag{5.3}$$

The initial particle system $\xi_0^{(N)} = (\xi_0^{(i,N)})_{0\leq i\leq N_0}$ is a sequence of N_0 i.i.d. random copies of X_0. Let $\mathcal{F}_k^N$ be the sigma-field generated by the particle approximation model from the origin, up to time k.

To simplify the presentation, when there is no confusion we suppress the population size parameter N, and we write ξ_k and ξ_k^i instead of $\xi_k^{(N)}$ and $\xi_k^{(i,N)}$. By construction, ξ_k is a genetic-type model with a neutral selection transition and a mutation type exploration

$$\xi_k \in E_k^{N_k} \xrightarrow{\text{Selection}} \widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1\leq i\leq \widehat{N}_k} \in E_k^{\widehat{N}_k} \xrightarrow{\text{Mutation}} \xi_{k+1} \in E_{k+1}^{N_{k+1}}, \tag{5.4}$$

with $\widehat{N}_k := N_{k+1}$.

During the selection transition, we select randomly N_{k+1} path-valued particles $\widehat{\xi}_k := (\widehat{\xi}^i_k)_{1\leq i\leq N_{k+1}}$ among the N_k path-valued particles $\xi_k = (\xi^i_k)_{1\leq i\leq N_k}$. Sometimes, this elementary transition is called a neutral selection transition in the literature on genetic population models. During the mutation transition $\widehat{\xi}_k \rightsquigarrow \xi_k$, every selected path-valued individual $\widehat{\xi}^i_k$ evolves randomly to a new path-valued individual $\xi^i_{k+1} = x$ randomly chosen with the distribution $M_{k+1}(\widehat{\xi}^i_k, x)$, with $1 \leq i \leq \widehat{N}_k$. By construction, every particle is a path-valued random variable defined by

$$\begin{aligned}\xi^i_k &:= \left(\xi^i_{0,k}, \xi^i_{1,k}, \ldots, \xi^i_{k,k}\right),\\ \widehat{\xi}^i_k &:= \left(\widehat{\xi}^i_{0,k}, \widehat{\xi}^i_{1,k}, \ldots, \widehat{\xi}^i_{k,k}\right) \in E_k := (E'_0 \times \cdots \times E'_k).\end{aligned}$$

By definition of the transition in path space, we also have that

$$\begin{aligned}\xi^i_{k+1} &= \left(\underbrace{\left(\xi^i_{0,k+1}, \xi^i_{1,k+1}, \ldots, \xi^i_{k,k+1}\right)}_{||}, \xi^i_{k+1,k+1}\right)\\ &= \left(\overbrace{\left(\widehat{\xi}^i_{0,k}, \quad \widehat{\xi}^i_{1,k}, \ldots, \quad \widehat{\xi}^i_{k,k}\right)}, \quad \xi^i_{k+1,k+1}\right) = \left(\widehat{\xi}^i_k, \xi^i_{k+1,k+1}\right),\end{aligned}$$

where $\xi^i_{k+1,k+1}$ is a random variable with distribution $M'_{k+1}(\widehat{\xi}^i_{k,k}, \cdot)$. In other words, the mutation transition $\widehat{\xi}^i_k \rightsquigarrow \xi^i_{k+1}$ simply consists in extending the selected path $\widehat{\xi}^i_k$ with an elementary move $\widehat{\xi}^i_{k,k} \rightsquigarrow \xi^i_{k+1,k+1}$ of the end point of the selected path.

From these observations, it is easy to check that the terminal random population model $\xi_{k,k} = (\xi^i_{k,k})_{1\leq i\leq N_k}$ and $\widehat{\xi}_{k,k} = (\widehat{\xi}^i_{k,k})_{1\leq i\leq N_{k+1}}$ is again defined as a genetic-type Markov chain defined as above by replacing the pair (E_k, M_k) by the pair (E'_k, M'_k), with $1 \leq k \leq n$. The latter coincides with the mean field particle model associated with the time evolution of the kth time marginals η'_k of the measures η_k on E'_k. Furthermore, the above path-valued genetic model coincides with the genealogical tree evolution model associated with the terminal state random variables.

Let η^N_k and $\widehat{\eta}^N_k$ be the occupation measures of the genealogical tree model after the mutation and the selection steps; that is, we have that

$$\eta^N_k := \frac{1}{N_k}\sum_{1\leq i\leq N_k} \delta_{\xi^i_k} \quad \text{and} \quad \widehat{\eta}^N_k := \frac{1}{\widehat{N}_k}\sum_{1\leq i\leq \widehat{N}_k} \delta_{\widehat{\xi}^i_k}.$$

In this notation, the selection transition $\xi_k, \rightsquigarrow \widehat{\xi}_k$ consists in choosing $\widehat{N}_k$ conditionally i.i.d. random paths $\widehat{\xi}^i_k$ with common distribution η^N_k. In other words, $\widehat{\eta}^N_k$ is the empirical measure associated with $\widehat{N}_k$ conditionally i.i.d. random paths $\widehat{\xi}^i_k$ with common distribution η^N_k. Also observe that η^N_k is the empirical measure associated with N_k conditionally i.i.d. random paths ξ^i_k with common distribution $\eta^N_{k-1}M_k$.

In practice, we can take $N_0 = N_1 = \cdots = N_n = N$ when we do not have any information on the variance of X_k. In the case when we know the approximate variance of X_k, we can take a large N_k when the variance of X'_k is large. To clarify the presentation, in the further development of the article we further assume that the particle model has a fixed population size $N_k = N$ for any $k \geq 0$.

In what follows, the simulation of the path-valued particle system $(\xi_k)_{0\leq k\leq n}$ will be called the *forward step* and is summarized in the following algorithm.

5.1.1. Forward algorithm.

Initialization At time step $k = 0$, generate N i.i.d. random copies of X_0 and set $\xi_0 = (\xi_0^i)_{0\leq i\leq N}$.

At each time step $k = 1, \dots, n$

1. **Selection**: For each $i = 1, \dots, N$, generate independently an index $I_i \in \{1, \dots, N\}$ with probability $\mathbb{P}(I_i = j) = 1/N$. Then set $\hat{\xi}_{k-1}^i = \xi_{k-1}^{I_i}$.
2. **Mutation**: For each $i = 1, \dots, N$, generate independently N i.i.d. random variables $(\xi_{k,k}^i)_{0\leq i\leq N}$ according to the transition kernel $M'_k(\hat{\xi}_{k-1,k-1}^i, \cdot)$. Then set $\xi_k^i = (\hat{\xi}_{k-1}^i, \xi_{k,k}^i)$.

5.2. Convergence analysis. For general mean field particle interpretation models (5.3), several estimates can be derived for the above particle approximation model (see, for instance, [10]). For instance, for any $n \geq 0$, $r \geq 1$, any $f_n \in \mathrm{Osc}_1(E_n)$, and any $N \geq 1$, we have the unbiased and the mean error estimates

$$\begin{aligned}
&\mathbb{E}\left(\eta_n^N(f_n)\right) = \eta_n(f_n) = \mathbb{E}\left(\widehat{\eta}_n^N(f_n)\right) \\
&\text{and}\quad \sqrt{N}\,\mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f_n)\right|^r\right)^{\frac{1}{r}} \leq 2\,a(r)\sum_{p=0}^{n}\beta(M_{p,n}),
\end{aligned} \tag{5.5}$$

with the Dobrushin ergodic coefficients

$$\beta(M_{p,n}) := \sup_{(x_p,y_p\in E_p)} \left\|M_{p,n}(x_p,\cdot) - M_{p,n}(y_p,\cdot)\right\|_{\mathrm{tv}},$$

and the collection of constants $a(p)$ defined in (2.7). Arguing as in (2.8), for time homogeneous population sizes $N_n = N$, for any functions $f \in \mathrm{Osc}_1(E_n)$, we conclude that

$$\begin{aligned}
&\mathbb{P}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right| \geq \frac{b(n)}{\sqrt{N}} + \epsilon\right) \leq \exp\left(-\frac{N\epsilon^2}{2b(n)^2}\right) \\
&\text{with}\quad b(n) := 2\sum_{p=0}^{n}\beta(M_{p,n}).
\end{aligned} \tag{5.6}$$

For the path space models (5.1), we have $\beta(M_{p,n}) = 1$ so that the estimates (5.5) and (5.6) take the form

$$\sqrt{N}\,\mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f_n)\right|^r\right)^{\frac{1}{r}} \leq 2\,a(r)\,(n+1) \tag{5.7}$$

and

$$\mathbb{P}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right| \geq \frac{2(n+1)}{\sqrt{N}} + \epsilon\right) \leq \exp\left(-\frac{N\epsilon^2}{8(n+1)^2}\right).$$

In the next lemma we extend these estimates to unbounded functions.

Lemma 5.1. *For any $p \geq 1$, we denote by p' the smallest even integer greater than p. In this notation, for any $k \geq 0$ and any function f, we have the almost sure estimate*

$$\sqrt{N}\mathbb{E}\left(\left|[\eta_n^N - \eta_{k-1}^N M_{k-1,n}](f)\right|^p \left|\mathcal{F}_{k-1}^N\right.\right)^{\frac{1}{p}} \leq 2a(p) \sum_{l=k}^{n} \left[\eta_{k-1}^N M_{k-1,l}(|M_{l,n}(f)|^{p'})\right]^{\frac{1}{p'}}. \tag{5.8}$$

In particular, for any $f \in \mathbb{L}_{p'}(\eta_n)$, we have the nonasymptotic estimates

$$\sqrt{N}\ \mathbb{E}\left(\left|[\eta_n^N - \eta_n](f)\right|^p\right)^{1/p} \leq 2\ a(p)\ \|f\|_{p',\eta_n}\ (n+1). \tag{5.9}$$

Proof. Writing $\eta_{-1}^N M_0 = \eta_0$, for any $k \geq 0$, we have the decomposition

$$[\eta_n^N - \eta_{k-1}^N M_{k,n}] = \sum_{l=k}^{n} [\eta_l^N - (\eta_{l-1}^N M_l)] M_{l,n},$$

with the semigroup

$$M_{k,n} = M_{k+1} M_{k+2} \ldots M_n.$$

Using the fact that

$$\mathbb{E}\left(\eta_l^N(f) \left|\eta_{l-1}^N\right.\right) = (\eta_{l-1}^N M_l)(f),$$

we obtain that

$$\mathbb{E}\left(\left|[\eta_l^N - (\eta_{l-1}^N M_l)](f)\right|^p \left|\mathcal{F}_{l-1}^N\right.\right)^{\frac{1}{p}} \leq \mathbb{E}\left(\left|[\eta_l^N - \mu_l^N](f)\right|^p \left|\mathcal{F}_{l-1}^N\right.\right)^{\frac{1}{p}},$$

where $\mu_l^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\zeta_l^i}$ stands for an independent copy of η_l^N given η_{l-1}^N. Using Khinchine-type inequalities, we have that

$$\begin{aligned}\sqrt{N}\ \mathbb{E}\left(\left|[\eta_l^N - \mu_l^N](f)\right|^p \left|\mathcal{F}_{l-1}^N\right.\right)^{\frac{1}{p}} &\leq 2\ a(p)\ \mathbb{E}\left(\left|f\left(\xi_l^1\right)\right|^{p'} \mid \mathcal{F}_{l-1}^N\right)^{\frac{1}{p'}} \\ &= 2\ a(p)\ \left[\eta_{l-1}^N M_l(|f|^{p'})\right]^{\frac{1}{p'}}.\end{aligned}$$

Using the unbias property of the particle scheme, we have that

$$\forall k \leq l \leq n, \qquad \mathbb{E}\left(\eta_l^N(f) \left|\mathcal{F}_{k-1}^N\right.\right) = (\eta_{k-1}^N M_{k-1,l})(f).$$

This implies that

$$\begin{aligned}\sqrt{N}\ \mathbb{E}\left(\left|[\eta_l^N - (\eta_{l-1}^N M_l)](f)\right|^p \left|\mathcal{F}_{k-1}^N\right.\right)^{\frac{1}{p}} &\leq 2\ a(p)\ \mathbb{E}\left(\eta_{l-1}^N M_l(|f|^{p'}) \left|\mathcal{F}_{k-1}^N\right.\right)^{\frac{1}{p'}} \\ &= 2\ a(p)\ \left[\eta_{k-1}^N M_{k-1,l}(|f|^{p'})\right]^{\frac{1}{p'}}.\end{aligned}$$

The end of the proof of (5.8) is now a direct application of Minkowski's inequality, while the proof of (5.9) is a direct consequence of (5.8). ∎

5.3. Particle approximations of the Snell envelope. In section 5.1, we have presented a genealogical based algorithm whose occupation measures η_n^N converge, as $N \uparrow \infty$, to the distribution η_n of the reference Markov chain $(X'_0, \ldots, X'_n)$ from the origin, up to the final time horizon n. Mimicking formula (5.1), we define the particle approximation of the Markov transitions M'_k as follows:

$$\widehat{M}'_{k+1}(f)(x) := \frac{\eta_n^N((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n^N((1_x \circ \pi_k))} := \frac{\sum_{1\leq i\leq N}\ 1_x(\xi^i_{k,n})\ f(\xi^i_{k+1,n})}{\sum_{1\leq i\leq N} 1_x(\xi^i_{k,n})}$$

for every state x in the support $\widehat{E}_{k,n}$ of the measure $\eta_n^N \circ \pi_k^{-1}$. Note that $\widehat{E}_{k,n}$ coincides with the collection of ancestors $\xi^i_{k,n}$ at level k of the population of individuals at the final time horizon. This random set can alternatively be defined as the set of states $\xi^i_{k,k}$ of the particle population at time k such that $\eta_n^N((1_{\xi^i_{k,k}} \circ \pi_k)) > 0$; more formally, we have

$$\widehat{E}_{k,n} := \bigcup_{1\leq i\leq N} \left\{\xi^i_{k,k}\ :\ \eta_n^N((1_{\xi^i_{k,k}} \circ \pi_k)) > 0\right\}. \tag{5.10}$$

It is interesting to observe that the random Markov transitions $\widehat{M}'_{k+1}$ coincide with the conditional distributions of the states X'_{k+1} given the current time states X'_k of a canonical Markov chain $X_n := (X'_0, \ldots, X'_n)$ with distribution η_n^N on the path space $E_n := (E'_0 \times \cdots \times E'_n)$. Thus, the flow of kth time marginal measures

$$\eta^N_{k,n} := \frac{1}{N}\sum_{i=1}^N\ \delta_{\xi^i_{k,n}}$$

are connected by the following formula:

$$\eta^N_{k,n}\widehat{M}'_{k,l} = \eta^N_{l,n} \qquad \forall\, k \leq l \leq n,$$

with the semigroup $\widehat{M}'_{k,l}$ associated with the Markov transitions $\widehat{M}'_{k+1}$ given by

$$\widehat{M}'_{k,l}(f)(x) = \widehat{M}'_{k+1}\widehat{M}'_{k+1}\ldots\widehat{M}'_l(f)(x) = \frac{\eta_n^N((1_x \circ \pi_k)\ (f \circ \pi_l))}{\eta_n^N((1_x \circ \pi_k))}, \tag{5.11}$$

for every state x in $\widehat{E}_{k,n}$. In connection with (5.10), we also have the following formula:

$$\eta^N_{k,n} = \frac{1}{N}\sum_{i=1}^N\ \left(N\ \eta_n^N\left(1_{\xi^i_{k,k}} \circ \pi_k\right)\right)\delta_{\xi^i_{k,k}} = \sum_{i=1}^N\ \eta_n^N\left(1_{\xi^i_{k,k}} \circ \pi_k\right)\delta_{\xi^i_{k,k}},$$

with the proportion $\eta_n^N(1_{\xi^i_{k,k}} \circ \pi_k)$ of individuals at the final time horizon having the common ancestor $\xi^i_{k,k}$ at level k. It is also interesting to observe that

$$\begin{aligned}\mathbb{E}\left(\eta^N_{k,n}(f)\,\middle|\mathcal{F}^N_k\right) &= \sum_{i=1}^N\ \mathbb{E}\left(\eta_n^N\left(1_{\xi^i_{k,k}} \circ \pi_k\right)\middle|\mathcal{F}^N_k\right) f(\xi^i_{k,k})\\ &= \sum_{i=1}^N \underbrace{\eta_k^N M_{k,n}\left(1_{\xi^i_{k,k}} \circ \pi_k\right)}_{=1/N} f(\xi^i_{k,k}) = \eta_k^N(f).\end{aligned}$$

The Snell envelope associated with this particle approximation model is defined by the following backward recursion:

$$\widehat{u}_k(x) = \begin{cases} f_k(x) \vee \widehat{M}'_{k+1}(u_{k+1})(x) & \forall x \in \widehat{E}_{k,n}, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the ancestors at level k, this recursion takes the following form:

$$\widehat{u}_k\left(\xi^i_{k,n}\right) = f_k\left(\xi^i_{k,n}\right) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})\left(\xi^i_{k,n}\right) \qquad \forall\, 1 \le i \le N.$$

In what follows, the computation of the Snell envelope approximation $(\hat{u}_k)_{0\le k\le n}$ will be called the *backward step* and is summarized in the following algorithm.

5.3.1. Backward algorithm.

Initialization At time step $k = n$, for all $i = 1, \ldots, N$, set $\hat{u}_n(\xi^i_{n,n}) = f(\xi^i_{n,n})$.
At each time step $k = n-1, \ldots, 0$, for all $i = 1, \ldots, N$, set

$$\hat{u}_k(\xi^i_{k,n}) = f_k(\xi^i_{k,n}) \vee \frac{\sum_{j=1}^N \hat{u}_{k+1}(\xi^j_{k+1,n})\, 1_{\xi^j_{k,n}=\xi^i_{k,n}}}{\sum_{j=1}^N 1_{\xi^j_{k,n}}}.$$

For later use in the further development of this section, we quote a couple of technical lemmas. The first one provides some $\mathbb{L}_p$ estimates of the normalizing quantities of the Markov transitions $\widehat{M}'_{k+1}$. The second one allows us to quantify the deviations of $\widehat{M}'_{k+1}$ around its limiting values M'_{k+1}, as $N \to \infty$.

Lemma 5.2. *For any $p \ge 1$ and $0 \le i \le N$, we have the following uniform estimate:*

$$\sup_{N\ge 1}\ \sup_{0\le l\le k\le n} \left\|\eta^N_k(1_{\xi^i_{l,k}} \circ \pi_l)^{-1}\right\|_p < \infty. \tag{5.12}$$

Lemma 5.3. *For any $p \ge 1$ and $0 \le i \le N$, we have the following uniform estimate:*

$$\sup_{0\le l\le n} \left\|\widehat{M}'_{l+1}(f)(\xi^i_{l,n}) - M'_{l+1}(f)(\xi^i_{l,n})\right\|_p \le c_p(n)/\sqrt{N}, \tag{5.13}$$

with some collection of finite constants $c_p(n) < \infty$ whose values depend only on the parameters p and n.

The proofs of these lemmas are rather technical; thus they are postponed to the appendices.

We are now in position to state and prove the main result of this section.

Theorem 5.4. *For any $p \ge 1$ and $0 \le i \le N$, we have the following uniform estimate:*

$$\sup_{0\le k\le n} \left\|(u_k - \widehat{u}_k)(\xi^i_{k,n})\right\|_p \le c_p(n)/\sqrt{N}, \tag{5.14}$$

with some collection of finite constants $c_p(n) < \infty$ whose values depend only on the parameters p and n.

Proof. First, we use the following decomposition:

$$|u_k - \widehat{u}_k| 1_{\widehat{E}_{k,n}} \le \sum_{k\le l\le n-1} \widehat{M}'_{k,l} |(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})|\ 1_{\widehat{E}_{k,n}}. \tag{5.15}$$

By construction, we have that

$$\widehat{M}'_{k,l}|(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})|1_{\widehat{E}_{(k,n)}} = \widehat{M}'_{k,l}|1_{\widehat{E}_{l,n}}(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})|1_{\widehat{E}_{k,n}}.$$

By (5.11), if we set

$$\widetilde{u}_{l+1} = |(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})|$$

on the set $\widehat{E}_{l,n}$, then we have that

$$\widehat{M}'_{k,l}(\widetilde{u}_{l+1})(\xi^i_{k,n}) = \frac{\eta^N_n((1_{\xi^i_{k,n}} \circ \pi_k)\ (\widetilde{u}_{l+1} \circ \pi_l))}{\eta^N_n((1_{\xi^i_{k,n}} \circ \pi_k))}.$$

For any $p \geq 1$, we have that

$$\left\|\widehat{M}'_{k,l}(\widetilde{u}_{l+1})(\xi^i_{k,n})\right\|_p \leq \left\|\eta^N_n((1_{\xi^i_{k,n}} \circ \pi_k))^{-1}\right\|_2^{1/p} \times \mathbb{E}\left(\eta^N_n((1_{\xi^i_{k,n}} \circ \pi_k)\ (\widetilde{u}_{l+1} \circ \pi_l)^{2p})\right)^{1/(2p)}.$$

This implies that

$$\left\|\widehat{M}'_{k,l}(\widetilde{u}_{l+1})(\xi^i_{k,n})\right\|_p \leq \left\|\eta^N_n((1_{\xi^i_{k,n}} \circ \pi_k))^{-1}\right\|_2^{1/p} \times \sup_{1\leq j\leq N} \left\|\widetilde{u}_{l+1}(\xi^j_{l,n})\right\|_{2p}.$$

The proof of (5.14) is now a clear consequence of Lemmas 5.2 and 5.3. ■

5.4. Bias analysis. To end this subsection, we will prove that just as with the bias of the Broadie–Glasserman-type estimators, the bias of the genealogical tree based estimator is always positive.

Note that, for any $0 \leq k \leq n$, function f on space E'_k, and any $i \in \{1, \ldots, N\}$, we have that

$$\mathbb{E}\left(f(\xi^i_{k+1,n})|\xi_{k,n}\right) = M_{k+1}f(\xi^i_{k,n}). \tag{5.16}$$

This is because in the neutral genealogical tree model, the selection steps are independent of the mutations steps. Here, $\xi_{k,n}$ contains all the information on the construction of the tree plus the information on the values of the nodes on this tree at instant k. Equation (5.16) comes from the fact that given the information $\xi_{k,n}$, the particle $\xi^i_{k+1,n}$ follows the distribution $M'_{k+1}(\xi^i_{k,n}, \cdot)$.

Theorem 5.5. *For any $0 \leq k \leq n$ and any $i \in \{1, \ldots, N\}$, we have that*

$$\mathbb{E}\left(\widehat{u}_k(\xi^i_{k,n})|\xi_{k,n}\right) \geq u_k(\xi^i_{k,n}). \tag{5.17}$$

Proof. To prove this, we will use a simple induction argument.

For $l = n$, $\widehat{u}_n = u_n$, we easily check that the following inequality is verified for all $i = 1, \ldots, N$:

$$\mathbb{E}\left(\widehat{u}_l(\xi^i_{l,n})|\xi_{l,n}\right) \geq u_l(\xi^i_{l,n}). \tag{5.18}$$

Assume that (5.18) is verified for all $i = 1, \ldots, N$, and let us prove that the same inequality is valid for instant $l - 1$.

With the elementary decomposition,

$$\mathbb{E}\left(\widehat{M}_l'(\widehat{u}_l)(\xi^i_{l-1,n})|\xi_{l-1,n}\right) = \mathbb{E}\left(\frac{\sum_{j=1}^N \widehat{u}_l(\xi^j_{l,n})1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}{\sum_{j=1}^N 1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}|\xi_{l-1,n}\right)$$
$$= \frac{\sum_{j=1}^N \mathbb{E}\left(\widehat{u}_l(\xi^j_{l,n})|\xi_{l-1,n}\right) 1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}{\sum_{j=1}^N 1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}.$$

By assumption (5.18) and (5.16), we have that

$$\mathbb{E}\left(\widehat{u}_l(\xi^j_{l,n})|\xi_{l-1,n}\right) \geq \mathbb{E}\left(u_l(\xi^j_{l,n})|\xi_{l-1,n}\right)$$
$$= M_l u_l(\xi^j_{l-1,n}).$$

Applying the preceding decomposition, it follows easily that

$$\mathbb{E}\left(\widehat{M}_l\widehat{u}_l(\xi^i_{l-1,n})|\xi_{l-1,n}\right) \geq \frac{\sum_{j=1}^N M_l u_l(\xi^i_{l-1,n})1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}{\sum_{j=1}^N 1_{\xi^j_{l-1,n}=\xi^i_{l-1,n}}}$$
$$= M_l u_l(\xi^i_{l-1,n}).$$

Then we can complete this proof by using Jensen's inequality, getting

$$\mathbb{E}\left(\widehat{u}_{l-1}(\xi^i_{l-1,n})|\xi_{l-1,n}\right) \geq f_{l-1}(\xi^i_{l-1,n}) \vee \mathbb{E}\left(\widehat{M}_l\widehat{u}_l(\xi^i_{l-1,n})|\xi_{l-1,n}\right)$$
$$\geq f_{l-1}(\xi^i_{l-1,n}) \vee M_l u_l(\xi^i_{l-1,n})$$
$$= u_{l-1}(\xi^i_{l-1,n}). \qquad \blacksquare$$

5.5. Numerical simulations. In this section, we give numerical examples to test the genealogical tree algorithm on two types of options from dimension 1 up to 6.

5.5.1. Price dynamics and options model. Our numerical examples are taken from Bouchard and Warin [7], who provided precise approximations of option values in their examples. The asset prices are modeled by a d-dimensional Markov process $(\tilde{X}_t)$ such that each component (i.e., each asset) follows a geometric Brownian motion under the risk-neutral measure; that is, for assets $i = 1, \ldots, d$,

$$\frac{d\tilde{X}_t(i)}{\tilde{X}_t(i)} = r dt + \sigma_i dz^i_t, \tag{5.19}$$

where z^i, for $i = 1, \ldots, d$, are independent standard Brownian motions. The interest rate r is set to 5% annually. We also assume that for all $i = 1, \ldots, d$, $\tilde{X}_{t_0}(i) = 1$ and $\sigma_i = 20\%$ annually.

We consider two different Bermudan options with maturity $T = 1$ year and 11 equally distributed exercise opportunities at dates $t_k = kT/n$ with $k = 0, 1, \ldots, n = 10$, associated with two different payoffs:

1. a geometric average put option with strike $K = 1$ and payoff $(K - \prod_{i=1}^d \tilde{X}_T(i))_+$,
2. an arithmetic average put option with strike $K = 1$ and payoff $(K - \frac{1}{d}\sum_{i=1}^d \tilde{X}_T(i))_+$.

Table 1

Benchmark values for the geometric and arithmetic put options (taken from [7]).

Number of assets	1	2	3	4	5	6
Geometric payoff	0.06033	0.07815	0.08975	0.09837	0.10511	0.11073
Arithmetic payoff	0.06033	0.03882	0.02947	0.02403	0.02046	0.01830

Note that the geometric average put payoff involves the process $\prod_{i=1}^{d}\tilde{X}(i)$ which can be identified with a one-dimensional nonstandard exponential Brownian motion. This trick was used in [7] to compute a precise benchmark option value by PDE techniques. We report in Table 1 the benchmark option values computed in [7] for both the geometric and the arithmetic put options (by using, respectively, the one-dimensional PDE method and the Longstaff–Schwartz method with $8 \times 10^6 \times d^2$ simulations and 10 basis functions for each direction).

5.5.2. State space discretization. The genealogical tree algorithm is designed for finite state spaces. Hence, before applying it to the aforementioned continuous space examples, we have to approximate the continuous state space Markov chain solution of (5.19) by a Markov chain with a finite state space. To this end, one can first discretize the state space using either a random tree or a stochastic mesh method, or a binomial tree or a quantization approach . . . . In our numerical simulations, the quantization discretization seemed to be the most efficient.

State space partitioning. Here, we propose using a quantization-like approach for the space discretization step. We simulate a first set of M i.i.d. paths at each $n+1$ possible exercise dates $t_0, \ldots, t_n$, $(\tilde{X}^i_{t_k})^{i=1,\ldots,M}_{k=0,\ldots,n}$ according to dynamic (5.19). Assume now that there exist two integers N' and P such that M can be written as the product $M = N'P$. Then, at each time step t_k, the particle set $\mathcal{S}_k = \{\tilde{X}^1_{t_k}, \ldots, \tilde{X}^M_{t_k}\}$ can be partitioned into N' localized subsets $\{\mathcal{S}^1_k, \ldots, \mathcal{S}^{N'}_k\}$ of P particles. Assume now that there exist d integers $(Q_1, \ldots, Q_d)$ such that N' can be written as the product $N' = Q_1 \ldots Q_d$. Assume for simplicity that $N' = Q^d$. One way to build this partition $\{\mathcal{S}^1_k, \ldots, \mathcal{S}^{N'}_k\}$ is then to apply the following procedure as in [7]:

1. sort the particles according to the first coordinate, and split the sorted particles into Q subsets containing the same number of particles $Q^{d-1}P$;
2. if $d \geq 2$, for each subset, sort the particles according to the second coordinate, and split the sorted particles into Q subsets containing the same number of particles $Q^{d-2}P$, which finally leads to Q^2 subsets containing the same number of particles $Q^{d-2}P$;
3. if $d \geq 3$, repeat this procedure recursively; in each direction, $i = 3, \ldots, d$.

This operation is realized with a complexity $O(dM\log(M))$ and produces a partition of $\mathcal{S}_k$ into $N' = Q^d$ subsets $\mathcal{S}^1_k, \ldots, \mathcal{S}^{N'}_k$ with the same number P of particles.

Now, for each subset $\mathcal{S}^j_k$, for $j = 1, \ldots, N'$, we compute a representative state, S^j_k, as the average particle over all the elements of $\mathcal{S}^j_k$. Then, at each time step t_k for $k = 1, \ldots, n$, we will consider the finite state space $E_k = \{S^1_k, \ldots, S^{N'}_k\}$ and we set $E_0 = \{X_{t_0}\}$. In what follows, the discrete points $S^1_k, \ldots, S^{N'}_k$ will be referred to as the *sites*.

Finite state space Markov chain. Assume now that a sequence of finite state spaces $E_k \subset \mathbb{R}^d$ is given for $k = 1, \ldots, n$ (for instance, by the above procedure). We define a finite state space Markov chain $(X'_k)_{k=0,\ldots,n}$ such that $X'_0 = \tilde{X}_{t_0}$ and for all $k = 1, \ldots, n$, the following hold:

- $X'_k \in E_k$;

- $\mathbb{P}(X'_k = S^j_k \,|\, X'_{k-1} = S^i_{k-1}) = \mathbb{P}(\tilde{X}_{t_k} \in V^j_k \,|\, \tilde{X}_{t_{k-1}} = S^i_{k-1})$, where V^j_k denotes the Voronoi cell associated with the site S^j_k in the discrete set E_k and $(\tilde{X}_{t_k})$ is the Markov process verifying (5.19) observed at the discrete times $t_0, \dots, t_n$.

To simulate a transition of the Markov chain $(X'_k)_{k=0,\dots,n}$ from the state $S^i_{k-1} \in E_{k-1}$ at the time step $k-1$ to the time step k, one can apply the following procedure:

1. Simulate a random variable $\tilde{X}_{t_k}$ according to $\tilde{M}_k(S^i_{k-1}, \cdot)$, where $\tilde{M}_k$ denotes the transition kernel of the continuous state space Markov chain verifying (5.19) from time t_{k-1} to t_k.
2. Set $X'_k = S^{i^*}_k$, where $S^{i^*}_k$ is the nearest neighbor of $\tilde{X}_{t_k}$ among the elements of E_k.

5.5.3. Complexity. In comparison with the quantization method proposed in [25], the genealogical algorithm based on the above space discretization needs only to simulate the finite state space Markov chain (X'_k) and avoids the time consuming computation of the transition probabilities.

In terms of complexity, the major part of the computing time is spent in the *forward step* described in section 5.1.1 for simulating the discrete space Markov chain (X'_k). More precisely, for each transition, one has to compute a nearest neighbor among N' sites which finally leads to a complexity of order $O(NN')$ by time step, when considering the whole set of N particles.

In terms of approximation error, we can decompose the error induced by the whole procedure, on the Snell envelope approximation, into the sum of two terms:

1. the state space discretization error which can be upper bounded, according to [25] or Proposition 3.6, by $\frac{c}{N'^{1/d}}$;
2. the error induced by the genealogical tree algorithm, which could be upper bounded, according to the proof of Theorem 5.4, by $c\frac{N'^{\beta}}{N^{1/2}}$, for a given positive real $\beta > 0$.

Hence, to minimize the resulting upper bound on the global error, one has to choose judiciously the number of sites N' as a function of the number of particles such that $N' = 0(N^{\frac{d}{2\beta d+2}})$. With this choice, the complexity of the global procedure is of order $O(N^{\frac{(1+2\beta)d+2}{2\beta d+2}})$, with an approximation error bounded by $\frac{c}{N^{\frac{1}{2\beta d+2}}}$. In our numerical simulations, we have set $\beta = 1/2$ so that the complexity grows with the dimension from $N^{4/3}, N^{3/2}, N^{8/5}, \dots, N^2$ for dimensions $d = 1, 2, 3, \dots, \infty$.

On the other hand, in the *backward step* (described in section 5.3.1), consisting in computing the Snell envelope, our algorithm requires only a complexity which is linear in the number of particles, N. Hence, for a given underlying price process, our approach can rapidly approximate several Bermudan options with different payoff functions.

5.5.4. Numerical results. For each example, we have performed the algorithm for different numbers of particles for $N = 5\times 10^3, 1\times 10^4, 2.5\times 10^4, 5\times 10^4, 1\times 10^5, 2\times 10^5, 4\times 10^5, 1\times 10^6, 2\times 10^6$. In each case, the sites were computed on the base of $M = \max(500000, 50\times N') = \max(500000, 50 \times N^{\frac{d}{d+2}})$ simulations. Many runs of the algorithm were performed to build boxplots for our estimates: 50 runs for $N < 10^6$ and 24 runs for $N = 1\times 10^6$ and $N = 2\times 10^6$.

Simulations results are reported in Figure 1 for the geometric put payoff and in Figure 2 for the arithmetic put payoff. First, notice that our algorithm has been implemented without any control variate technique. Moreover, our implementation has not been optimized. In particular, we have not investigated in this article any parallel implementations of our algorithm.

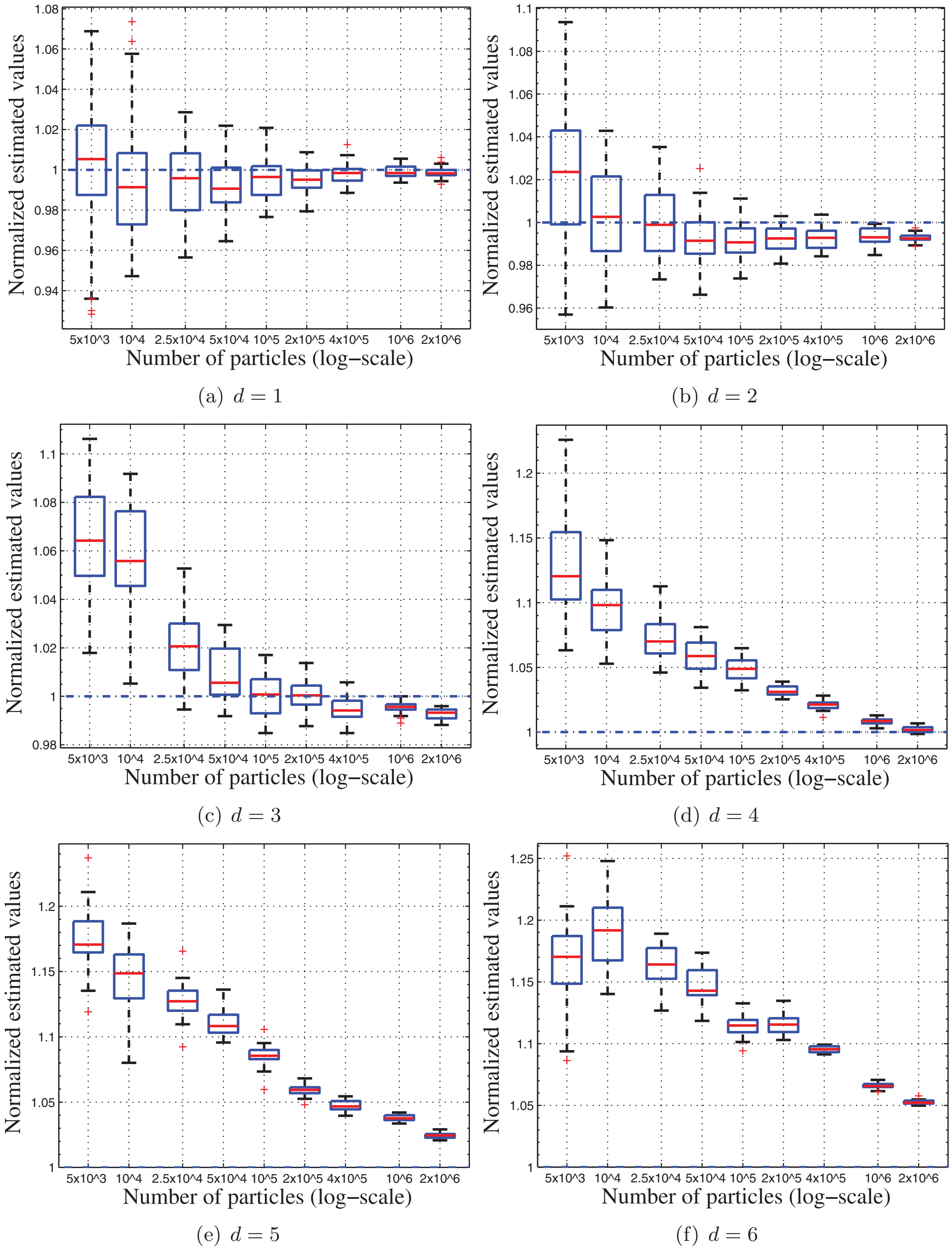
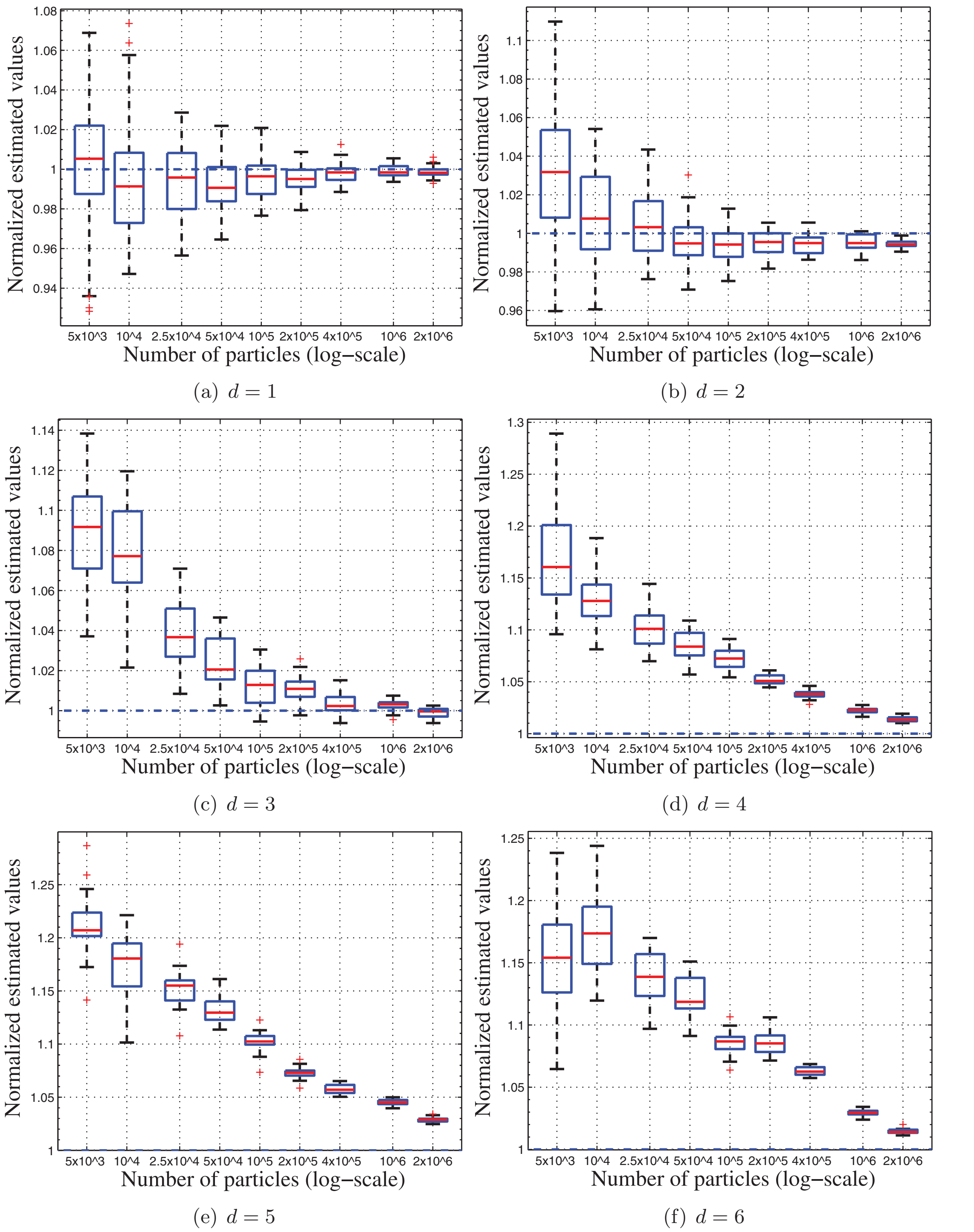


Figure 1. *Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the* geometric *put payoff. The box stretches from the* 25*th percentile to the* 75*th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within* 1.5 *IQR (interquartile range), and red crosses indicate outliers.*



(a) $d = 1$ (b) $d = 2$

(c) $d = 3$ (d) $d = 4$

(e) $d = 5$ (f) $d = 6$

Figure 2. *Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the* arithmetic put payoff*. The box stretches from the* 25*th percentile to the* 75*th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within* 1.5 *IQR (interquartile range), and red crosses indicate outliers.*

Table 2

Error (in % of the option value in Table 1) of the genealogical algorithm with $N = 25000$ particles and $N' = N^{\frac{d}{d+2}}$ sites, and within parentheses of the quantization algorithm with $N = 25600$ quantization points (taken from [7]), for the geometric and arithmetic put options.

Number of assets	$d = 3$	$d = 4$	$d = 5$	$d = 6$
Geometric put error (in % of the option value)	2 (2)	7 (8)	14 (15)	17 (22)
Arithmetic put error (in % of the option value)	3.5 (3.5)	10 (8)	15 (16)	14 (17)

Thus, it does not seem relevant to report any running time measurements on the paper. However, the algorithm complexity gives a good indication of the number of operations required by our algorithm. Moreover, the estimates reported in our graph correspond to the *backward estimate* provided by Algorithm A2 in Bouchard and Warin [7] and should be compared to that type of estimate. We could also obtain a *forward estimate* with our genealogical approach by applying the backward induction on the stopping times (just as in the Longstaff–Schwartz algorithm) with probably better performances than the *backward estimator*, but this is not the subject of the present paper.

Hence, to compare the estimation errors of the *backward estimate* provided by our algorithm to a corresponding approach, we have reported, in Table 2, the estimation errors obtained with the genealogical algorithm using $N = 25000$ particles and $N' = N^{\frac{d}{d+2}}$ sites, in valuing the geometric put (on the first line) and the arithmetic put (on the second line) and, within parentheses, the performances of the *backward estimate* provided by the quantization approach [2] implemented in [7], with 25600 quantization points for the same options. One can observe that both algorithms achieve similar performances for approximately the same number N of quantization points (for the quantization algorithm) and particles (for the genealogical algorithm).

Now, notice that the complexity (per time step) of the genealogical algorithm is of order $NN' = N^{\frac{2d+2}{d+2}}$ for the construction of the genealogical tree and of order N for the backward induction on the prices, which is slightly smaller than the complexity of the quantization approach of order N^2 for the backward induction on prices (without taking into account the complexity related to the construction of the quantization tree and to the computation of the transition probabilities). Hence, we can conclude that our new algorithm is competitive with respect to comparable algorithms.

Notice that one can observe on the graph that for $d = 2$ or 3 the bias of our estimator can be negative. However, this is not in contradiction to Theorem 5.5. Indeed, recall that our estimator cumulates two kinds of approximations:

1. The first approximation is the discretization of the Markov chain which can induce a negative bias.
2. The second is the backward genealogical algorithm to compute the Snell envelope of the discrete Markov chain which (by theorem) induces a positive bias.

Looking into further applications, this algorithm is also well suited for Bermudan options with path dependent payoff. Indeed, by construction, the genealogical tree algorithm is defined in terms of the historical process; then it is able to compute conditional expectations with respect to the whole past of the process with no additional complexity.

In the same vein, we believe that this algorithm and the related convergence result could be extended, with slight modifications, to the more general case of reflected backward stochastic differential equations with a nonzero driver that does not depend on the z variable and which satisfies suitable regularity conditions.

Finally, in further research, it could also be interesting to extend this algorithm for the computation of price sensitivities for hedging purposes.

Appendix A. Proof of Lemma 5.2. Set

$$\delta_{l,n}(N) := \inf_{x\in E'_l} \eta_n^N(g_{l,x}),$$

with the function $g_{l,x}$ defined in (B.2). Note that

$$\mathbb{P}\left(\delta_{l,n}(N)=0\right) \leq \sum_{x\in E'_l} \mathbb{P}\left(\eta_n^N(g_{l,x})=0\right).$$

On the other hand, for any $\epsilon \in [0,1)$ we have

$$\mathbb{P}\left(\eta_n^N(g_{l,x})=0\right) \leq \mathbb{P}\left(\left|\eta_n^N(g_{l,x})-\eta_n(g_{l,x})\right| > \epsilon\ \eta_n(g_{l,x})\right).$$

Arguing as in (5.7), for any $x \in E'_l$ s.t. $\eta_n(g_{l,x})\ (=\mathbb{P}\left(X'_l = x\right)) > 0$ we prove that

$$\sqrt{N}\ \mathbb{E}\left(\left|\eta_n^N(g_{l,x})-\eta_n(g_{l,x})\right|^r\right)^{\frac{1}{r}} \leq 2\ a(r)\ (n+1)\ \eta_n(g_{l,x})^{-1} \tag{A.1}$$

and therefore

$$\mathbb{P}\left(\left|\eta_n^N(g_{l,x})-\eta_n(g_{l,x})\right| \geq \left(\frac{2(n+1)}{\sqrt{N}}+\epsilon\right)\eta_n(g_{l,x})\right) \leq \exp\left(-\frac{N\epsilon^2}{8(n+1)^2}\right).$$

For any $N \geq (2(n+1)/(1-\epsilon))^2$, this implies that

$$\mathbb{P}\left(\delta_{l,n}(N)=0\right) \leq \mathrm{Card}(E'_l)\ \exp\left(-\frac{N\epsilon^2}{8(n+1)^2}\right).$$

If we choose $\epsilon = 1/2$ and $N \geq (4(n+1))^2$, we conclude that

$$\mathbb{P}\left(\delta_{l,n}(N)=0\right) \leq \mathrm{Card}(E'_l)\ \exp\left(-\frac{N}{32(n+1)^2}\right).$$

On the other hand, by construction we have the almost sure estimate

$$\eta_n^N(g_{l,\xi^i_{l,n}}) = \sum_{x\in E'_l} \eta_n^N(g_{l,x})\ 1_{\xi^i_{l,n}=x} \geq \delta_{l,n}(N)\ 1_{\delta_{l,n}(N)>0} + \frac{1}{N}\ 1_{\delta_{l,n}(N)=0},$$

from which we find that

$$\eta_n^N(g_{l,\xi^i_{l,n}})^{-1} \leq \delta_{l,n}(N)^{-1}\ 1_{\delta_{l,n}(N)>0} + N\ 1_{\delta_{l,n}(N)=0}.$$

Therefore, we have

$$\begin{aligned}\left|\left|\eta_n^N(g_{l,\xi^i_{l,n}})^{-1}\right|\right|_p &\leq ||\delta_{l,n}(N)^{-1}\ 1_{\delta_{l,n}(N)>0}||_p + N||1_{\delta_{l,n}(N)=0}||_p\\ &\leq \sum_{x\in E'_l} ||\eta_n^N(g_{l,x})^{-1}\ 1_{\eta_n^N(g_{l,x})>0}||_p + N\ \mathbb{P}(\delta_{l,n}(N)=0)^{1/p}.\end{aligned}$$

If we set $\overline{g}_{l,n}(x) = g_{l,x}/\eta_n(g_{l,x})$, using the fact that

$$\frac{1}{1-u} = 1+u+u^2+\frac{u^3}{1-u},$$

for any $u \neq 1$, and $\eta_n^N(\overline{g}_{l,x})^{-1}\ 1_{\eta_n^N(g_{l,x})>0} \leq N\ \eta_n(g_{l,x})$, we find that

$$\begin{aligned}\eta_n^N(\overline{g}_{l,x})^{-1}\ 1_{\eta_n^N(g_{l,x})>0} &\leq 1+\left|1-\eta_n^N(\overline{g}_{l,x})\right| + \left(1-\eta_n^N(\overline{g}_{l,x})\right)^2\\ &\quad + N\ \eta_n(g_{l,x})\ \left|1-\eta_n^N(\overline{g}_{l,x})\right|^3.\end{aligned}$$

Combining this estimate with (A.1), for any $p \geq 1$ we prove the following upper bound:

$$\begin{aligned}||\eta_n^N(\overline{g}_{l,x})^{-1}\ 1_{\eta_n^N(g_{l,x})>0}||_p &\leq 1+\frac{1}{\sqrt{N}}\ 2a(p)(n+1) + (2a(2p)(n+1))^2\frac{1}{N}\\ &\quad + \frac{1}{\sqrt{N}}(2a(3p)(n+1))^3,\end{aligned}$$

from which we find the rather crude estimates

$$||\eta_n^N(\overline{g}_{l,x})^{-1}\ 1_{\eta_n^N(g_{l,x})>0}||_p \leq 1+\frac{3}{\sqrt{N}}\ a'(p)\ (n+1)^3,$$

with the collection of finite constants $a'(p) := 2a(p) + (2a(2p))^2 + (2a(3p))^3$. Using the above exponential inequalities, we find that

$$\left|\left|\eta_n^N(g_{l,\xi^i_{l,n}})^{-1}\right|\right|_p \leq \sum_{x\in E'_l} \frac{1}{\eta_n(g_{l,x})}\left[1+\frac{3}{\sqrt{N}}\,a'(p)\,(n+1)^3\right] + N\ \mathrm{Card}(E'_l)^{1/p}\ \exp\left(-\frac{N}{32p(n+1)^2}\right),$$

completing the proof of the lemma. ∎

Appendix B. Proof of Lemma 5.3. By construction, we have

$$\forall x \in \widehat{E}_{l,n}, \qquad M'_{l+1}(f)(x) = \frac{\eta_l^N M_{l,n}((1_x\circ\pi_l)\ (f\circ\pi_{l+1}))}{\eta_l^N M_{l,n}((1_x\circ\pi_l))}. \tag{B.1}$$

Thus, by (B.1), we have

$$\widehat{M'_{l+1}}(f)(x) - M'_{l+1}(f)(x) := \frac{\eta_n^N(g_{l,x}f_{l+1})}{\eta_n^N(g_{l,x})} - \frac{\eta_l^N M_{l,n}(g_{l,x}f_{l+1})}{\eta_l^N M_{l,n}(g_{l,x})},$$

for any $x \in \widehat{E}_{l,n}$, with the collection of functions

$$g_{l,x} := 1_x \circ \pi_l \quad \text{and} \quad f_{l+1} := f \circ \pi_{l+1}. \tag{B.2}$$

It is readily checked that

$$\widehat{M}'_{l+1}(f)(x) - M'_{l+1}(f)(x) = \frac{1}{\eta_n^N(\bar{g}_{l,x}^N)} \left[\eta_n^N(\bar{f}_{l+1,x}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,x}^N)\right],$$

for any $x \in \widehat{E}_{l,n}$, with the pair of $\mathcal{F}_l^N$-measurable functions

$$\bar{f}_{l+1,x}^N := \frac{g_{l,x}}{\eta_l^N M_{l,n}(g_{l,x})} \left[f_{l+1} - \frac{\eta_l^N M_{l,n}(g_{l,x} f_{l+1})}{\eta_l^N M_{l,n}(g_{l,x})}\right] \quad \text{and} \quad \bar{g}_{l,x}^N = \frac{g_{l,x}}{\eta_l^N M_{l,n}(g_{l,x})}.$$

It is also important to observe that as $g_{l,x}$ varies only on E'_l, then

$$\eta_l^N M_{l,n}(g_{l,x}) = \eta_l^N(g_{l,x}) \leq 1.$$

In this notation, for any $0 \leq i \leq N$ and any $p \geq 1$, we have

$$\begin{aligned} &\left\|\widehat{M}'_{l+1}(f)(\xi_{l,n}^i) - M'_{l+1}(f)(\xi_{l,n}^i)\right\|_p \\ &\leq \left\|\eta_n^N(g_{l,\xi_{l,n}^i})^{-1}\right\|_{2p} \left\|\eta_n^N(\bar{f}_{l+1,\xi_{l,n}^i}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^i}^N)\right\|_{2p}. \end{aligned} \tag{B.3}$$

The collection of random functions $\bar{f}_{l+1,\xi_{l,l}^j}^N$ is well defined, and we have

$$\begin{aligned} &\left(\eta_n^N(\bar{f}_{l+1,\xi_{l,n}^i}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^i}^N)\right)^\beta \\ &= \frac{1}{\eta_l^N\left(g_{l,\xi_{l,n}^i}\right)} \frac{1}{N}\sum_{j=1}^N \left[\eta_n^N(\bar{f}_{l+1,\xi_{l,l}^j}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,l}^j}^N)\right]^\beta 1_{\xi_{l,l}^j = \xi_{l,n}^i} \end{aligned}$$

for any $\beta \geq 0$. Combining the above formula for $\beta = 2p$ and Holder's inequality, we prove that

$$\begin{aligned} &\left\|\eta_n^N(\bar{f}_{l+1,\xi_{l,n}^i}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^i}^N)\right\|_{2p} \\ &\leq \left\|\eta_l^N\left(g_{l,\xi_{l,n}^i}\right)^{-1}\right\|_q^{1/(2p)} \times \sup_{1\leq j\leq N} \left\|\eta_n^N(\bar{f}_{l+1,\xi_{l,l}^j}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,l}^j}^N)\right\|_{2pq'}, \end{aligned}$$

for any $q, q' \geq 1$, with $\frac{1}{q} + \frac{1}{q'} = 1$.

We observe that, as $(\xi_{l,l}^j, (\xi_{l,l}^i)_{0\leq i\leq N}, (\xi_{l,n}^i)_{0\leq i\leq N})$ have the same distribution, for any $1 \leq j \leq N$, then for any function h and any $1 \leq j, j' \leq N$ we have that

$$\mathbb{E}\left(h(\xi_{l,l}^j, (\xi_{l,l}^i)_{0\leq i\leq N}, (\xi_{l,n}^i)_{0\leq i\leq N})\right) = \mathbb{E}\left(h(\xi_{l,l}^{j'}, (\xi_{l,l}^i)_{0\leq i\leq N}, (\xi_{l,n}^i)_{0\leq i\leq N})\right),$$

which implies that

$$\sup_{1\le j\le N}\left\|\eta_n^N(\bar f^N_{l+1,\xi^j_{l,l}})-\eta_l^N M_{l,n}(\bar f^N_{l+1,\xi^j_{l,l}})\right\|_{2pq'} = \left\|\eta_n^N(\bar f^N_{l+1,\xi^j_{l,l}})-\eta_l^N M_{l,n}(\bar f^N_{l+1,\xi^j_{l,l}})\right\|_{2pq'}.$$

As this equation works for any $1\le j\le N$, in the further development we take $j=1$ to simplify the notation.

Using Lemma 5.1, and recalling that $\eta_l^N M_{l,n}(g_{l,x})=\eta_l^N(g_{l,x})$, for any $1\le j\le N$ we prove the almost sure estimate

$$\begin{aligned}
\sqrt{N}\,\mathbb{E}&\left(\left|[\eta_n^N-\eta_l^N M_{l,n}](\bar f^N_{l+1,\xi^1_{l,l}})\right|^{2pq'}\,\middle|\mathcal{F}_l^N\right)^{\frac{1}{2pq'}}\\
&\le 2\,a(2pq')(n-l)\,\left[\eta_l^N M_{l,n}\left(\left|\bar f^N_{l+1,\xi^1_{l,l}}\right|^{2pq'}\right)\right]^{\frac{1}{2pq'}}\\
&\le 4\,a(2pq')(n-l)\,\|f_{l+1}\|\,\left(\eta_l^N M_{l,n}(g_{l,\xi^1_{l,l}})\right)^{\frac{1}{2pq'}-1}.
\end{aligned}$$

This yields that

$$\sqrt{N}\mathbb{E}\left(\left|[\eta_n^N-\eta_l^N M_{l,n}](\bar f^N_{l+1,\xi^1_{l,l}})\right|^{2pq'}\,\middle|\mathcal{F}_l^N\right)^{\frac{1}{2pq'}} \le 4\,a(2pq')(n-l)\|f_{l+1}\|\,\eta_l^N(g_{l,\xi^1_{l,l}})^{-1},$$

and therefore

$$\begin{aligned}
\sqrt{N}&\left\|\eta_n^N(\bar f^N_{l+1,\xi^i_{l,n}})-\eta_l^N M_{l,n}(\bar f^N_{l+1,\xi^i_{l,n}})\right\|_{2pq'}\\
&\le 4\,a(2pq')(n-l)\|f_{l+1}\|\,\left\|\eta_l^N\left(g_{l,\xi^i_{l,n}}\right)^{-1}\right\|_q^{1/(2p)}\left\|\eta_l^N(g_{l,\xi^1_{l,l}})^{-1}\right\|_{2pq'}.
\end{aligned}$$

Finally, by (B.3), we conclude that

$$\begin{aligned}
\sqrt{N}\,&\left\|\widehat{M}'_{l+1}(f)(\xi^i_{l,n})-M'_{l+1}(f)(\xi^i_{l,n})\right\|_p\\
&\le 4\,a(2pq')(n-l)\|f_{l+1}\|\,\left\|\eta_n^N(g_{l,\xi^i_{l,n}})^{-1}\right\|_{2p}\left\|\eta_l^N\left(g_{l,\xi^i_{l,n}}\right)^{-1}\right\|_q^{1/(2p)}\\
&\times\left\|\eta_l^N(g_{l,\xi^1_{l,l}})^{-1}\right\|_{2pq'}.
\end{aligned}$$

We prove (5.13), by taking $q = 1 + 2p$ and $q' = 1 + 1/(2p)$, so that $q = 2pq' \geq 2p$

$$\sqrt{N} \ \left|\left|\widehat{M}'_{l+1}(f)(\xi^i_{l,n}) - M'_{l+1}(f)(\xi^i_{l,n})\right|\right|_p$$

$$\leq 4\ a(1+2p)(n-l)\|f_{l+1}\| \ \sup_{l\leq k\leq n} \left|\left|\eta^N_k(g_{l,\xi^1_{l,k}})^{-1}\right|\right|^{2+1/(2p)}_{1+2p}.$$

This end of proof is now a direct consequence of Lemma 5.2. ∎

Acknowledgments. We are grateful to Laurent Plagne for his tremendous help in accelerating our numerical simulation implementation. We are particularly grateful to the anonymous referees for their very stimulating remarks and comments.

REFERENCES

[1] A. N. Avramidis and H. Matzinger, *Convergence of the stochastic mesh estimator for pricing Bermudan options*, J. Comput. Finance, 7 (2004), pp. 73–91.

[2] V. Bally, G. Pagès, and J. Printems, *A quantization tree method for pricing and hedging multidimensional American options*, Math. Finance, 15 (2005), pp. 119–168.

[3] V. Bally and D. Talay, *The law of the Euler scheme for stochastic differential equations:* I. *Convergence rate of the distribution function*, Probab. Theory Related Fields, 104 (1996), pp. 43–60.

[4] V. Bally and D. Talay, *The law of the Euler scheme for stochastic differential equations:* II. *Approximation of the density*, Monte Carlo Methods Appl., 2 (1996), pp. 93–128.

[5] M. Broadie and P. Glasserman, *A stochastic mesh method for pricing high-dimensional American options*, J. Comput. Finance, 7 (2004), pp. 35–72.

[6] B. Bouchard and N. Touzi, *Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations*, Stochastic Process Appl., 111 (2004), pp. 175–206.

[7] B. Bouchard and X. Warin, *Monte-Carlo valorisation of American options: Facts and new algorithms to improve existing methods*, in Numerical Methods in Finance, R. Carmona, P. Del Moral, P. Hu, and N. Oudjane, eds., Springer-Verlag, Berlin, to appear.

[8] J. Bucklew and G. Wise, *Multidimensional asymptotic quantization theory with rth power distortion measures*, IEEE Trans. Inform. Theory, 28 (1982), pp. 239–247.

[9] P. Del Moral, *Measure valued processes and interacting particle systems. Application to nonlinear filtering problems*, Ann. Appl. Probab., 8 (1998), pp. 1254–1278.

[10] P. Del Moral, *Feynman-Kac Formula: Genealogical and Interacting Particle Systems with Applications*, Prob. Appl. (N.Y.), Springer-Verlag, New York, 2004.

[11] P. Del Moral, A. Doucet, and A. Jasra, *On adaptive resampling procedures for sequential Monte Carlo methods*, Bernoulli, to appear.

[12] P. Del Moral, A. Doucet, and S. S. Singh, *A backward particle interpretation of Feynman-Kac formulae*, M2AN Math. Model. Numer. Anal., 44 (2010), pp. 947–975.

[13] P. Del Moral and A. Guionnet, *On the stability of interacting processes with applications to filtering and genetic algorithms*, Ann. Inst. H. Poincaré Probab. Statist., 37 (2001), pp. 155–194.

[14] P. Del Moral, B. Rémillard, and S. Rubenthaler, *Monte Carlo approximations of American options that preserve monotonicity and convexity*, SSRN eLibrary no. 1703906, 2010.

[15] P. Del Moral and E. Rio, *Concentration inequalities for mean field particle models*, Ann. Appl. Probab., to appear.

[16] P. Del Moral, L. Miclo, F. Patras, and S. Rubenthaler, *The convergence to equilibrium of neutral genetic models*, Stoch. Anal. Appl., 28 (2010), pp. 123–143.

[17] P. Dupuis and H. Wang, *On the convergence from discrete to continuous time in an optimal stopping problem*, Ann. Appl. Probab., 15 (2005), pp. 1339–1366.

[18] D. Egloff, *Monte Carlo algorithms for optimal stopping and statistical learning*, Ann. Appl. Probab., 15 (2005), pp. 1–37.

[19] D. Egloff, M. Kohler, and N. Todorovic, *A dynamic look-ahead Monte Carlo algorithm for pricing Bermudan options*, Ann. Appl. Probab., 17 (2007), pp. 1138–1171.

[20] P. Glasserman and B. Yu, *Number of paths versus number of basis functions in American option pricing*, Ann. Appl. Probab., 14 (2004), pp. 1–30.

[21] E. Gobet, J.-P. Lemor, and X. Warin, *A regression-based Monte-Carlo method for backward stochastic differential equations*, Ann. Appl. Probab., 15 (2005), pp. 2172–2202.

[22] S. Graf and H. Luschgy, *Foundations of Quantization for Probability Distributions*, Lecture Notes in Math. 1730, Springer-Verlag, Berlin, 2000.

[23] F. Le Gland and N. Oudjane, *Stability and uniform approximation of nonlinear filters using the Hilbert metric and application to particle filters*, Ann. Appl. Probab., 14 (2004), pp. 144–187.

[24] G. Liu and L. J. Hong, *Revisit of stochastic mesh method for pricing American options*, Oper. Res. Lett., 37 (2009), pp. 411–414.

[25] G. Pagès, *A space vector quantization method for numerical integration*, J. Comput. Appl. Math., 89 (1997), pp. 1–38.

[26] G. Pagès, H. Pham, and J. Printems, *Optimal quantization and applications to numerical problems in finance*, in Handbook of Computational and Numerical Methods in Finance, S. T. Rachev, ed., Birkhäuser, Boston, 2004, pp. 253–297.

[27] G. Pagès and J. Printems, *Functional quantization for numerics with an application to option pricing*, Monte Carlo Methods Appl., 11 (2005), pp. 407–446.

[28] R. Van Handel, *Uniform time average consistency of Monte Carlo particle filters*, Stochastic Process. Appl., 119 (2009), pp. 3835–3861.