The Snell envelope and analysis of various approximation schemes

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Joint works with P. Del Moral, N. Oudjane & B. Rémillard
P. Del Moral, P. HU, N. Oudjane and B. Rmillard, "On the Robustness of the Snell Envelope"[40p], SIAM J. Finan. Math., Vol. 2, pp. 587-626 , 2011.
P. Del Moral, P. HU and N. Oudjane, "Snell envelope with small probability criteria"[22p], preprint inria-00507794 [submitted], 2010.

Outline

1. Introduction
2. Broadie-Glasserman Models
3. Genealogical/Ancestral tree based model
4. Snell envelope with small probability criteria

Summary

1. Introduction
   - Some notation
   - Path space models
   - Snell envelope
   - Preliminary
   - Examples
   - Path dependent case
   - Exponential concentration inequalities
2. Broadie-Glasserman Models
3. Genealogical/Ancestral tree based model
4. Snell envelope with small probability criteria

Some notation

E state space, $\mathcal{P}(E)$ proba. on E & $\mathcal{B}(E)$ bounded functions

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $M(x, dy)$ **integral operator over** E

$$\begin{aligned} M(f)(x) &= \int M(x, dy) f(y) \\ [\mu M](dy) &= \int \mu(dx) M(x, dy) \qquad (\Longrightarrow [\mu M](f) = \mu[M(f)]\ ) \end{aligned}$$

Markov chain X_n with transitions $M_n(x_{n-1}, dx_n)$ from E_{n-1} to E_n

$$\mathbb{E}_{\mathbb{P}_{\eta_0}} \{f_n(X_n) | X_0, \ldots, X_k\} = M_{k,n}(f_n)(X_k) := \int_{E_n} M_{k,n}(X_k, dx_n)\ f_n(x_n)$$

with

$$M_{k,n}(x_k, dx_n) = (M_{k+1} M_{k+2} \ldots M_n)(x_k, dx_n) = \mathbb{P}\left(X_n \in dx_n | X_k = x_k\right)$$

Path space notations

- Given a elementary X'_k Markov chain with transitions $M'_k(x'_{k-1}, dx'_k)$ from E'_{k-1} into E'_k.
- The historical process $X_k = (X'_0, \ldots, X'_k) \in E_k = (E'_0 \times \cdots \times E'_k)$ can be seen as a Markov chain with transitions $M_k(x_{k-1}, dx_k)$

Snell envelope

Description

- For $0 \leq k \leq n$, some process Z_k (gain) with $\mathcal{F}_k$ available infomation on k, $\mathcal{T}_k$ set of stopping times taking value in(k,k+1…n)
- **Purpose:** find $\sup_{\tau \in \mathcal{T}_k} \mathbb{E}(Z_\tau | \mathcal{F}_k)$
- Y_k the **Snell envelope** of Z_k :

$$\begin{aligned} Y_n &= Z_n \\ Y_k &= Z_k \vee \mathbb{E}(Y_{k+1} | \mathcal{F}_k) \end{aligned}$$

- Main property of the Snell envelope:

$$Y_k = \sup_{\tau \in \mathcal{T}_k} \mathbb{E}(Z_\tau | \mathcal{F}_k) = \mathbb{E}(Z_{\tau_k^*} | \mathcal{F}_k) \quad \tau_k^* = \min \{ k \leq j \leq n \ : \ Y_j = Z_j \} \in \mathcal{T}_k$$

Assumption

- Some Markov chain $(X_k)_{0\leq k\leq n}$, with $\eta_0 \in \mathcal{P}(E_0)$, $M_n(x_{n-1}, dx_n)$ from E_{n-1} to E_n on filtered space $(\Omega, \mathcal{F}, \mathbb{P}_{\eta_0})$, $\mathcal{F}_k$ associated natural filtration.
- For $f_k \in \mathcal{B}(E_k)$, assume $Z_k = f_k(X_k)$ (payoff)
- Then $Y_k = u_k(X_k)$

Snell envelope recursion:

$$u_k = f_k \vee M_{k+1}(u_{k+1}) \quad \text{with} \quad u_n = f_n$$

A NSC for the existence of the Snell envelope

$M_{k,l}f_l(x) < \infty$ for any $1 \leq k \leq l \leq n$, and any state $x \in E_k$. To check this claim, we simply notice that

$$f_k \leq u_k \leq f_k + M_{k+1}u_{k+1} \implies f_k \leq u_k \leq \sum_{k\leq l\leq n} M_{k,l}f_l$$

Numerical solution

- Replacing $(f_k, M_k)_{0\le k\le n}$ by some approximation model $(\widehat{f}_k, \widehat{M}_k)_{0\le k\le n}$ on some possibly reduced measurable subsets $\widehat{E}_k \subset E_k$.
- $\widehat{u}_k = \widehat{f}_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1})$ with terminal condition $\widehat{u}_n = \widehat{f}_n$ for $0 \le k \le n$

A robustness/continuity lemma

For any $0 \le k < n$, on the state space $\widehat{E}_k$, we have that

$$|u_k - \widehat{u}_k| \quad \le \quad \sum_{l=k}^{n} \widehat{M}_{k,l}|f_l - \widehat{f}_l| + \sum_{l=k}^{n-1} \widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})u_{l+1}|$$

Proof: By inequality $|(a \vee b) - (a' \vee b')| \le |a \vee a'| + |b \vee b'|$ and induction.

Application examples of the lemma

Deterministic models

- Cut-off type models
- Euler approximation models
- Interpolation type models
- Quantization tree models

Monte Carlo models (stoch. N-grid approximation)

- ▷ Broadie-Glasserman models [N^2]
  ▷ BG type adapted mean-field particle model [N^2]
  ▷ Importance sampling model for path dependent case [N^2]
- Genealogical tree based model [N]

Problematic

- Given gain functions $(f_k)_{0 \leq k \leq n}$ and **obstacle functions** $(G_k)_{0 \leq k \leq n}$
- Snell envelope of $f_k(X_k) \prod_{p=0}^{k-1} G_p(X_p)$?
- Impossible to compute if G_k too small

New recursion

- Original Snell envelope :
  $\mathbf{u}_k(X_0, \ldots, X_k) = f_k(X_k) \prod_{p=0}^{k-1} G_p(X_p) \vee \mathbb{E}(\mathbf{u}_{k+1}(X_0, \ldots, X_{k+1}) | \mathcal{F}_k)$
  with $\quad \mathbf{u}_n(X_0, \ldots, X_n) = f_n(X_n) \prod_{p=0}^{n-1} G_p(X_p)$
- We provide a **new recursion** $v_k = f_k \vee (G_k M_{k+1}(v_{k+1}))$ with $v_n = f_n$
- $\mathbf{u}_k(x_0, \ldots, x_k) = v_k \prod_{p=0}^{k-1} G_p(x_p)$

Exponential concentration inequalities

Important constants

$\forall p \geq 0 \qquad a(2p)^{2p} = (2p)_p \; 2^{-p} \quad \text{and} \quad a(2p+1)^{2p+1} = \frac{(2p+1)_{p+1}}{\sqrt{p+1/2}} \; 2^{-(p+1/2)}$

Proposition

If we have a Khinchine's type $\mathbb{L}_p$-mean error bounds in the following form:
$\forall$ *integer* $p \geq 1$ *and constant* c

$$\sqrt{N} \sup_{x \in E_k} ||u_k(x) - \widehat{u}_k(x)||_{\mathbb{L}_p} \leq a(p)\; c$$

then we have the following exponential concentration inequality

$$\sup_{x \in E_k} \mathbb{P}\left(|u_k(x_k) - \widehat{u}_k(x_k)| > \frac{c}{\sqrt{N}} + \epsilon\right) \leq exp\left(-N\epsilon^2/c^2\right)$$

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2. Broadie-Glasserman Models
   - Original Broadie-Glasserman
   - BG adapted mean-field particle model
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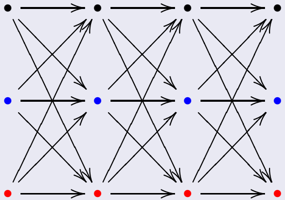
M. Broadie and P. Glasserman. A Stochastic Mesh Method for Pricing High- Dimensional American Options *Journal of Computational Finance* (04)

Original Broadie-Glasserman Models (hyp : $M'_k \ll \eta_k$)

$\eta_k \simeq \widehat{\eta}_k = \frac{1}{N}\sum_{i=1}^{N}\delta_{\xi_k^i}$ where $\xi_k := (\xi_k^i)_{1\leq i\leq N} \sim$ i.i.d. N-grid η_k on $\widehat{E}'_k = E'_k$

$$
\begin{aligned}
M'_{k+1}(x'_k, dx'_{k+1}) \simeq \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) &= \widehat{\eta}_{k+1}(dx'_{k+1})\ \underbrace{R_{k+1}(x'_k, x'_{k+1})} \\
&= \widehat{\eta}_{k+1}(dx'_{k+1})\ \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1})
\end{aligned}
$$

$(N = 3\ n = 3)$



$\rightsquigarrow$ N^2 *computations / time units*

By Khintchine's inequality we notice:

$$\sqrt{N}\ \left|\left|\left[M'_{l+1}-\widehat{M}'_{l+1}\right](f)(x'_l)\right|\right|_{\mathbb{L}_p} \leq 2\ a(p)\ \eta_{l+1}\left[(R_{l+1}(x'_l,.)f)^p\right]^{\frac{1}{p}}$$

We provide the following non asymptotic convergence estimate

Theorem

For any integer $p \geq 1$, we denote by p' the smallest even integer greater than p. Then for any time horizon $0 \leq k \leq n$, and any $x'_k \in E'_k$, we have

$$\sqrt{N}\,||u'_k(x'_k)-\widehat{u}'_k(x'_k)||_{\mathbb{L}_p}$$

$$\leq 2a(p)\sum_{k\leq l<n}\left\{\int M'_{k,l}(x'_k,dx'_l)\eta_{l+1}\left[(R_{l+1}(x'_l,.)u_{l+1})^{p'}\right]\right\}^{\frac{1}{p'}}$$

BG adapted mean-field particle model

New (N^2) algorithm with the choice $\eta_k = \mathrm{Law}(X'_k) = \eta_{k-1} M'_k$

Description (hyp. : $M'_k \ll \lambda_k$)

$\eta_k \simeq \widehat{\eta}_k = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_k^i}$ with i.i.d. copies ξ_k^i of X'_k

$$M'_{k+1}(x'_k, dx'_{k+1}) \simeq \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) \quad = \quad \widehat{\eta}_{k+1}(dx'_{k+1}) \, \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\,.\,, x'_{k+1}))}$$

with

$$(H)_0 \qquad H_n(x'_{n-1}, x'_n) = \frac{dM'_n(x'_{n-1}, \,.\,)}{d\lambda_n}(x'_n)$$

BG adapted mean-field particle model

Snell envelope

- Set by recursion $\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left(\int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1})\ \widehat{u}'_{k+1}(x'_{k+1}) \right)$ with terminal condition $\widehat{u}'_n = f'_n$

Theorem

$(H)_1$

$$\|M'_{l+1}(h^{2p}_{l+1})\| < \infty \text{ with } \sup_{x'_l, y'_l \in E'_l} \frac{H_{l+1}(x'_l, x'_{l+1})}{H_{l+1}(y'_l, x'_{l+1})} \leq h_{l+1}(x'_{l+1})$$

$$\|M'_{l+1}(u^{2p}_{l+1})\| < \infty$$

Under the conditions $(H)_0$ and $(H)_1$ stated above, for any even integer $p > 1$, any $0 \leq k \leq n$, and $x'_k \in E'_k$, we have

$$\sqrt{N}\ ||u'_k(x'_k) - \widehat{u}'_k(x'_k)||_{\mathbb{L}_p} \leq 2a(p) \sum_{k \leq l < n} \left(\|M'_{l+1}(h^{2p}_{l+1})\|\ \|M'_{l+1}(u^{2p}_{l+1})\| \right)^{\frac{1}{2p}}$$

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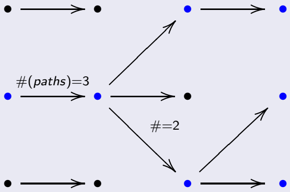
Evolution example of genealogical tree

$(N = 3\ n = 3)$



$\oplus$ Snell envelope computation on the N-stochastic grid

Genealogical tree based model

notation

- The k-th coordinate mapping

$$\pi_k \;:\; x_n = (x'_0, \ldots, x'_n) \in E_n = (E'_0 \times \ldots \times E'_n) \mapsto \pi_k(x_n) = x'_k \in E'_k$$

- $\forall 0 \leq k < n$, $x'_k \in E'_k$ and any function $f \in \mathcal{B}(E'_{k+1})$, we have

$$\eta_n = \mathrm{Law}(X'_0, \ldots, X'_n) \quad \text{and} \quad M'_{k+1}(f)(x) := \frac{\eta_n((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n((1_x \circ \pi_k))}$$

- Remark $\eta_n = \eta'_0 \times M'_1 \times \cdots \times M'_n = \eta_{n-1} M_n$

Particle system = Neutral genetic particle algorithm

- Markov chain taking values in the product state spaces E_k^N.
- Initial system $\bar{X}_0 = (\bar{X}_0^i)_{1\leq i\leq N}$ i.i.d. random copies of X_0
- Evolution

$$\bar{X}_k \in E_k^N \xrightarrow{\text{Selection}} \widehat{X}_k := \left(\widehat{X}_k^i\right)_{1\leq i\leq N} \in E_k^N \xrightarrow{\text{Mutation}} \bar{X}_{k+1} \in E_{k+1}^N$$

Structure = Ancestral lines

$$\bar{X}_k = \begin{bmatrix} \bar{X}_k^1 \\ \vdots \\ \bar{X}_k^i \\ \vdots \\ \bar{X}_k^N \end{bmatrix} = \begin{bmatrix} (\bar{X}_{0,k}^1, \bar{X}_{1,k}^1, \ldots, \bar{X}_{k,k}^1) \\ \vdots \\ (\bar{X}_{0,k}^i, \bar{X}_{1,k}^i, \ldots, \bar{X}_{k,k}^i) \\ \vdots \\ (\bar{X}_{0,k}^N, \bar{X}_{1,k}^N, \ldots, \bar{X}_{k,k}^N) \end{bmatrix}$$

Remark

$$\begin{aligned} \bar{X}_{k+1}^i &= \Big(\underbrace{(\bar{X}_{0,k+1}^i, \bar{X}_{1,k+1}^i, \ldots, \bar{X}_{k,k+1}^i)}_{||}, \bar{X}_{k+1,k+1}^i \Big) \\ &= \Big(\overbrace{(\widehat{X}_{0,k}^i, \widehat{X}_{1,k}^i, \ldots, \widehat{X}_{k,k}^i)}, \bar{X}_{k+1,k+1}^i \Big) = \Big(\widehat{X}_k^i, \bar{X}_{k+1,k+1}^i \Big) \end{aligned}$$

Genealogical tree based model

Occupation measures

$$\eta_k^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\bar{X}_k^i} \quad \text{and} \quad \widehat{\eta}_k^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\widehat{X}_k^i}$$

- $\widehat{\eta}_k^N$: empirical meas. of $\widehat{X}_k^i \overset{c-iid}{\sim} \eta_k^N$
- η_k^N : empirical meas. of $\bar{X}_k^i \overset{c-iid}{\sim} \widehat{\eta}_{k-1}^N M_k$

Genealogical tree based model

With elementary decomposition

$$[\eta_n^N - \eta_{k-1}^N M_{k,n}] = \sum_{l=k}^{n} [\eta_l^N - (\eta_{l-1}^N M_l)] M_{l,n}$$

and Khintchine's inequality, by induction we have following estimates

Lemma

For any $p \geq 1$, p' the smallest even integer greater than p. In this notation, for any $k \geq 0$ and any function f, we have the almost sure estimate

$$\sqrt{N}\, \mathbb{E}\left(\left| [\eta_n^N - \eta_{k-1}^N M_{k-1,n}](f) \right|^p \left| \mathcal{F}_{k-1}^N \right. \right)^{\frac{1}{p}} \leq 2a(p) \sum_{l=k}^{n} \left[\eta_{k-1}^N M_{k-1,l}(|M_{l,n}(f)|^{p'}) \right]^{\frac{1}{p'}}$$

Approximation of the Markov transitions M'_{k+1}

$$\widehat{M}'_{k+1}(f)(x) := \frac{\eta_n^N((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n^N((1_x \circ \pi_k))} := \frac{\sum_{1 \le i \le N}\ 1_x(\bar{X}^i_{k,n})\ f(\bar{X}^i_{k+1,n})}{\sum_{1 \le i \le N} 1_x(\bar{X}^i_{k,n})}$$

Construction of Model

$$\widehat{u}_k(x) = \begin{cases} f_k(x) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})(x) & \forall x \in \widehat{E}_{k,n} \\ 0 & \text{otherwise} \end{cases}$$

In terms of the ancestors at level k, this recursion takes the following form

$$\forall 1 \le i \le N \qquad \widehat{u}_k\left(\bar{X}^i_{k,n}\right) = f_k\left(\bar{X}^i_{k,n}\right) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})\left(\bar{X}^i_{k,n}\right)$$

Applying the local error given by precedent lemma and the robustness lemma, we finally get

Theorem

For any $p \geq 1$, and $0 \leq i \leq N$ we have the following uniform estimate

$$\sup_{0 \leq k \leq n} \left\| (u_k - \widehat{u}_k)(\bar{X}^i_{k,n}) \right\|_{\mathbb{L}_p} \leq c_p(n)/\sqrt{N}$$

with some collection of finite constants $c_p(n) < \infty$ whose values only depend on the parameters p and n.

Numerical examples

Asset modeling

$$\frac{dX'(i)_t}{X'(i)_t} = rdt + \sigma_i dz_t^i, \quad i = 1, \ldots, d = 6.$$

- z^i independent standard Brownian motions
- r=5% annually
- $X'_0(i) = 1$ and $\sigma_i = 20\%$ annually

Bermudan options

Maturity $T = 1$ year and 11 equally distributed exercise opportunities:

- geometric average put with payoff $(K - \prod_{i=1}^{d} X'(i)_T)_+$, $K = 1$
- arithmetic average put with payoff $(K - \frac{1}{d}\sum_{i=1}^{d} X'(i)_T)_+$, $K = 1$

Numerical examples

Benchmark

Nb. assets	1	2	3	4	5	6
Geometric	0.06033	0.07815	0.08975	0.09837	0.10511	0.11073
Arithmetic	0.0603331	0.03881	0.02945	0.02403	0.02070	0.01895

Figure: Benchmark values for the geometric and arithmetic put options (taken from B. Bouchard and X. Warin, Monte-Carlo valorisation of American options: facts and new algorithms to improve existing methods, To appear in *Numerical Methods in Finance*, ed. R. Carmona, P. Del Moral, P. Hu and N. Oudjane (2011).

State space discretization

Methods : random tree, stochastic mesh, Binomial tree, quantization approach or quantization-like approach:

State space partitioning

1. Simulate N i.i.d. paths according to asset dynamic
2. At each time step, partition the particles into M subsets
3. For each subset, compute the representative state $(S_k^j)_{1\leq j\leq M, 1\leq k\leq n}$ as average of particles

Finite state space Markov chain

1. Define $\tilde{E}_k = \{S_k^1, \dots, S_k^M\}$ as new finite state space.
2. The dynamic of new Markov chain $\tilde{X}_k$:

$$\mathbb{P}\left(\tilde{X}_k = S_k^j \,|\, \tilde{X}_{k-1} = S_{k-1}^i\right) = \mathbb{P}\left(X_k \in V_k^j \,|\, X_{k-1} = S_{k-1}^i\right)$$

Complexity and errors

Complexity : Forward step $O(MN)$, Backward step $O(N)$

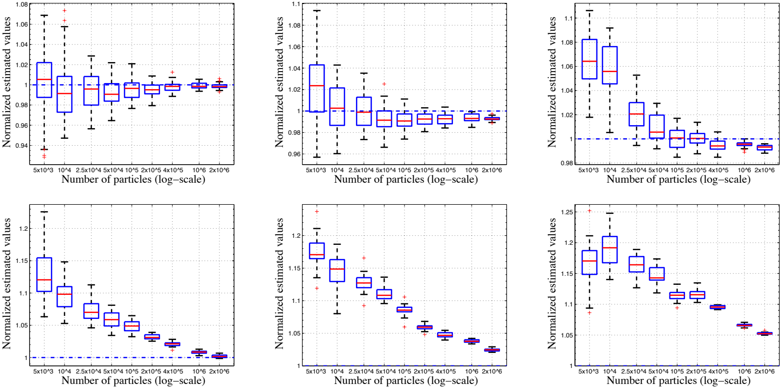
- State discretization error bounded by $\frac{c}{M^{\frac{1}{d}}}$
- G.T algorithm error bounded by $c\frac{M^{\beta}}{N}$, for $\beta > 0$

Optimization

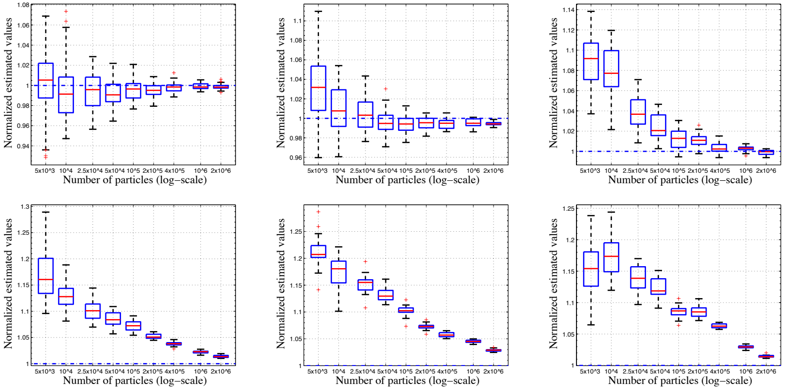
Set $M = O(N^{\frac{d}{2\beta d+2}})$

- Global complexity of order $N^{\frac{(1+2\beta)d+2}{2\beta d+2}}$
- Approximation error bounded by $\frac{c}{N^{\frac{1}{2\beta d+2}}}$

In following example, we set $\beta = 1/2$ so that the complexity grows with the dimension from $N^{4/3}, N^{3/2}, N^{8/5}, \cdots, N^2$ for dimensions $d = 1, 2, 3, \cdots, \infty$.



Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **geometric** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.



Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **arithmetic** put-payoff.

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Problematic

- Markov chain $(X_k)_{0\leq k\leq n}$ on $(E_k, \mathcal{E}_k)_{0\leq k\leq n}$ with $(M_k)_{0\leq k\leq n}$
- $(\mathbb{P}_k)_{0\leq k\leq n} \sim (X_0, \ldots, X_k)_{0\leq k\leq n}$
- To calculate, with $v_n = f_n$:
  - $v_k = f_k \vee (G_k M_{k+1}(v_{k+1})) \quad 0 \leq G_k \leq 1$
    (Barrier options.)
  - $v_k = f_k \vee (M_{k+1}(v_{k+1}))$ but f_k are localized in a small region
    (Deep out of money options.)

Change of measure

- Natural and optimal choice to reduce variance:

$$d\mathbb{Q}_n = \frac{1}{Z_n}\left[\prod_{k=0}^{n-1} G_k\right] d\mathbb{P}_n, \quad \text{with} \quad Z_n = \mathbb{E}\left(\prod_{k=0}^{n-1} G_k(X_k)\right) = \prod_{k=0}^{n-1} \eta_k(G_k)$$

where

$$\eta_k(f) := \frac{\mathbb{E}\Big(f(X_k)\prod_{p=0}^{k-1} G_p(X_p)\Big)}{\mathbb{E}\Big(\prod_{p=0}^{k-1} G_p(X_p)\Big)}$$

- Remark $\eta_k(f) = \dfrac{\eta_{k-1}(G_{k-1}M_k(f))}{\eta_{k-1}(G_{k-1})}$ and define $\eta_k = \Phi_k(\eta_{k-1})$

Notation & Hyp.

- $Q_k(f)(x_{k-1}) := \int G_{k-1}(x_{k-1})M_k(x_{k-1}, dx_k)f(x_k)$
- $M_k(x_{k-1}, dx_k) = H_k(x_{k-1}, x_k)\lambda_k(dx_k)$

Lemma on the change of measure

- For any measure η on E_k, recursion of v_k can be rewritten:

$$v_k(x_k) = f_k(x_k) \vee Q_{k+1}(v_{k+1})(x_k) = f_k(x_k) \vee \Phi_{k+1}(\eta)\left(\frac{dQ_{k+1}(x_k,\cdot)}{d\Phi_{k+1}(\eta)}v_{k+1}\right),$$

for any $x_k \in E_k$, where

$$\frac{dQ_{k+1}(x_k,\cdot)}{d\Phi_{k+1}(\eta)}(x_{k+1}) = \frac{G_k(x_k)H_{k+1}(x_k, x_{k+1})\eta(G_k)}{\eta(G_k H_{k+1}(\cdot, x_{k+1}))}$$

for any $(x_k, x_{k+1}) \in E_k \times E_{k+1}$.

With small probability criteria

Algorithm

- $\xi_0 = \left(\xi_0^i\right)_{1 \le i \le N}$ i.i.d. random copies of X_0
- Evolution

$$\xi_k \in E_k^N \xrightarrow[S_{k,\eta_k^N}]{\text{Selection}} \widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1 \le i \le N} \in E_k^N \xrightarrow[M_{k+1}]{\text{Mutation}} \xi_{k+1} \in E_{k+1}^N$$

With small probability criteria

Selection S_{k,η_k^N}

- First step

$$\begin{tikzcd}[column sep=huge, row sep=large]
\xi_k^i \arrow[r, "\epsilon G_k(\xi_k^i)"] \arrow[dr, "1-\epsilon G_k(\xi_k^i)"'] & \xi_k^i \\
 & 2^{nd} step
\end{tikzcd}$$

$\forall \epsilon \ s.t. \ \epsilon G_k \leq 1$

- Second step $\xi_k^i \sim \widehat{\xi}_k^i$ with proba $\frac{G_k(\xi_k^i)}{\sum_{j=1}^N G_k(\xi_k^j)}$

Occupation measure

$$\eta_k^N = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_k^i}$$

Proposition

For any integer $p \geq 1$, we denote by p' the smallest even integer greater than p. In this notation, for any $0 \leq k \leq n$ and any integrable function f on space E_{k+1}, we have:

$$\mathbb{E}\left(\eta_{k+1}^N(f)|\mathcal{F}_k^N\right) = \Phi_{k+1}(\eta_k^N)(f)$$

and

$$\sqrt{N}\,\mathbb{E}\left(\left|\left[\eta_{k+1}^N - \Phi_{k+1}(\eta_k^N)\right](f)\right|^p |\mathcal{F}_k^N\right)^{\frac{1}{p}} \leq 2\ a(p)\ \left[\Phi_{k+1}(\eta_k^N)(|f|^{p'})\right]^{\frac{1}{p'}}$$

With small probability criteria

Approximation of the transition operator Q_{k+1}

$$\begin{aligned}
\widehat{Q}_{k+1}(f)(x_k) &:= \int_{E_{k+1}^N} \eta_{k+1}^N(dx_{k+1}) \frac{dQ_{k+1}(x_k,\cdot)}{d\Phi_{k+1}(\eta_k^N)}(x_{k+1}) f(x_{k+1}) \\
&= \int_{E_{k+1}^N} \eta_{k+1}^N(dx_{k+1}) \frac{G_k(x_k) H_{k+1}(x_k, x_{k+1}) \eta_k^N(G_k)}{\eta_k^N(G_k H_{k+1}(\cdot, x_{k+1}))} f(x_{k+1}) \\
&= \sum_{1 \le i \le N} \frac{G_k(x_k) H_{k+1}(x_k, \xi_{k+1}^i) \sum_{1 \le j \le N} G_k(\xi_k^j)}{\sum_{1 \le l \le N} G_k(\xi_k^l) H_{k+1}(\xi_k^l, \xi_{k+1}^i))} f(\xi_{k+1}^i)
\end{aligned}$$

Remark

- $Q_{k+1}(x_k, dx_{k+1}) := \Phi_{k+1}(\eta_k^N)(dx_{k+1}) \frac{dQ_{k+1}(x_k,\cdot)}{d\Phi_{k+1}(\eta_k^N)}(x_{k+1})$ No Error!
- $\widehat{Q}_{k+1}(x_k, dx_{k+1}) := \eta_{k+1}^N(dx_{k+1}) \frac{dQ_{k+1}(x_k,\cdot)}{d\Phi_{k+1}(\eta_k^N)}(x_{k+1})$

With small probability criteria

Construction of Model

$$\widehat{v}_k(x) = \begin{cases} f_k(x) \vee \widehat{Q}_{k+1}(\widehat{u}_{k+1})(x) & \forall x \in E_k^N \\ 0 & \text{otherwise} \end{cases}$$

Theorem

For any $0 \leq k \leq n$ and any integer $p \geq 1$, we have

$$\sup_{x \in E_k} \|(\widehat{v}_k - v_k)(x)\|_{L_p} \leq \sum_{k<l<n} \frac{2\ a(p)}{\sqrt{N}} q_{k,l} \ \left[Q_{k,l+1}(h_{l+1}^{p'-1} v_{l+1}^{p'})(x) \right]^{\frac{1}{p'}}$$

with a collection of constants $q_{k,l}$ and functions h_k defined as

$$q_{k,l} := \left[\|G_l\| \ \|h_{k+1}\| \prod_{m=k}^{l-1} \|G_m\| \right]^{\frac{p'-1}{p'}} \quad \text{and} \quad h_k(x_k) := \sup_{x,y \in E_{k-1}} \frac{H_k(x, x_k)}{H_k(y, x_k)}$$

Numerical results

Prices dynamics

$$dS_t(i) = S_t(i)(rdt + \sigma dz_t^i) \ ,$$

with $r = 10\%,\ \sigma = 20\%,\ T = 1,\ n = 11$, and $S_{t_0}(i) = 1$, for $i = 1, 2, 3$.

Options Model

1. Geometric average put option with payoff $(K - \prod_{i=1}^{d} S_T(i))_+$,
2. Arithmetic average put option with payoff $(K - \frac{1}{d}\sum_{i=1}^{d} S_T(i))_+$,

Choice of potential functions

$$\begin{cases} G_0(x_1) = (f_1(x_1) \vee \varepsilon)^\alpha \ , \\ \\ G_k(x_k, x_{k+1}) = \frac{(f_{k+1}(x_{k+1}) \vee \varepsilon)^\alpha}{(f_k(x_k) \vee \varepsilon)^\alpha} \ , \quad \text{for all } k = 1, \cdots, n-1 \ , \end{cases}$$

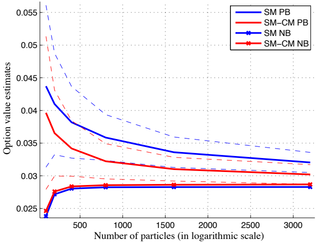
where f_k are the payoff functions and $\alpha \in (0,1]$ and $\varepsilon > 0$ are parameters fixed in our simulations to the values $\alpha = 1/5$ and $\varepsilon = 10^{-7}$.

SM vs. SMCM

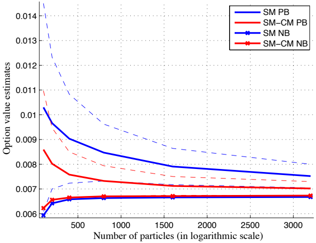
Payoff	K	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Geometric Put	0.95	1 (1%)	1 (3%)	1 (6%)	1 (9%)	1 (10%)
	0.85	5 (2%)	8 (6%)	6 (11%)	4 (14%)	3 (14%)
	0.75	18 (6%)	28 (11%)	18 (17%)	16 (18%)	11 (16%)
Arithmetic Put	0.95	1 (1%)	3 (2%)	3 (7%)	4 (13%)	5 (18%)
	0.85	5 (2%)	13 (6%)	24 (19%)	56 (24%)	100 (20%)
	0.75	18 (6%)	71 (15%)	363 (14%)	866 (16%)	− (−)

Table: Variance ratio ($\frac{Var(\hat{v}_{SM})}{Var(\hat{v}_{SMCM})}$) and Bias ratio ($\frac{\mathbb{E}(\hat{v}_{SM})-\mathbb{E}(\hat{v}_{SMCM})}{E(\hat{v}_{SM})}$) (within parentheses) computed over 1000 runs for $N = 3200$ mesh points. (For the arithmetic put, when $d = 5$ and $K = 0.75$, the 1000 estimates provided by the standard SM algorithm were all equal to zero, hence the associated variance ratio has not been reported).

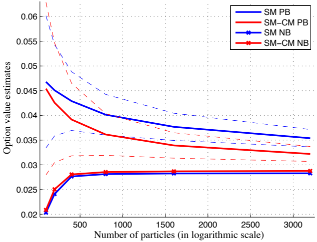
Numerical results

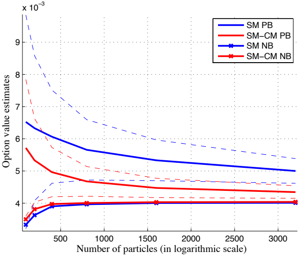


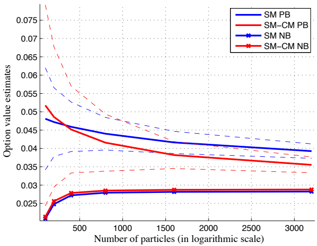
(a) Geometric Put with $d = 3$



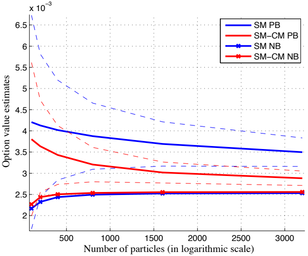
(b) Arithmetic Put with $d = 3$







(e) Geometric Put with $d = 5$



(f) Arithmetic Put with $d = 5$

Figure: Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the *SM* algorithm (in blue line) and the *SMCM* algorithm (in red line), as a function of the number of mesh points for geometric (on the left column) and arithmetic (on the right column) put options with **strike** $K = 0.95$.

Thank you!