Non-backtracking spectrum of random graphs

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Joint work with Marc Lelarge and Laurent Massoulié

Extremal eigenvalues of graphs

Take a finite graph $G = (V, E)$ and define a local operator, e.g. a discrete analog of a differential operator.

Which properties of the graph are contained in the extremal eigenvalues and their eigenvectors ?

In this talk : non-backtracking matrices.

Adjacency matrix

Take a finite, simple, non-oriented graph $G = (V, E)$.

Adjacency matrix : symmetric, indexed on vertices, for $u, v \in V$,

$$A_{uv} = \mathbf{1}(\{u, v\} \in E).$$

Perron eigenvalue

If $|V| = n$, the (real) eigenvalues of A are

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$$

From Perron-Frobenius Theorem : if G is connected, then

$$\lambda_1 > \lambda_2 \quad \text{and} \quad \lambda_1 \geqslant -\lambda_n.$$

Moreover, $\lambda_1 = -\lambda_n$ is equivalent to G bipartite.

Regular graphs

Assume $\deg(v) = d$ for all $v \in V$.

Then

$$\lambda_1 = d$$

with associated eigenvector

$$\psi_1 = (1, \ldots, 1)^\top / \sqrt{n}.$$

Spectral gap

Largest non-trivial eigenvalue

$$\lambda = \max\{|\lambda_k| : |\lambda_k| \neq d\}.$$

Theorem (Alon-Boppana (1991))

$$\lambda \geqslant 2\sqrt{d-1} - \frac{c_d}{\log n}.$$

Ramanujan graphs

A d-regular is Ramanujan if

$$\lambda \leqslant 2\sqrt{d-1}$$

Existence of infinite sequence of bipartite Ramanujan graphs

- $d = p^k + 1$, p prime : *Lubotzky, Phillips & Sarnak (1988), Margulis (1988), Morgenstern (1994),*
- any $d \geqslant 3$: *Marcus, Spielman, Srivastava (2013).*

Spectral gap and Diameter

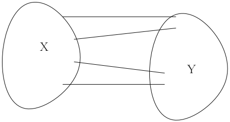
Theorem (Chung (1989))

$$\mathrm{diam}(G) \leqslant \frac{\log(n-1)}{\log d - \log \lambda} + 2.$$

Spectral gap and Expansion

For $X, Y \subset V$, define

$$E(X,Y) = \sum_{x\in X, y\in Y} \mathbf{1}(\{u,v\} \in E).$$



Isoperimetric constant :

$$h(G) = \min_{X\subset V} \frac{E(X,X^c)}{\min(|X|,|X^c|)}.$$

Theorem (Cheeger's Inequality)

$$\frac{h(G)^2}{2d} \leqslant d - \lambda_2 \leqslant 2h(G).$$

Random regular graph

Theorem (Friedman (2004))

Fix integer $d \geqslant 3$. Let G_n is a sequence of uniformly distributed d-regular graphs on n vertices, then with high probability,

$$\lambda = 2\sqrt{d-1} + o(1).$$

Most regular graphs are nearly Ramanujan !!

Non-regular graphs

It is not straightforward to extend the previous notions to non-regular graphs. *Lubotzky (1995), Hoory (2005).*

Eigenvectors of extremal eigenvalues tend to localize on large degree vertices.

Non-regular graphs

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Eigenvectors of extremal eigenvalues tend to localize on large degree vertices.

For example, if G is an Erdős-Rényi graph with parameter α/n, for any fixed $k \geqslant 1$, with high probability,

$$\lambda_k \sim \sqrt{\max_{v \in V}{}^{[k]} \deg(v)} \sim \sqrt{\frac{\log n}{\log \log n}},$$

Sudakov & Krivelevich (2003).

Oriented edge set :

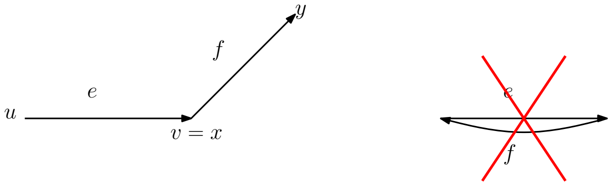
$$\vec{E} = \{uv : \{u, v\} \in E\},$$

hence, $|\vec{E}| = 2|E|$.

If $e = uv, f = xy$ are in $\vec{E}$,

$$B_{ef} = \mathbf{1}(v = x)\mathbf{1}(u \neq y),$$

defines a $|\vec{E}| \times |\vec{E}|$ matrix on the oriented edges.



Perron eigenvalue

A closed non-backtracking path $p = (v_1 \dots v_n)$ is a path such that $v_{i-1} \neq v_{i+1}$. If $e = uv$,

$$\|B^\ell \delta_e\|_1 = \text{nb of NB paths starting with } vu \text{ of length } \ell + 1.$$

Perron eigenvalue

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$$\|B^\ell \delta_e\|_1 = \text{nb of NB paths starting with } vu \text{ of length } \ell + 1.$$

If G is 2-connected (any vertex or pair of vertices are part of a cycle) then B is irreducible and

$$\lambda_1 = \lim_{\ell \to \infty} \|B^\ell \delta_e\|_1^{1/\ell} = \text{growth rate of the universal cover of } B.$$

Hashimoto's Identity

Let Q the diagonal matrix with $Q_{vv} = \deg(v) - 1$. We have

$$\det(z - B) \;=\; (z^2 - 1)^{|E|-|V|} \det(z^2 - Az + Q)$$

Hashimoto's Identity

Let Q the diagonal matrix with $Q_{vv} = \deg(v) - 1$. We have

$$\det(z - B) \quad = \quad (z^2 - 1)^{|E|-|V|} \det(z^2 - Az + Q)$$

If G is d-regular, then $Q = (d-1)I$ and

$$\sigma(B) = \{\pm 1\} \cup \left\{ \lambda : \lambda^2 - \lambda\mu + (d-1) = 0 \text{ with } \mu \in \sigma(A) \right\}.$$

Angel, Friedman, Hoory (2007), Terras (2011)

Non-Backtracking matrix of regular graphs

For a d-regular graph, $\lambda_1 = d - 1$,

- ⋆ Alon-Boppana bound : $\max_{k\neq 1} \mathfrak{Re}(\lambda_k) \geqslant \sqrt{\lambda_1} - o(1)$.
- ⋆ Ramanujan (non bipartite) : $|\lambda_2| = \sqrt{\lambda_1}$
- ⋆ Friedman's thm : $|\lambda_2| \leqslant \sqrt{\lambda_1} + o(1)$ if G random uniform.

Ihara-Bass Formula

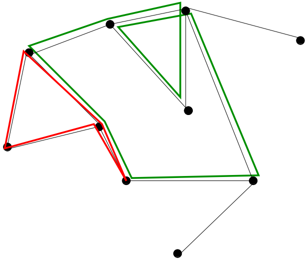
Theorem (Ihara-Bass Formula)
Let ζ_G be the Ihara's zeta function. We have

$$\frac{1}{\zeta_G(z)} = \det(I - Bz) = (1 - z^2)^{|E|-|V|} \det(I - Az + Qz^2).$$

The poles of the zeta function are the reciprocal of eigenvalues of B.

Non-Backtracking Walks

A closed non-backtracking path $p = (v_1, \cdots, v_n)$ is a closed path such that $v_{i-1} \neq v_{i+1} \bmod(n)$.



A closed non-backtracking path is prime if it cannot be written as $p = (q, q, \cdots, q)$ with q closed non-backtracking path.

Equivalence class $p \sim p'$ if $v'_i = v_{i+k} \bmod(n)$.

Ihara's Zeta Function (1966)

$$\zeta_G(z) = \prod_{p:\ \text{prime eq. class}} \left(1 - z^{|p|}\right)^{-1}.$$

Ihara-Bass Formula :

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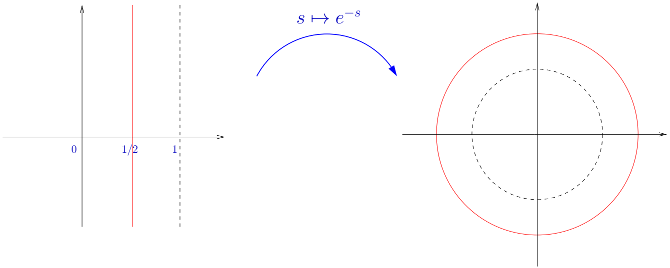
Riemann hypothesis for graphs

With $s = -\ln(z)$ and $N(p) = e^{|p|}$,

$$\zeta_G(z) = \prod_p \left(1 - z^{|p|}\right)^{-1} = \prod_p \left(1 - N(p)^{-s}\right)^{-1}.$$

For $\mathfrak{Re}(s) > 1$,

$$\zeta(s) = \sum_{n \geqslant 1} n^{-s} = \prod_{p\,:\ \text{prime}} \left(1 - p^{-s}\right)^{-1}.$$



Graph analog of RH = poles on a circle = Ramanujan! *(Stark & Terras)*

Non-Backtracking matrix of arbitrary graph

"In general graphs, the condition $|\lambda_2| \leqslant \sqrt{\lambda_1}$ *is one of the possible analogs of a Ramanujan property".*

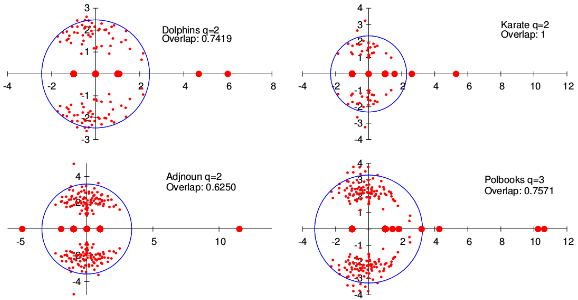
BUT

- ⋆ No Alon-Boppana lower bound.
- ⋆ No Cheeger-type isoperimetric inequality.
- ⋆ No Chung-type diameter inequality.

A more satisfactory analog was proposed by Lubotzky (1995).

Community detection

"Eigenvalues/eigenvectors such that $|\lambda_k| > \sqrt{\lambda_1}$ should contain relevant global information on the graph".

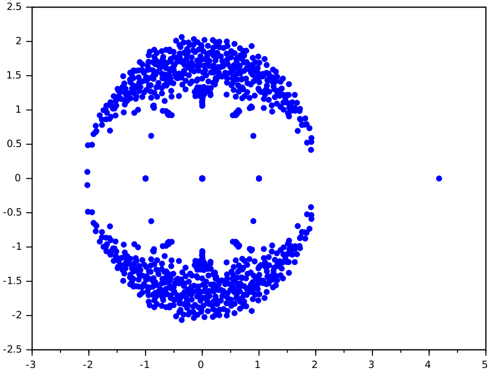


Krzakala/Moore/Mossel/Neeman/Sly/Zdeborová/Zhang (2013)

Non-backtracking spectrum of random graphs

Simulation for Erdős-Rényi Graph

Eigenvalues of B for an Erdős-Rényi graph $\mathcal{G}(n, \alpha/n)$ with $n = 500$ and $\alpha = 4$.



Erdős-Rényi Graph

$$\lambda_1 \geqslant |\lambda_2| \geqslant \dots.$$

Theorem

Let $\alpha > 1$ *and* G *with distribution* $\mathcal{G}(n, \alpha/n)$. *With high probability,*

$$\begin{aligned} \lambda_1 &= \alpha + o(1) \\ |\lambda_2| &\leqslant \sqrt{\alpha} + o(1). \end{aligned}$$

Stochastic Block Model

Consider a set of types $[r] = \{1, \cdots, r\}$ and assign type $\sigma_n(v)$ to vertex v. We assume that

$$\pi_n(i) = \frac{1}{n} \sum_{v=1}^{n} \mathbf{1}(\sigma_n(v) = i) = \pi(i) + O(n^{-\gamma}),$$

for some probability vector π.

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for some probability vector π.

If $\sigma(u) = i, \sigma(v) = j$, the edge $\{u, v\}$ is present independently with probability

$$\frac{W_{ij}}{n} \wedge 1,$$

where W is a symmetric matrix.

(Inhomogeneous random graph, Chung-Lu random graph, . . .)

Stochastic Block Model

If $\sigma(v) = j$, mean number of type i neighbors is

$$\pi(i)W_{ij} + O(1/n).$$

Mean progeny matrix

$$M = \mathrm{diag}(\pi)W.$$

We assume that the average degree is homogeneous, for all $j \in [r]$,

$$\sum_{i=1}^{r} M_{ij} = \alpha > 1.$$

Assume that M is strongly irreducible and we order its real eigenvalues

$$\alpha = \mu_1 > |\mu_2| \geqslant \cdots \geqslant |\mu_r|.$$

Stochastic Block Model

Model used in community detection. Notably for $r = 2$,

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and, with $a > b$,

$$W = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

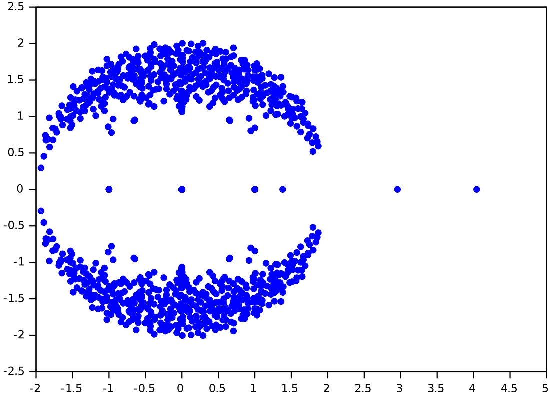
Then

$$\mu_1 = \frac{a+b}{2} \quad \text{and} \quad \mu_2 = \frac{a-b}{2}.$$

Decelle, Krzakala, Moore, Mossel, Neeman, Sly, Zdeborová, Zhang

Stochastic Block Model

$$n = 500, \quad a = 7, \quad b = 1, \quad \mu_1 = 4, \quad \mu_2 = 3.$$



Stochastic Block Model

Let $1 \leqslant r_0 \leqslant r$ such that

$$\alpha = \mu_1 > |\mu_2| \geqslant \cdots \geqslant |\mu_{r_0}| > \sqrt{\mu}_1 \geqslant |\mu_{r_0+1}| \geqslant \cdots \geqslant |\mu_r|.$$

Theorem
Let $\alpha > 1$ and G a stochastic block model as above. With high probability, up to reordering the eigenvalues of B,

$$\begin{array}{rcll} \lambda_k & = & \mu_k + o(1) & \text{if } k \in [r_0] \\ |\lambda_k| & \leqslant & \sqrt{\alpha} + o(1) & \text{if } k \notin [r_0]. \end{array}$$

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(+ a description of the eigenvectors of λ_k, $k \in [r_0]$, if the μ_k are distinct, In particular, they are asymptotically orthogonal).

Stochastic Block Model

Assume

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad W = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

If $(a-b)^2 > 2(a+b)$, with high probability, we may reconstruct correctly a proportion larger than $1/2 + \varepsilon$ of the types from the second largest eigenvector of B.

If $(a-b)^2 < 2(a+b)$, no algorithm can perform that *(Neeman, Mossel & Sly (2012))*.

Some ideas of proofs

Perron eigenvalue

Let us restrict ourselves to the Erdős-Rényi case.

We zoom and consider the matrix B^ℓ where for some well chosen $0 < \kappa < 1/2$,

$$\ell \sim \kappa \log_\alpha n.$$

If $e = uv \in \vec{E}$ and $\chi(f) = 1$ for all $f \in \vec{E}$,

$$\langle \delta_e, B^\ell \chi \rangle = \text{ nb of NB paths of length } \ell \text{ starting from } v \text{ in } G \backslash e$$

is close to the population Z_ℓ at generation ℓ in a Galton-Watson process with $\mathrm{Poi}(\alpha)$ distribution.

Perron eigenvalue

Seneta-Heyde thm, conditionned on non-extinction, a.s.

$$\frac{Z_\ell}{\alpha^\ell} \to M \in (0, \infty).$$

Hence, conditionned on non-extinction, a.s.

$$\frac{Z_{2\ell}}{\alpha^\ell Z_\ell} \to 1.$$

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The vector

$$\varphi = \frac{B^\ell \chi}{\|B^\ell \chi\|}$$

should be close to an eigenvector of B^ℓ associated to α^ℓ.

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Also, if $x \in \mathbb{R}^{\vec{E}}$ has positive entries, $(B^\ell x)/\|B^\ell x\|$ should be nearly aligned with the Perron eigenvector.

Strategy of proof

If $x \in \mathbb{R}^{\vec{E}}$, set $\check{x}(e) = x(e^{-1})$,

$$\zeta = \frac{B^\ell \check{\varphi}}{\|B^\ell \check{\varphi}\|} = \frac{B^\ell B^{*\ell} \chi}{\|B^\ell B^{*\ell} \chi\|} \quad \text{and} \quad \theta = \|B^\ell \check{\varphi}\|.$$

The statement : $\lambda_1 = \alpha + o(1)$ with eigenvector asymptotically aligned to ζ and $|\lambda_2| \leqslant \sqrt{\alpha} + o(1)$ is implied by

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Proposition (Near eigenvector)

With high probability,

$$\langle \zeta, \check{\varphi} \rangle > c_0 \quad \textit{and} \quad c_0 \alpha^\ell < \theta < c_1 \alpha^\ell.$$

Proposition (Small norm in the complement)

With high probability,

$$\sup_{x : \langle x, \check{\varphi} \rangle = 0} \|B^\ell x\| \leqslant (\log n)^c \alpha^{\ell/2} \|x\|.$$

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Standard issue : the graph contains a clique of size m with proba larger than $n^{-m^2/2}$,

$$\mathbb{E}(B^\ell)_{ee} \geqslant (m-1)^\ell n^{-m^2/2} = e^{(\kappa \log(m-1) - m^2/2)\log n}.$$

Polynomially small event may have a big influence in expectation.

Small norm in the complement

With high probability, the graph is ℓ-tangled free that is : no vertex has more than two distinct cycles in its ℓ neighborhood.

Small norm in the complement

With high probability, the graph is ℓ-tangled free that is : no vertex has more than two distinct cycles in its ℓ neighborhood.

We may replace B^{ℓ} by

$$
\begin{aligned}
(B^{(\ell)})_{ef} &= \text{nb of NB tangle free paths } \gamma \text{ of length } \ell \text{ from } e \text{ to } f \\
&= \sum_{\gamma} \prod_{s=0}^{\ell+1} A_{\gamma_s,\gamma_{s+1}},
\end{aligned}
$$

where the sum is over NB tangle free paths of length ℓ from e to f in the complete graph.

Friedman (2004), Neeman-Sly-Mossel (2013), ...

Small norm in the complement

Consider the centered matrix

$$\Delta^{(\ell)}_{ef} = \sum_{\gamma} \prod_{s=0}^{\ell+1} \Big(A_{\gamma_s, \gamma_{s+1}} - \frac{\alpha}{n} \Big),$$

where the sum is over NB tangle free paths of length ℓ from e to f.

After a tricky decomposition,

$$\|B^\ell x\|_2 \leqslant \|\Delta^{(\ell)}\| + \frac{\alpha}{n} \sum_{t=1}^{\ell-1} \|\Delta^{(t-1)}\chi\|_2 \Big| \langle (B^*)^{\ell-t-1} \chi, x \rangle \Big| + \dots$$

which we should estimate over $\langle \check{\varphi}, x \rangle = \langle (B^*)^{\ell} \chi, x \rangle = 0$.

Massoulié (2013)

Small norm in the complement

$$\|B^\ell x\|_2 \leqslant \|\Delta^{(\ell)}\| + \frac{\alpha}{n}\sum_{t=1}^{\ell-1} \|\Delta^{(t-1)}\chi\|_2 \left|\langle (B^*)^{\ell-t-1}\chi, x\rangle\right|$$

From the Galton-Watson tree comparison
$\langle (B^*)^\ell \chi, \delta_e\rangle \simeq \alpha^{\ell-t}\langle (B^*)^t\chi, \delta_e\rangle$,

$$\max_{\langle (B^*)^\ell \chi, x\rangle = 0} \left|\langle (B^*)^t\chi, x\rangle\right| \leqslant (\log n)^c \sqrt{n}\,\alpha^{t/2}\|x\|_2.$$

Small norm in the complement

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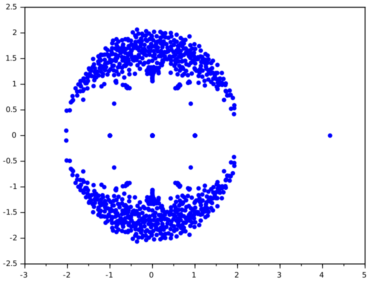
By the method of moments, with $m \simeq \log n / \log\log n$,

$$\|\Delta^{(t)}\| \leqslant \left(\mathrm{Tr}\left(\Delta^{(t)}\Delta^{(t)*}\right)^m\right)^{1/m} \leqslant (\log n)^c \alpha^{t/2}$$

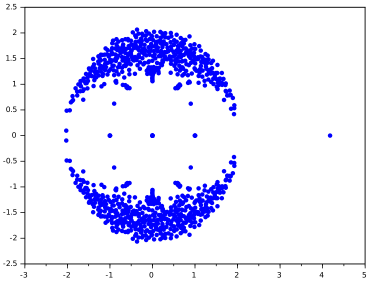
$$\|\Delta^{(t)}\chi\| \leqslant (\log n)^c \sqrt{n}\,\alpha^{t/2}.$$

Final comments



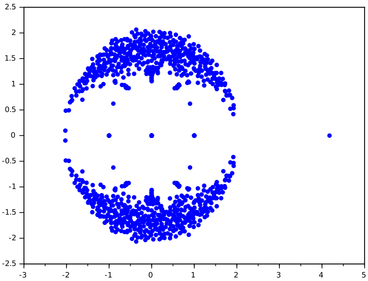
- ⋆ No lower bound on $|\lambda_2|$.

Final comments



- $\star$ No lower bound on $|\lambda_2|$.
- $\star$ Limit empirical distribution of eigenvalues ?

Final comments



- ⋆ No lower bound on $|\lambda_2|$.
- ⋆ Limit empirical distribution of eigenvalues ?
- ⋆ Without the homogeneous mean degree assumption ? (also open for random lifts of irregular graphs).

Final comments

⋆ A new proof of Friedman's Theorem ?

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- ⋆ In a general graph can we relate the condition $|\lambda_2| \leqslant \sqrt{\lambda_1} + o(1)$ to something geometric ?

Final comments

- ⋆ A new proof of Friedman's Theorem ?
- ⋆ In a general graph can we relate the condition $|\lambda_2| \leqslant \sqrt{\lambda_1} + o(1)$ to something geometric ?
- ⋆ Generally a good idea to study non-Hermitian local operators.

Thank you for your attention!

Near eigenvector

Proposition (Near eigenvector)
With high probability,

$$\langle \zeta, \check{\varphi} \rangle > c_0 \quad \text{and} \quad c_0 \alpha^\ell < \theta < c_1 \alpha^\ell .$$

It requires to prove convergence of expressions of the form

$$\alpha^{-2\ell} \langle \delta_e, B^{2\ell} B^{*\ell} \chi \rangle$$

toward a limit random variable.

Near eigenvector for SBM

For the stochastic block model, if ϕ_k is the left eigenvector of M with eigenvalue μ_k, we set,

$$\chi_k(e) = \phi_k(\sigma(e_2)).$$

If $|\mu_k| > \sqrt{\mu}_1$, the candidate eigenvector is ζ_k defined as

$$\varphi_k = \frac{B^\ell \chi_k}{\|B^\ell \chi_k\|}, \quad \theta_k = \|B^\ell \check{\varphi}_k\|, \quad \zeta_k = \frac{B^\ell B^{*\ell} \check{\chi}_k}{\|B^\ell B^{*\ell} \check{\chi}_k\|}.$$

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We now deal with a multi-type Galton-Watson tree, the condition $|\mu_k| > \sqrt{\mu}_1$, is Kesten-Stigum condition and after tedious computations, we find notably that for $k \neq j \in [r_0]$,

$$|\langle \zeta_j, \check{\varphi}_j \rangle| \geqslant c_0, \quad \langle \zeta_j, \check{\varphi}_k \rangle = o(1) \quad \text{and} \quad \langle \zeta_j, \zeta_k \rangle = o(1).$$

Kesten-Stigum Theorem (1966)

Consider the multi-type Galton-Watson process with mean progeny matrix M (+ finite second moment).

Let $Z_\ell \in \mathbb{N}^r$ is the population vector at generation ℓ,

If $|\mu_k| > \sqrt{\mu_1}$, then, for some centered M_k, a.s. and in L^2,

$$\frac{\langle Z_\ell, \phi_k \rangle}{\mu_k^\ell} - \langle Z_0, \phi_k \rangle \to M_k.$$

If $|\mu_k| < \sqrt{\mu_1}$, then, for some M_k, in L^2,

$$\frac{\langle Z_\ell, \phi_k \rangle}{\mu_1^{\ell/2}} \to M_k.$$