Quantum Ergodicity for sub-Riemannian Laplacians

(joint work in progress with Luc Hillairet and Emmanuel Trélat)

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1. What is Q(uantum)E(rgodicity)?

Let us consider a positive self-adjoint operator Δ with a compact resolvent on the Hilbert space $L^2(X, \mu)$ (with X a closed manifold). Let (ϕ_n, λ_n) be an eigen-decomposition of Δ and $N(\lambda) := \#\{\lambda_n \leq \lambda\}$.

We say that the eigen-basis $(\phi_n)_{n\in\mathbb{N}}$ satisfies QE if there exists

- A probability measure ν

- A density one sub-sequence (λ_{n_j}) of λ_n so that

$$\text{weak-}\lim_{j\to\infty} |\phi_{n_j}|^2 \mu = \nu \ .$$

Density 1 means that

$$\lim_{\lambda\to\infty} \frac{\#\{\lambda_{n_j} \leq \lambda\}}{N(\lambda)} = 1 \ .$$

Remark: the measure ν could be $\neq \mu$ and even singular w.r.t. to μ.

The measure ν comes from a *local Weyl formula*

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leq \lambda} \int_X |\phi_n|^2 f d\mu = \int f d\nu \ .$$

Existence of a local Weyl formula is often given, using the heat kernel $p_t(x, y)$, by looking at the asymptotics of the integrals $\int_X p_t(x, x) f(x) d\mu$ as $t \to 0^+$.

In fact, a *microlocal Weyl formula* giving the limit of the averages $\langle A\phi_n|\phi_n\rangle$ with $A = \mathrm{Op}(a)$ a ΨDO is needed in general:

$$\mathrm{Trace}_{\triangle}(\mathrm{Op}(a)) := \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n\le\lambda} \langle \mathrm{Op}(a)\phi_n|\phi_n\rangle = \int_{S^\star X} a d\tilde{\nu}$$

where the image of $\tilde{\nu}$ by the projection $\pi : S^\star X \to X$ is ν.

Equivalently, one needs to evaluate the behaviour, as $t \to 0^+$,

$$\int_{X\times X} p_t(x,y) k_A(x,y) d\mu_x d\mu_y \ .$$

For this we need to know $p_t(x,y)$ *near* the diagonal.

The historical example of QE is due to A. Shnirelman (74'): **if (X, g) is a closed Riemannian manifold whose geodesic flow is ergodic, QE holds for any eigen-basis of the Laplace-Beltrami operator with ν the normalized Riemannian volume.** This applies in particular if the curvature of (X, g) is < 0.

This result has been extended to many cases: manifolds with boundaries, discontinuous metrics, semi-classical Schrödinger operators, large regular graphs. To our knowledge, nothing was known before our work in the sR case.

Path to QE Theorems: One need to prove

- A *microlocal Weyl* formula

$$\mathrm{Trace}_{\Delta}(A) := \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} \langle A\phi_n | \phi_n \rangle = \int_{S^\star X} a d\tilde{\nu}$$

  with $A = \mathrm{Op}(a)$ and $\pi_\star \tilde{\nu} = \nu$ ($\pi : T^\star X \to X$ the canonical projection).

- A *vanishing Theorem for the variance*: if $\int_{T^\star X} a d\tilde{\nu} = 0$, then

$$\mathrm{Var}_{\Delta}(A) := \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} |\langle A\phi_n | \phi_n \rangle|^2 = 0 \ .$$

Usually the vanishing of the variance comes from ergodicity assumptions for a suitable dynamics preserving $\tilde{\nu}$.

2. Sub-Riemannian Laplacians

Given

- X a closed manifold with a smooth measure μ
- A smooth distribution $E \subset TX$ satisfying the bracket generating condition
- A smooth metric g on E,

we define a sR Laplacian $\Delta_{g,\mu}$ as the self-adjoint operator on $L^2(X, \mu)$ which is the Friedrichs extension of the closure of the quadratic form $D(f) := \int_X \|df\|_g^2 d\mu$.

Locally, if $(e_1, e_2, \cdots, e_l)$ is a local orthonormal frame for (E, g),

$$\Delta_{g,\mu} = D_1^\star D_1 + D_2^\star D_2 + \cdots$$

where the adjoint are taken w.r. to μ.

The principal symbol of $\Delta_{g,\mu}$ is the co-metric $g^\star$ defined by $g^\star(x,\xi) = \|\xi_{|E_x}\|_{g_x}^2$. The sub-principal symbol vanishes and all choices of μ give operators unitarily equivalent up to a bounded operator.

It follows from Hörmander's Theorem that $\Delta_{g,\mu}$ is sub-elliptic and has a compact resolvent, hence a discrete spectrum and a spectral decomposition (ϕ_n, λ_n). We are interested in micro-local Weyl formulae and QE Theorems for sR Laplacians $\Delta_{g,\mu}$.

3. Statement of the main result: a QE theorem in the 3D contact case

Let us start with a 3D closed manifold with a smooth measure μ and an oriented *contact* distribution, i.e. $E = \ker \alpha$ with $\alpha \wedge d\alpha$ non vanishing. Let us give a smooth sR metric g on E. There exists a unique contact form β so that $d\beta(e_1, e_2) = 1$ for any positive orthonormal frame (e_1, e_2) of E for g. Let us denote by Z the *Reeb vector field* of β (i.e. $\beta(Z) = 1$, $d\beta(Z, .) = 0$). Then the Popp measure $dP = |\beta \wedge d\beta|$ is $Z-$invariant. The Weyl formula reads

$$N(\lambda) \sim \frac{\int_X dP}{32} \lambda^2 \ .$$

Let ν the probability given by $\nu = \frac{1}{\int_X dP} dP$.

An Hamiltonian interpretation of Z: If Σ is the symplectic subcone of $T^\star X$ generated by α and $q : \Sigma \to \mathbb{R}$ is the positively homogeneous function with value 1 on $\pm\beta$, the Hamiltonian vector field $\mathcal{X}_q$ on Σ projects onto $\pm Z$.

Our result is as follows

Theorem 1 .– If the dynamics of the Reeb vector field Z is ergodic for the Popp volume, QE holds for any eigenbasis of the sR Laplacians $\Delta_{g,\mu}$ with the measure ν given by the normalized Popp volume.

Example: X is the unit cotangent bundle of a 2D closed Riemannian manifold (M, h), $E = \ker \alpha$ with α the Liouville form. Then we can choose g so that the Reeb vector field is the geodesic flow of (M, h).

Question: in this case, what is the link of our result and Shnirelman's Theorem?

Remark 1: The Reeb dynamics and the geodesics

All geodesics in $T^\star X$ with Cauchy data $(x_0, \xi_0 + \tau\alpha(x_0)) \in T^\star X$ with $\tau \in \mathbb{R}$ have the same Cauchy data in TX. As $\tau \to \pm\infty$, they spiral around the trajectories of $\pm Z$.

Remark 2: the Popp volume is the asymptotic Hopf invariant introduced by V. Arnold.

If $X = S^3$ and Z is a divergence free vector field, Arnold introduced an invariant measuring the average asymptotic linking number of two long trajectories of Z. In our case, this is exactly $1/\int_X dP$. Hence, Weyl formula shows that the Arnold invariant is a spectral invariant.

4a. Ideas of the proof: micro-local Weyl formula via heat expansions

Estimation of $\Delta-$traces will follow from estimating the asymptotic behaviour of traces

$$\text{Trace}\left(Ae^{-t\Delta}\right)$$

as $t \to 0^+$, where A is a ΨDO of order 0. This needs to know the asymptotic behaviour of the heat kernel not only on the diagonal, but near the diagonal. The needed estimates can be found in the work of Davide Barilari: he gives a good approximation of the rescaled heat kernel by the Heisenberg kernel, known at least since a paper of B. Gaveau.

The result is as follows, if $A = \mathrm{Op}(a)$ with a homogeneous of degree 0:

$$\mathrm{Trace}\left(Ae^{-t\Delta}\right) \sim_{t\to 0^+} \frac{1}{8\pi^2 t^2}\int_\Sigma \frac{q}{\sinh q} a dL \ ,$$

with L the Liouville measure on Σ.

It follows that

$$\mathrm{Trace}_\Delta(A) = \frac{1}{2}\int_X \left(a(x,\alpha(x)) + a(x,-\alpha(x))\right) d\nu \ .$$

In average, the eigenfunctions microlocally concentrate on the contact cone Σ.

4b. Ideas of the proof: normal forms and Reeb dynamics

The idea of normal forms in dynamical systems goes back at least to Birkhoff. In our context, a normal form was proposed by Richard Melrose in 1984. This normal form is a formal normal form along Σ: this is enough in view of the microlocal Weyl formula.

This normal form on the level of classical mechanics is as follows: it is given by the symplectic cone $\Sigma \times \mathbb{R}^2$ with the identification of $\Sigma \subset T^\star X$ to $\Sigma \times 0$. The geodesic Hamiltonian takes the form

$$g^\star = q\omega + O_\Sigma(\infty)$$

where ω is an harmonic oscillator.

A corresponding ΨDO normal form takes place reducing along Σ the Laplace operator to something like $\Delta \sim Q\Omega$ with Ω a quantum harmonic oscillator and $[Q, \Omega] = 0$.

Let us see how the proof works assuming $\Delta = Q\Omega$. We need to show that the variance $V(f)$ vanishes as soon as $\int_X f dP = 0$.

The proof follows the Shnirelman's proof replacing Δ by Q.

Using the microlocal Weyl formula, we can replace f by a $\Psi DO \quad A$ commuting with Ω. Let us denote by $A_t := e^{-itQ}Ae^{itQ}$; the principal symbol of A_t is $a \circ \Phi_t$ where Φ_t is the flow of $\mathcal{X}_q$ which is ergodic on each level set $q = cte$. By decomposing ϕ_n into eigenfunctions of Ω, one sees that

$$\langle A_t \phi_n | \phi_n \rangle = \langle A \phi_n | \phi_n \rangle = \langle A_T \phi_n | \phi_n \rangle$$

with $A_T := \frac{1}{T} \int_0^T A_t dt$.

Hence $\mathsf{Var}_\Delta(A - A_T) = 0$ and

$$\mathsf{Var}_\Delta(A) = \mathsf{Var}_\Delta(A_T) \leq \mathsf{Trace}_\Delta(A_T^\star A_T) \ .$$

The last expression is given by the integral on $\Sigma \cap \{q = 1\}$ of a_T^2. This integral is going to 0 as $T \to \infty$ thanks to the ergodicity and the von Neumann ergodic Theorem.

5. Other sR geometries. There is a large zoo of sR distributions. Important facts are

- The presence of singularities in the horizontal distribution: the simplest case is the Martinet case.

- The possible presence of anormal geodesics

- The possibility of resonances in the normal forms.

Let us discuss the Martinet case:

We consider the 3D case where $E = \ker \alpha$ with $\alpha \wedge d\alpha$ vanishing with a non zero differential on a sub-manifold $Y \subset X$ (locally $\alpha = dx - z^2 dy$). On $X \setminus Y$, we have a contact sR metric and a Popp volume dP. Locally, if Y is defined by $F = 0$, we have $dP = \nu \otimes d \log |F| + 0(1)$ where ν is a measure on Y. Our conjecture is the following local Weyl formula

$$\sum_{\lambda_n \leq \lambda} \int f \phi_n^2 d\mu \sim \frac{\int_Y f d\nu}{32} \lambda^2 \log \lambda \ .$$

This implies that, in average, the eigenfunctions concentrate on Y. Concentration of a density 1 sub-sequence would follow from a QE Theorem (to be found).

We can prove our conjecture in some “separable cases” and also prove a weaker form using some heat kernel estimates due to A. Sanchez-Calle.

This would give another form of the results of R. Montgomery in the nice paper “Hearing the zero locus of a magnetic field”.

What remains to do?

The general case is certainly difficult to handle. Even a local Weyl formula is not available at the moment. In the regular case, approach by heat kernels will probably give the answer in a more or less explicit way. This will work for example in the contact case or in the Engel case.

In the non regular case, the desingularization proposed by Rotschild-Stein could be used.

Concerning the dynamics leading to QE, the situation is less clear, even in the 5D contact case. In general, the singular geodesics could enter into the game!

Any help will be welcome!