Orthogonal polynomials on infinite gap sets

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Centre for Mathematical Sciences, Lund University

Spectral theory and its applications

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Outline

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- **Introduction**

  — Jacobi matrices and orthogonal polynomials

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  — of Parreau–Widom type

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  — Remling's theorem

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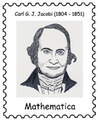
  — Remling's theorem

- **Szegő class theory**

  — and discussion of two conjectures



Jacobi operators

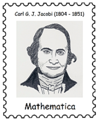
Suppose that J is a bounded selfadjoint operator on $\ell^2(\mathbb{N})$. If J has a cyclic vector ψ, that is,

$$\{J^n \psi\}_{n=0}^{\infty} \text{ is dense in } \ell^2(\mathbb{N}),$$

then there is an appropriate basis such that J is represented by a matrix of the form

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with real entries in the diagonal and positive entries above/below.



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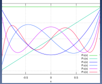
Moreover, there is a probability measure $d\mu$ on $\sigma(J)$ so that J is unitarily equivalent to the operator of multiplication by the identity function in the Hilbert space $L^2(\mathbb{R}, d\mu)$.



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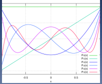

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The polynomials $\{P_n\}_{n\geq 0}$ generated by the three-term recurrence relation

$$P_0(x) = 1, \quad a_1 P_1(x) = x - b_1,$$

$$xP_n(x) = a_{n+1} P_{n+1}(x) + b_{n+1} P_n(x) + a_n P_{n-1}(x), \quad n > 1$$

are orthonormal with respect to the measure $d\mu$, that is,

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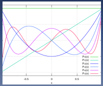

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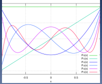
Fact: There is a one-one correspondence between bounded Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ and nontrivial probability measures $d\mu$ on $\mathbb{R}$ with compact support.



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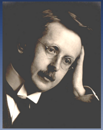
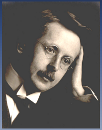
In spectral theory, one seeks to relate properties of the Jacobi parameters to properties of the measure of orthogonality, and vice versa.



The Weyl m-function

A key role is played by the m-function defined by

$$m(x) := m_\mu(x) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-x}, \quad x \in \mathbb{C} \setminus \operatorname{supp}(d\mu).$$



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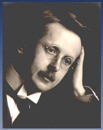
$$m(x) = -1/x + \mathcal{O}(x^{-2}) \quad \text{near } \infty.$$



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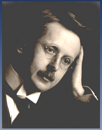

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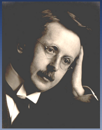

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Hence, isolated mass points of $d\mu$ are poles of the m-function.

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Almost periodic parameters tend to produce Cantor spectrum.

In this talk, we focus on inverse spectral theory for a certain class of infinite gap sets.

Infinite gap sets

In this talk, we consider infinite gap sets of the form

$$\mathsf{E} = [\alpha, \beta] \setminus \bigcup_j (\alpha_j, \beta_j),$$

where $\bigcup_j$ is a countable union of disjoint open subintervals.

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[Remove the middle ε_1 part of $[0,1]$, the middle ε_2 part of the two remaining intervals, etc]

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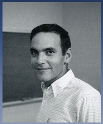
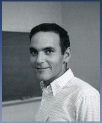
By definition, this means there is an $\varepsilon > 0$ so that

$$|(t - \delta, t + \delta) \cap \mathrm{E}| \geq \delta\varepsilon \text{ for all } t \in \mathrm{E} \text{ and all } \delta < \mathrm{diam}(\mathrm{E}).$$



Parreau–Widom sets

A more general class of infinite gap sets is defined as follows.



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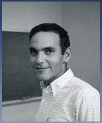
Here, $d\mu_{\mathsf{E}}$ is the equilibrium measure of E and Cap denotes the logarithmic capacity.



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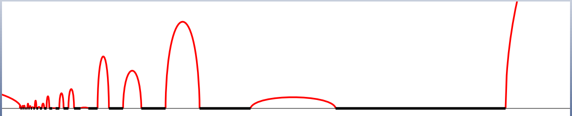
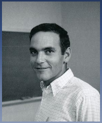
While g vanishes on E, it is concave on (α_j, β_j) for each j. So there is precisely one critical point c_j per gap in E.





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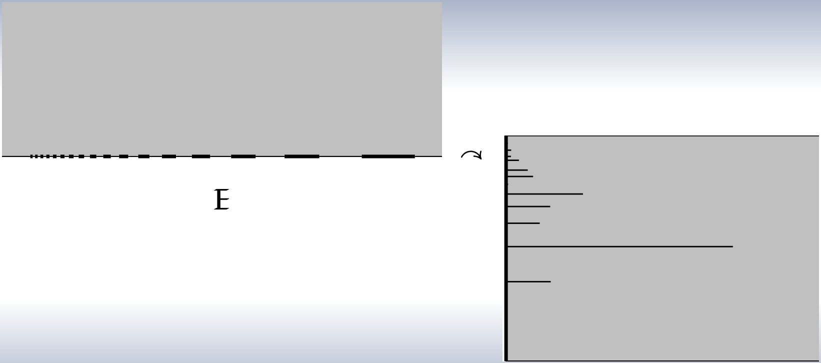
Defn. We say that E is a *Parreau–Widom set* if

$$\textstyle\sum_j g(c_j) < \infty$$

Comb-like domains



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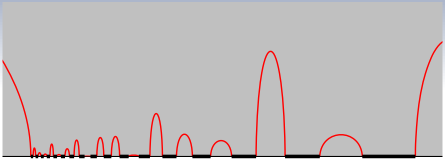

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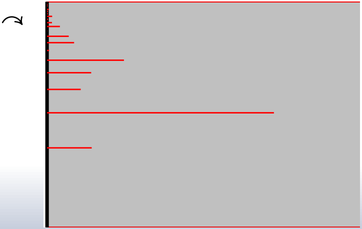

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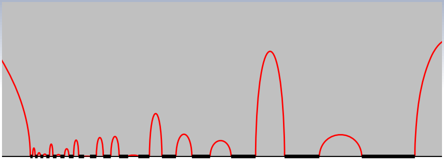

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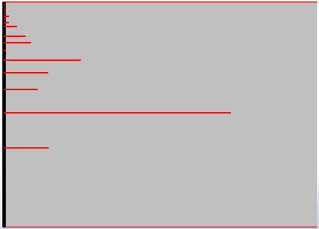

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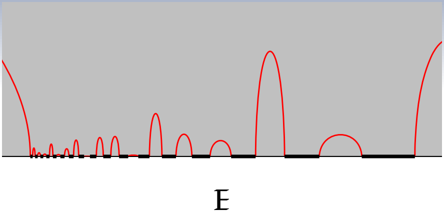

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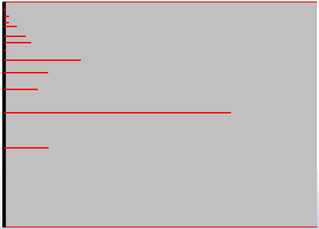

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This is always the case when $\sum_j g(c_j) < \infty$ (i.e., E is PW).

The isospectral torus

We denote by $\mathcal{T}_{\mathrm{E}}$ the set of all two-sided matrices $J' = \{a'_n, b'_n\}_{n=-\infty}^{\infty}$ that are *reflectionless* on E and for which $\sigma(J') = \mathrm{E}$.

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$$J' = \left(\begin{array}{ccc|ccc} \ddots & \ddots & & & & \\ \ddots & b'_{n-1} & a'_{n-1} & & & \\ & a'_{n-1} & b'_n & a'_n & & \\ \hline & & a'_n & b'_{n+1} & a'_{n+1} & \\ & & & a'_{n+1} & b'_{n+2} & \ddots \\ & & & & \ddots & \ddots \end{array} \right)$$

Equivalently,

$$(a'_n)^2 m_n^+(t + i0)\, \overline{m_n^-(t + i0)} = 1 \quad \text{for a.e. } t \in \mathsf{E} \text{ and all } n,$$

where m_n^+ is the m-function for $J_n^+ = \{a'_{n+k}, b'_{n+k}\}_{k=1}^{\infty}$ and m_n^- the m-function for $J_n^- = \{a'_{n-k}, b'_{n+1-k}\}_{k=1}^{\infty}$.

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Hence, $\mathcal{T}_{\mathrm{E}}$ is the natural limiting object associated with E.

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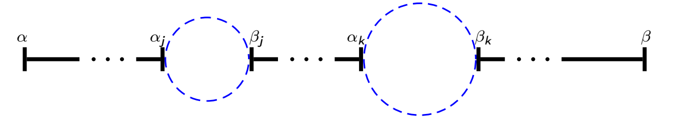

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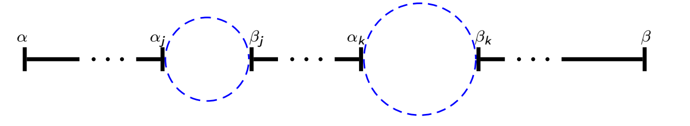

We shall equip $\mathcal{D}_{\mathrm{E}}$ with the product topology.

A map $\mathcal{T}_{\mathrm{E}} \to \mathcal{D}_{\mathrm{E}}$

When $J' \in \mathcal{T}_{\mathrm{E}}$, we know that $G(x) := \left\langle \delta_0, (J' - x)^{-1}\delta_0 \right\rangle$ is analytic on $\mathbb{C} \smallsetminus \mathrm{E}$ and has purely imaginary boundary values a.e. on E.

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As m^+ and $1/m^-$ have no common poles, this in turn allows us to define a map $\mathcal{T}_{\mathrm{E}} \to \mathcal{D}_{\mathrm{E}}$.

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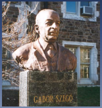
Q: What can we say about a_n and b_n when $J \in \mathrm{Sz}(\mathsf{E})$?



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When J belongs to the Szegő class for E, we always have

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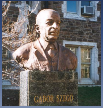

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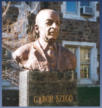

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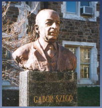

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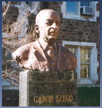
But can't we say more about a_n and what about b_n?



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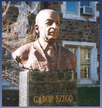

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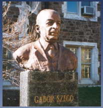

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Consequently, a_n and b_n are asymptotically almost periodic.

Moreover, if $d\mu'$ is the spectral measure of J' restricted to $\ell^2(\mathbb{N})$, then

$$P_n(x, d\mu) \,/\, P_n(x, d\mu')$$

has a limit for all $x \in \overline{\mathbb{C}} \setminus \mathbb{R}$.

Hence, $\prod(a_n/a_n')$ and $\sum(b_n - b_n')$ converge conditionally.

Two conjectures

Our goal is to prove the following two conjectures:

<u>Conj. 1</u> If $\sum |a_n - a_n'| + |b_n - b_n'| < \infty$ for some $J' \in \mathcal{T}_{\mathsf{E}}$, then J belongs to $\mathrm{Sz}(\mathsf{E})$.

<u>Conj. 2</u> If J lies in $\mathrm{Sz}(\mathsf{E})$, then $\sum (a_n - a_n')^2 + (b_n - b_n')^2 < \infty$ for a unique $J' \in \mathcal{T}_{\mathsf{E}}$.

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In this talk, I shall merely focus on the first conjecture.

The critical bound

Let $\{x_k\}$ be the eigenvalues in $\mathbb{R} \setminus \mathsf{E}$ of some $J \in \mathrm{Sz}(\mathsf{E})$.

If we can prove that

$$\sum_k g(x_k) \leq C_1 + C_2 \sum |a_n - a_n'| + |b_n - b_n'|,$$

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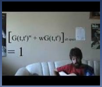
Unfortunately, we don't know how to get this critical bound (which was obtained by Frank–Simon for finite gap sets).



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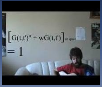

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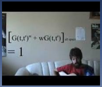

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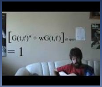

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There seems to be at least two possibilities for generalization:

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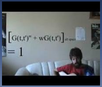

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The first step is to control the (Dirichlet) Green's function

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For finite gap sets, the appropriate estimate is

$$|G_{nn}(\lambda)| \le C\,|\lambda - \alpha_j|^{-1/2}, \quad \alpha_j < \lambda \le c_j < \beta_j.$$

There seems to be at least two possibilities for generalization:

$$(1)\ |G_{nn}(\lambda)| \le C/g(\lambda), \quad (2)\ |G_{nn}(\lambda)| \le C\,g'(\lambda).$$

Assume for now that we can prove the latter (to be discussed a little later).

Lieb-Thirring bounds

For $p > 0$, define

$$f(x) := \int_{\alpha_j}^{x} g(\lambda)^{p-1} d\lambda \text{ for } \alpha_j \leq x \leq c_j < \beta_j.$$

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As the conditions on J lead to bounds on the number of eigenvalues, we deduce that

$$\int_{\alpha_j}^{c_j} f'(\lambda) g'(\lambda) d\lambda < \infty \implies \sum_{x_k \in (\alpha_j, c_j)} f(x_k) < \infty.$$

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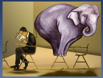
This works for *all* gaps (also at the right ends) and

$$\int f' g' = \int g^{p-1} g' = \sum_j \int_0^{h_j} x^{p-1} dx = \sum_j {h_j}^p, \quad h_j = g(c_j).$$



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When E is homogeneous, we have $g \in \mathrm{Lip}\,\gamma$ for some $\gamma \leq 1/2$.

But even in the simple case $\alpha_j = 1/2^j$ and $\beta_j = \alpha_j + 1/2^{j+1}$, one can show that $\gamma < 1/2$.

Estimating G_{nn}

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This in turn is equivalent to boundedness of the outer function

$$\exp\left\{ \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(\frac{f^*(\mathbf{x}(e^{i\theta}))}{f_{J^+}(\mathbf{x}(e^{i\theta}))} \right) \frac{d\theta}{4\pi} \right\}, \quad z \in \mathbb{D}$$

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where f^* is the a.c. part of a suitable reference measure.

The bound should be independent of f_{J^+} (as J' varies on $\mathcal{T}_{\mathsf{E}}$).

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We believe this is true for fat Cantor sets, but do not have a rigorous proof.

However, it seems to fail for the simple homogeneous set given by $\alpha_j = 1/2^j$ and $\beta_j = \alpha_j + 1/2^{j+1}$.

Merci beaucoup pour votre attention!

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