Scattering theory of the Hodge-Laplacian

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Spectral theory and its applications

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This talk is about the paper:

Francesco Bei & Batu Güneysu & Jörn Müller: *Scattering theory of the Hodge-Laplacian under a conformal perturbation*. Preprint (2014), available from arxiv.

- Let M be an arbitrary smooth m-manifold
- One of the most fundamental problems in geometry is the determination of the spectrum of the Laplace operator corresponding to a Riemannian metric g on M, in particular the one of the Hodge-Laplace operator $\Delta_g^{(j)}$ which acts on differential j-forms
- If M is compact, then the spectrum $\sigma(\Delta_g^{(j)})$ of $\Delta_g^{(j)}$ consists of eigenvalues with a finite multiplicity and thus the situation is (analytically) very simple
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However, there is a "perturbative" way to control the *absolutely continuous* part $\sigma_{\mathrm{ac}}(\Delta_g^{(j)})$ of $\sigma(\Delta_g^{(j)})$ in the noncompact case:

Assume that there is a quasi-isometric metric $\tilde{g}$ on M such that we have some good information about the absolutely continuous part $(\Delta_{\tilde{g}}^{(j)})_{\mathrm{ac}}$ of $\Delta_{\tilde{g}}^{(j)}$. Then once we can show that the wave operators $W_{\pm}(\Delta_g^{(j)}, \Delta_{\tilde{g}}^{(j)}, I)$ exist and are complete, they induce unitary equivalences

$$(\Delta_{\tilde{g}}^{(j)})_{\mathrm{ac}} \sim (\Delta_g^{(j)})_{\mathrm{ac}}, \text{ so that } \sigma_{\mathrm{ac}}(\Delta_{\tilde{g}}^{(j)}) = \sigma_{\mathrm{ac}}(\Delta_g^{(j)}).$$

Here $I = I_{g,\tilde{g}} : \Omega(M, g) \to \Omega(M, \tilde{g})$ is the canonical identification $\alpha \mapsto \alpha$.

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Here $I = I_{g,\tilde{g}} : \Omega(M,g) \to \Omega(M,\tilde{g})$ is the canonical identification $\alpha \mapsto \alpha$.

The question we address here is:

In what sense do $\tilde{g}$ and g have to be close to each other to ensure that $W_{\pm}\big(\Delta_g, \Delta_{\tilde{g}}, I\big)$ exist and are complete?

From calculating $\Delta_{\tilde{g}} - \Delta_g$ in the (particulary important) case where one metric arises a conformal perturbation of the other, we expect the correct assumption to be a *first order* control in the deviation of g and $\tilde{g}$.

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- no systematic treatment (if at all) for k-forms in the literature
- For functions= 0-forms: Classically, people have considered special topologies $M = (0, \infty) \times \mathbb{S}^{m-1}$ with warped metrics. Then the problem is typically unitarily equivalent to a scattering problem for Sturm-Liouville operators, which is a rather old (but not necessarily easy) story
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- The first *entirely global* result for functions has been established by W. Müller/G. Salomonsen. Their result essentially reads as follows:

Theorem (Müller/Salomonsen 2007, JFA 253)

Let g, $\tilde{g}$ be complete metrics on M with $|\mathrm{sec}_g|, |\mathrm{sec}_{\tilde{g}}| \leq L$, such that the covariant C^2-deviation $x \mapsto {}^2|g - \tilde{g}|_g(x)$ of g from $\tilde{g}$ is bounded from above by some $\beta : M \to (0, \infty)$ of moderate decay, in a way such that for appropriate constants a, b, c, C one has

$$\beta^a \in \mathsf{L}^1(M, g), \ \left|\beta^b(x)\widetilde{\mathrm{inj}}_g(x)^c\right| \leq C \ \text{ for all } x,$$

where $\widetilde{\mathrm{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \ \mathrm{inj}_g(x)\right)$. *Then* $W_\pm\left(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)}\right)$ *exist and are complete.*

Although a breakthrough at that time, this result is certainly not optimal: The required control is of *second order* in the deviation of g and $\tilde{g}$.

Indeed, using the harmonic radius function (later...) $x \mapsto r_g(x)$ with a certain Sobolev control, one can do much better:

Theorem (Hempel/Weder/Post 2013, JFA 266)

Let g, $\tilde{g}$ be complete quasi-isometric metrics on M with

$$\int_M \mathrm{d}(g,\tilde{g})(x) h^{-(m+2)}(x) \mu_g(\mathrm{d}x) < \infty, \tag{1}$$

where $\mathrm{d}(g,\tilde{g}) : M \to (0,\infty)$ is a certain function (later...) which measures a zeroth order deviation of the metrics, and where $h : M \to (0,1]$ is an arbitrary lower bound on

$$M \ni x \longmapsto \max\big(\min(r_g(x),1), \min(r_{\tilde{g}}(x),1)\big) \in (0,1].$$

Then $W_\pm\big(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)}\big)$ exist and are complete.

This *zeroth order* result should be the state of the art for *functions*.

We were interested in extending the latter result to differential forms.

Here, for some entirely algebraic reasons, we have restricted ourselves to conformal perturbations.

Given a Riemannian metric g we have

- ∇_g: the Levi-Civita connection
- $\mathrm{Q}_g : \wedge^2 \mathrm{T}M \to \wedge^2 \mathrm{T}M$ the s.a. curvature endomorphism
- for a smooth 1-form α, $\mathrm{int}_g(\alpha) = \mathrm{ext}^{\dagger_g} : \wedge \mathrm{T}^* M \to \wedge \mathrm{T}^* M$ is interior multiplication with α
- the codifferential $\delta_g := \mathrm{d}^{\dagger_g} : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Dirac type operator $D_g := \mathrm{d} + \delta_g : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Hodge-Laplacian $\Delta_g := D_g^2 : \Omega_{\mathsf{C}^\infty}(M) \to \Omega_{\mathsf{C}^\infty}(M)$
- the Friedrichs realization H_g of Δ_g in $\Omega_{\mathsf{L}^2}(M, g)$
- the resolvents $R_{g,\lambda} := (H_g + \lambda)^{-1}$, $\lambda > 0$.
- everything filters through the form degree; notation: $\Omega_{\mathsf{L}^2}(M) = \bigoplus_{j=0}^m \Omega^{(j)}_{\mathsf{L}^2}(M, g)$, $H_g = \bigoplus_{j=0}^m H_g^{(j)}$ etc.

Our main technical tool will be harmonic coordinates with Sobolev control:

Definition (Cheeger/Anderson)

Let $p \in (m, \infty)$, $q \in (1, \infty)$, $x \in M$. Then the *$\mathrm{W}^{1,p}_g$-harmonic radius at x with Euclidean distortion q*, $r_g(x, p, q) \in (0, \infty]$, is defined to be supremum of all $r > 0$ such that there is a $\Delta^{(0)}_g$-harmonic chart $\Phi : \mathrm{B}_g(x, r) \to U \subset \mathbb{R}^m$ which, with respect to the Φ-coordinates, satisfies the estimates

$$q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq q(\delta_{ij}) \text{ as symmetric bilinear forms,} \tag{2a}$$

$$r^{1-\frac{m}{p}} \left(\int_U |\partial_k g_{ij}(y)|^p \mathrm{d}y \right)^{1/p} \leq q - 1 \text{ for all } i, j, k \in \{1, \dots, m\}. \tag{2b}$$

It is not obvious at all that $r_g(x, p, q) > 0$. Anyway, one has the following elementary fact:

Proposition (B/G/M)

For any fixed p, q, the function $x \mapsto \min(1, r_g(x, p, q))$ is 1-Lipschitz w.r.t. g.

The fact that indeed $r_g(x, p, q) > 0$ as claimed in the definition follows from applying the following results near x:

Proposition (Cheeger/Anderson 90%; B/G/M)

Assume that $\mathrm{Ric}_g(x) \geq -\frac{1}{\beta^2}$ *and* $\mathrm{inj}_g(x) \geq \tilde{h}(x)$, *where* $\beta > 0$ *is a constant and* $\tilde{h} : M \to (0, \infty)$ *is a continuous. Then:*

a) *If* $\tilde{h}$ *is* g*-Lipschitz, then for any* p, q *there is* $C = C(m,p,q) > 0$ *such that for all* $x \in M$ *one has*

$$\min(r_g(x,p,q), 1) \geq C \min\left(1, \frac{\tilde{h}(x)}{1 + \|\mathrm{d}\tilde{h}\|_{\infty,g}}, \beta\right).$$

b) *If there is a* $x_0 \in M$, *and* $c_1 > 0$, $c_2 \geq 0$ *such that* $\tilde{h} \geq c_1 \mathrm{e}^{-c_2 \mathrm{d}_g(\cdot, x_0)}$, *then for any* p, q *there is* $C = C(m,p,q) > 0$ *such that for all* $x \in M$ *one has*

$$\min(r_g(x,p,q), 1) \geq C \min\left(1, \frac{c_1}{\mathrm{e}^{c_2}} \mathrm{e}^{-c_2 \mathrm{d}_g(x,x_0)}, \beta\right).$$

The importance of Sobolev harmonic coordinates: By embedding theorems, we get a Hölder control on g_{ij}. To make an effective use of this observation in the form-case, we add:

Definition

For any $K > 0$ and any function $h : M \to (0, 1]$, let

$$\mathscr{M}_{K,h}(M) := \Big\{ \tilde{g} \;\Big|\; \tilde{g} \textit{ is a complete metric on } M \textit{ with } \mathrm{Q}_{\tilde{g}} \geq -K \textit{ and } \min(1, r_g(\cdot, p, q)) \geq h \textit{ for some } p \in (m, \infty), q \in (1, \sqrt{2}) \Big\}.$$

Carleman type resolvent estimate:

Theorem (B/G/M)

Assume that $g \in \mathscr{M}_{K,h}(M)$ for some pair (K,h). Then for all $n \in \mathbb{N}$ with $n \geq m/4+2$ there is a $C = C(m,n) > 0$, such that for all $\lambda > K \max_{j=0,\dots,m} j(m-j)+1$, the operator $R^n_{g,\lambda}$ is an integral operator, with a Borel integral kernel

$$M \times M \ni (x,y) \longmapsto R^n_{g,\lambda}(x,y) \in \mathrm{Hom}\left(\wedge \mathrm{T}^*_y M, \wedge \mathrm{T}^*_x M\right)$$

which satisfies the estimate

$$\int_M \left| R^n_{g,\lambda}(x,y) \right|^2_{\mathscr{J}^2} \mu_g(\mathrm{d}y) \leq C h(x)^{-m} \text{ for all } x \in M.$$

The proof is rather complicated. The key observations are:

- $V_g^{(j)} := \Delta_g^{(j)} - \nabla_{g,j}^{\dagger}\nabla_{g,j}$ is zeroth order and s.a. by Weitzenböck's formula
- The Gallot-Meyer estimate states that under $\mathrm{Q}_g \geq -K$ one has $V_g^{(j)} \geq -K \cdot j(m-j)$
- Now one can use probabilistic domination results for covariant Schrödinger semigroups $\mathrm{e}^{-t(\nabla^{\dagger}\nabla+V)}$ (e.g. my paper in JFA 262) to control $R_{g,\lambda}^{(j),n}$ by $R_{g,1}^{(0),n}$. The latter can be controlled by $\min(1, r_g(\cdot,p,q))$

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- Given a smooth function $\psi : M \to \mathbb{R}$ let g_ψ denote the conformally equivalent metric $g_\psi := \mathrm{e}^{2\psi} g$. Then g and g_ψ are quasi-isometric if and only if ψ is bounded and then we have the canonical identification operator $I = I_{g,g_\psi} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,g_\psi)$.
- Given a Borel function $h : M \to (0,\infty)$ and a smooth function $\psi : M \to \mathbb{R}$ define

$$\mathrm{d}(g,\psi)(x) := \max\big\{\sinh(2|\psi(x)|), |\mathrm{d}\psi(x)|_g\big\}, \quad x \in M,$$

$$\mathrm{d}_h(g,\psi) := \int_M \mathrm{d}(g,\psi)(x) h(x)^{-(m+2)}\, \mu_g(\mathrm{d}x) \quad \in [0,\infty].$$

We call ψ a *h-scattering perturbation of g*, if one has $\mathrm{d}_h(g,\psi) < \infty$.

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We call ψ a *h-scattering perturbation of* g, if one has $\mathrm{d}_h(g,\psi) < \infty$.

Now we can formulate our main result for forms:

Theorem (B/G/M)

Let $\psi : M \to \mathbb{R}$ be smooth with ψ, $|\mathrm{d}\psi|_g$ bounded, and assume that $g, g_\psi \in \mathscr{M}_{K,h}(M)$ for some pair (K, h), in a way such that ψ is a h-scattering perturbation of g. Then the wave operators

$$W_\pm(H_{g_\psi}, H_g, I) = \underset{t\to\pm\infty}{s\text{–}\lim}\, \mathrm{e}^{\mathrm{i}tH_{g_\psi}} I \mathrm{e}^{-\mathrm{i}tH_g} P_{\mathrm{ac}}(H_g)$$

exist and are complete, and everything filters (a posteriori...$\rightsquigarrow$ total forms and Dirac type operators!) through the form degree.

Corollary

Assume that g is complete with $\mathrm{Q}_g \geq -K$ for some $K > 0$ and that $\tilde{g}$ is a metric on M which is conformally equivalent to g and which coincides with g at infinity. Then the assumptions of our main result are satisfied.

Indeed, since ψ is compactly supported by assumption, we can take

$$h(x) := \min(1, r_g(x,p,q), r_{g_\psi}(x,p,q)) \text{ for all } p > m,\ 1 < q < \sqrt{2},$$

which is a positive continuous function, to make ψ a h-scattering perturbation of g.

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Corollary

Assume that $\psi : M \to \mathbb{R}$ is smooth and bounded, g is complete such that $|\mathrm{sec}_g|, |\mathrm{sec}_{g_\psi}| \leq L$ for some $L > 0$, that there is some $\beta : [0,\infty) \to (0,\infty)$ exponentially bounded from below, and a point $x_0 \in M$ such that with $\beta(x) := \beta(1 + \mathrm{d}_g(x, x_0))$ one has:

(i) *There are constants $b \in (0,1)$ with $\beta^b \in \mathsf{L}^1(M,g)$, and $C_1 > 0$ such that for all $x \in M$,*

$$\widetilde{\mathrm{inj}}_g(x) := \min\Big(\frac{\pi}{12\sqrt{L}}, \mathrm{inj}_g(x)\Big) \geq C_1 \cdot \beta(x)^{\frac{1-b}{m+2}}. \tag{3}$$

(ii) *For some constant $C > 0$ one has*

$$^1|g - g_\psi| := |g - g_\psi|_g + |\nabla_g - \nabla_{g_\psi}|_g \leq C \cdot \beta \tag{4}$$

Then the assumptions of our main result are satisfied.

The latter result can be considered as a generalization in the conformal case of the initial Müller/Salomonsen result to forms. Note however that, being a *first order result*, it is better even on functions.

Some steps in the proof of our main result...
An essential tool is to use a decomposition formula (the algebra of which forced us to restrict ourselves to the conformal case) efficiently with harmonic coordinates:

Proposition

Assume that ψ and $|\mathrm{d}\psi|_g$ are bounded, let $\lambda > 0$, $n \geq 1$ and let g (and thus g_ψ) be complete. Then the bounded operator

$$R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda} : \Omega_{\mathsf{L}^2}(M, g) \longrightarrow \Omega_{\mathsf{L}^2}(M, g_\psi)$$

can be decomposed as

$$\begin{aligned} &R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda} = \\ &R^n_{g_\psi,\lambda}\Big(D_{g_\psi} \cdot 2\sinh(2\psi) I D_g + D_{g_\psi} I (1 - \mathrm{e}^{-2\psi})\mathrm{d} - \mathrm{d} \circ (1 - \mathrm{e}^{2\psi}) I D_g \\ &\qquad + D_{g_\psi} \mathrm{int}_{g_\psi}(\mathrm{d}\psi)\, \tau\, I - \tau\, \mathrm{int}_g(\mathrm{d}\psi) D_g\Big) R^n_{g,\lambda}. \quad (5) \end{aligned}$$

Here $\tau := \bigoplus_{j=0}^m (m - 2j) 1_{\wedge^j \mathrm{T}^* M} : \wedge \mathrm{T}^* M \longrightarrow \wedge \mathrm{T}^* M$.

- Now we combine the latter decomposition formula with our Carleman type resolvent estimate and the commutator relations $[A, R^n_{g,\lambda}] = 0$, where $A \in \{D_g, \mathrm{d}, \delta_g\}$, to get that $R^n_{g_\psi,\lambda}(H_{g_\psi} I - I H_g) R^n_{g,\lambda}$ is trace class for large $n \rightsquigarrow$ the assumptions of Belopol'skii-Birman's theorem are satisfied
- The decomposition formula heavily requires that the underlying Hamiltonians are of the form $L^* L$. That is why we work with *total forms and the Dirac type operator* D_g and $H_g = D_g^2 = D_g^* D_g$ instead of on a fixed form degree. On functions, all of this is very simple as $\Delta^{(0)} = \mathrm{d}^{\dagger_g} \mathrm{d}$ where the differential d does not depend on the metric (and this leads to a zeroth order condition in this case).

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- I am sure that one can drop our curvature assumption $\mathrm{Q}_g \geq -K$
- I am almost sure that one can drop the restriction to conformal perturbations. But you have to solve the following entirely algebraic problem: Give yourself two metrics g and $\tilde{g}$ on M. You can always write the one as a multiplicative perturbation of the other. But: How do you calculate $\delta_{\tilde{g}} = F_{\tilde{g},g}(\delta_g)$? In the conformal case, there are somewhat accessible perturbative formulae.
- Further studies for functions: Local Dirichlet forms (not clear at all)? Weighted infinite graphs (this should at least admit a clear formulation in terms of the edge and vertex weight functions)?

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Thank you for listening!