Davies Lemma on Graphs

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Outline

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Davies lemma on manifolds

Lemma (Davies-Gaffney-Grigor'yan)

Let M be a complete Riemannian manifold, then for any measurable subsets $U_1, U_2 \subset M$, $t > 0$,

$$\int_{U_1}\int_{U_2} p_t(x,y)d\mathrm{vol}(x)d\mathrm{vol}(y) \leq \sqrt{\mathrm{vol}(U_1)\mathrm{vol}(U_2)}\exp(-\Lambda t) \times \exp\left(-\frac{d^2(U_1,U_2)}{4t}\right),$$

where Λ is the bottom of the L^2 spectrum of Laplacian.

Coulhon-Sikora

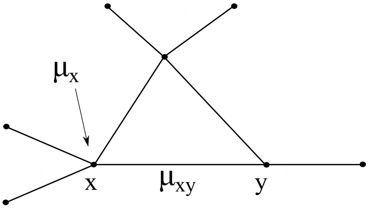
On any metric measure space, Coulhon-Sikora proved that the following are equivalent:

- The solutions to the wave equation have finite propagation speed.
  $(u_{tt} - \Delta u = 0)$
- The classical Davies Lemma holds.
  $(u_t - \Delta u = 0)$

Setting

Let $G = (V, E)$ be a graph with the vertex set V and edge set E.

- The triple (V, E, μ) is called a weighted graph where $\mu : E \to \mathbb{R}_+, E \ni (x, y) \mapsto \mu_{xy}$ is a weight function on E.
- This induces a (degree) weight on V, $\mu : V \to \mathbb{R}_+, x \mapsto \mu_x = \sum_{y \sim x} \mu_{xy}$.



Definition (Laplacian on graphs)

$$\begin{aligned}\Delta f(x) &:= \frac{1}{\mu_x}\sum_{y\sim x}\mu_{xy}(f(y)-f(x)) \\ &= \frac{1}{\mu_x}\sum_{y\sim x}\mu_{xy}f(y)-f(x).\end{aligned}$$

Thinking of infinite matrices, one can rewrite it as

$$\Delta = P - Id,$$

where $P_{xy} = \frac{\mu_{xy}}{\mu_x}$, $\sum_{y\in V} P_{xy} = 1$, which corresponds to the transition matrix for the simple random walk on G.

Couterexample by Friedman-Tillich

- Friedman-Tillich '04 showed that the finite propagation speed property of wave equations fails on graphs.
- Proof. Fix $x_0 \in V$, for any $x \in V$, set $d(x, x_0) = n$, then

$$\begin{aligned}
& \cos(t\sqrt{-\Delta})\delta_{x_0}(x) \\
= \;& \delta_{x_0}(x) + \frac{t^2}{2}\Delta\delta_{x_0}(x) + \cdots + \frac{t^{2k}}{(2k)!}\Delta^k\delta_{x_0}(x) + o(t^{2k}) \\
= \;& \frac{t^{2n}}{(2n)!}\Delta^n\delta_{x_0}(x) + o(t^{2n})
\end{aligned}$$

Davies Lemma on graphs

Theorem (Bauer-H.-Yau preprint '14)

There is a constant $\gamma \geq 1$ *such that for any graph* $G = (V, E, \mu)$*, and* $U_1, U_2 \subset V$, $t \geq 0$,

$$\sum_{x \in U_1} \sum_{y \in U_2} p_t(x,y)\mu_x\mu_y \leq \sqrt{\mu(U_1)\mu(U_2)}e^{-\frac{1}{2}\Lambda t} \times \exp\left(-\zeta(\gamma t + 1, d(U_1, U_2))\right),$$

where $\zeta(t,d) = d\,\mathrm{arcsinh}\left(\frac{d}{t}\right) - \sqrt{d^2 + t^2} + t$.

- For $t \gg d(x,y)$,

$$\zeta(t, d(x,y)) \sim \frac{d^2(x,y)}{2t}.$$

Proof of Davies Lemma

Lemma (Integral maximum principle)

Let $u(t,x)$ solve the heat equation on $\Omega \subset V$ with Dirichlet boundary condition and $\mu_1(\Omega)$ be the first Dirichlet eigenvalue of Ω. Suppose that there are positive and decreasing function in t, $K(t,x)$, and $\delta \in [0,1]$ such that for any $t \geq 0$, $x \sim y \in V$

$$\begin{aligned} &\left(K(t,x) + K(t,y) - 2(1-\delta)\sqrt{K(t,x)K(t,y)}\right)^2 \\ \leq\ &\left(K_t(t,x) - 2\delta K(t,x)\right)\left(K_t(t,y) - 2\delta K(t,y)\right), \end{aligned}$$

then

$$t \mapsto e^{2(1-\delta)\mu_1(\Omega)t} \sum_{x\in\Omega} K(t,x)u^2(t,x)\mu(x),$$

is nonincreasing in $t \in [0,\infty)$.

Proof of Davies Lemma

Lemma

For any $0 < \delta \leq 1$ *there exists a constant* $\gamma(\delta) \geq 1$ *such that*

$$K(t,x) := e^{2\zeta(\gamma t + \frac{1}{2}, d(x))}$$

satisfies the equation before where $d(x) = d(x,B)$ *for any* $B \subset V$.

In the lemma,

$$\zeta(t,d) = d\,\mathrm{arcsinh}\left(\frac{d}{t}\right) - \sqrt{d^2 + t^2} + t$$

is defined by the Legendre associate

$$\zeta(t,d) = \max_{s \geq 0}\{d \cdot s - (\cosh(s) - 1)t\}.$$

Ricci curvature on graphs

Definition (Lin-Yau '12)

We say a graph G satisfy $CD(n,\kappa)$ condition, if for any function $f : V \to \mathbb{R}$,

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2 + \langle \nabla(\Delta f), \nabla f\rangle + \kappa|\nabla f|^2.$$

Definition (Bauer-Horn-Lin-Lippner-Mangoubi-Yau '13)

$CDE(n,\kappa)$ condition on a graph is defined as:
For any $x \in V$, any positive function satisfying $\Delta f(x) < 0$, the following holds,

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2 + \langle \nabla(\Delta f), \nabla f\rangle + \left\langle \nabla \frac{|\nabla f|^2}{f}, \nabla f \right\rangle + \kappa|\nabla f|^2.$$

Li-Yau gradient estimate

Theorem (Bauer etc. '13)

Let G satisfy $CDE(n,0)$ condition and $u(t,x)$ be a positive solution to heat equation on $B_{2R}(x_0)$. Then for any $0 < \alpha < 1$,

$$(1-\alpha)\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2(1-\alpha)t} + \frac{C(n)}{(1-\alpha)R} \quad \text{on } B_R(x_0)$$

$$\text{(Riemannian case)} \qquad \cdots + \frac{C(n)}{(1-\alpha)R^2}.$$

Corollary

Let G satisfy $CDE(n,0)$ condition, then any harmonic function satisfying $u = o(\sqrt{d(\cdot,x_0)})$ is constant.

Gaussian upper bounds

- Davies '93 proved the following heat kernel estimate: For any infinite graph G,

$$p_t(x,y) \le \frac{1}{\sqrt{\mu_x \mu_y}} e^{-d(x,y)\log\frac{d(x,y)}{6t}}, \quad x, y \in V, t > 0.$$

Theorem (Bauer-H.-Yau preprint '14)

Let G satisfy $CDE(n,0)$ condition, then there exists C_1, C_2 such that for $t \ge d(x,y) \vee 1$,

$$p_t(x,y) \le \frac{C_1 \exp(-\frac{1}{2}\Lambda t)}{\sqrt{\mu(B_x(\sqrt{t}))\mu(B_y(\sqrt{t}))}} \exp\left(-\frac{d^2(x,y)}{C_2 t}\right).$$

Eigenvalue estimates

For compact manifold M, by using Davies Lemma Chung-Grigor'yan-Yau(1996) proved

$$\lambda_2 \leq \frac{C_1}{d(A_1, A_2)^2} \left(\log \frac{C_2 \mathrm{vol}(M)}{\sqrt{\mathrm{vol}(A_1)\mathrm{vol}(A_2)}} \right)^2,$$

where A_1, A_2 are two disjoint subsets of M.

Theorem

Let G be a finite graph and $A_1, A_2, \cdots, A_k$ be disjoint subsets on G and $\delta := \min_{i \neq j} d(A_i, A_j)$. Then

$$\lambda_k \leq \frac{1}{\delta} \max_{i \neq j} \frac{\log \frac{2\mu(V)}{\sqrt{\mu(A_i)\mu(A_j)}}}{h\left(\frac{2}{\delta} \log \frac{2\mu(V)}{\sqrt{\mu(A_i)\mu(A_j)}} \right)} \sim \frac{C}{\delta^2} \quad \delta \to \infty,$$

where $h(t)$ is the inverse function of $\zeta(t, 1)$.