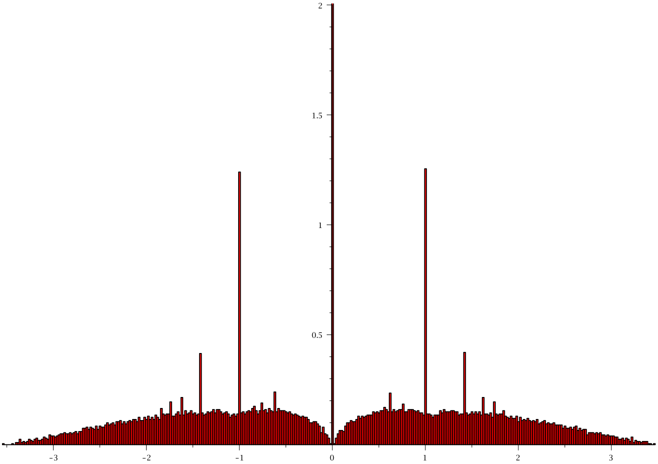
Atoms in the limiting spectrum of sparse graphs

Justin Salez (LPMA)



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Empirical spectral distribution of a graph

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Question: How does μ_G **typically** look when G is large ?

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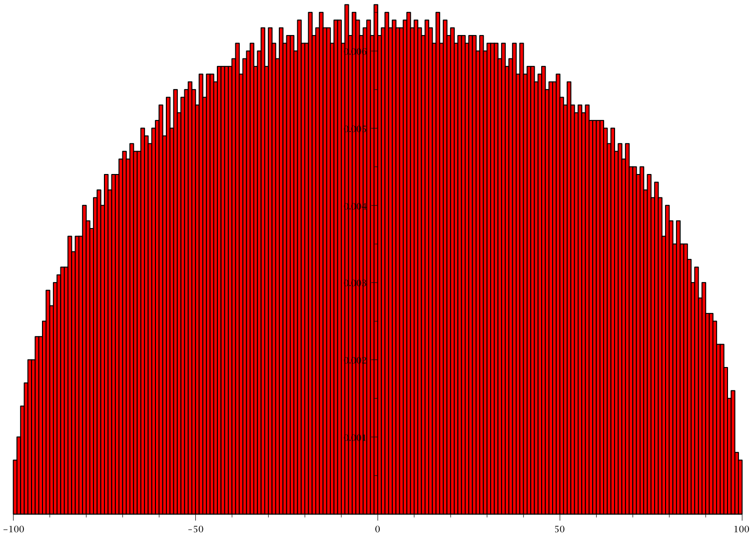

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- What about **sparse graphs**: $|E| \asymp |V|$?

Graph with average degree 3 on 1000 nodes

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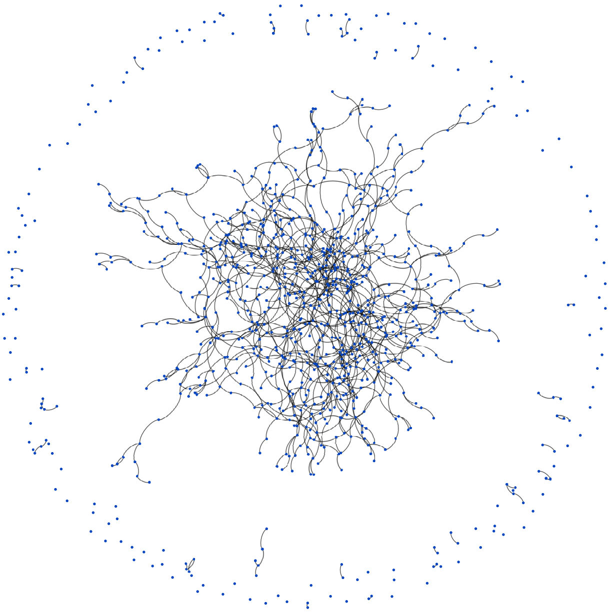

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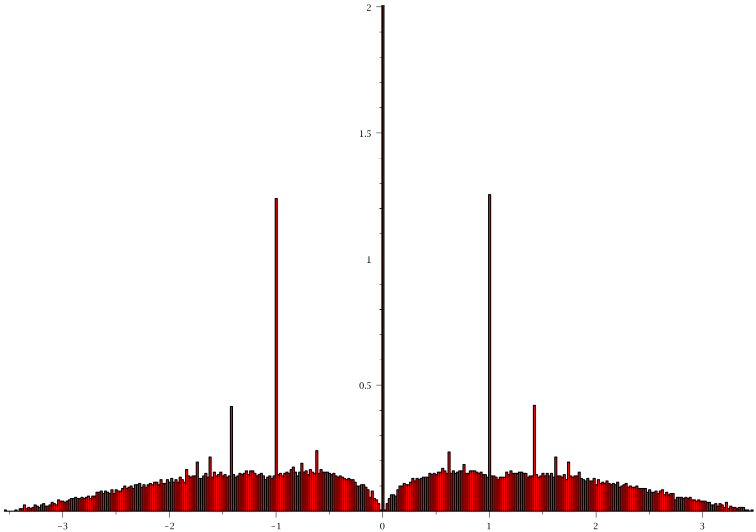
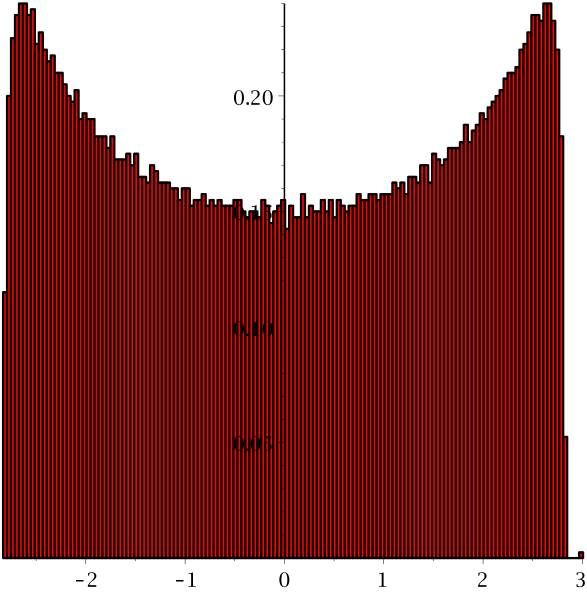
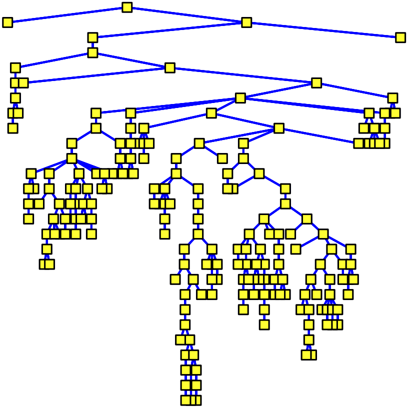


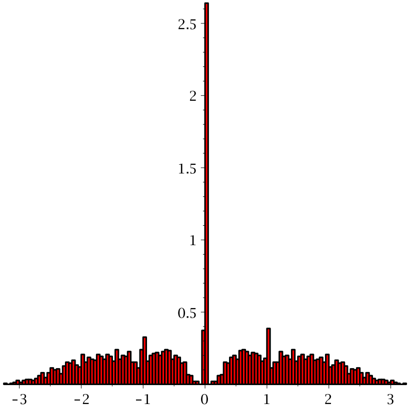
random 3-regular graph on 10000 nodes

RANDOM 3-REGULAR GRAPH ON 10000 NODES



UNIFORM RANDOM TREE ON 250 NODES





SPECTRA OF SPARSE GRAPHS

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This phenomenon is just one of the many consequences of the fact that the **underlying local geometry** converges !

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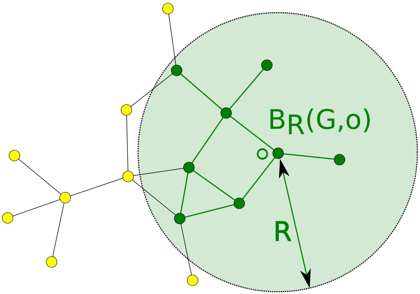

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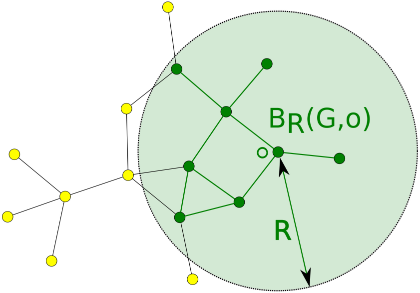

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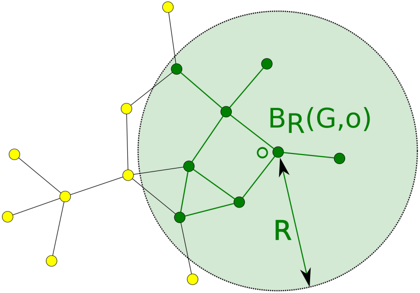

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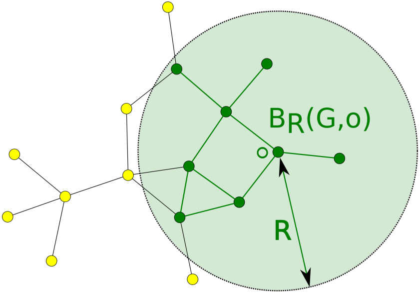

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▷ $\mathcal{L}$ describes the local geometry of G_n around a random node.

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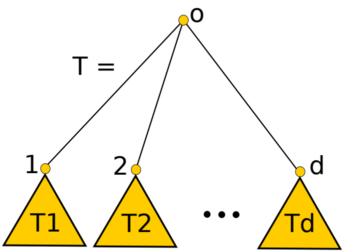
Fact: $$\boxed{G_n \xrightarrow[n\to\infty]{loc.} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \mu_{\mathcal{L}}}$$

RECURSION IN THE CASE OF TREES



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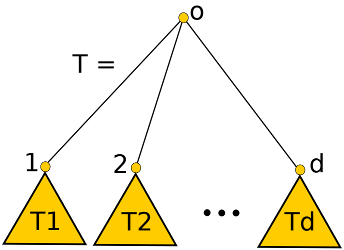

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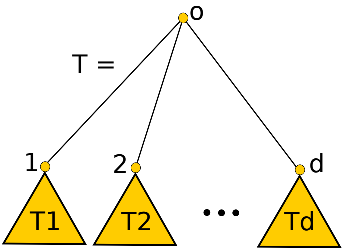

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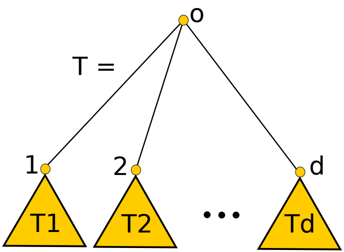

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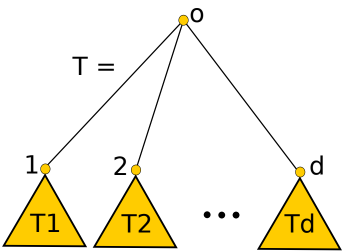

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- In principle, this equation contains everything about $\mu_{\mathcal{L}}$

ILLUSTRATION: THE NULLITY OF SPARSE GRAPHS

Illustration: the nullity of sparse graphs

Conjecture (Bauer-Golinelli '01). For G_n : Erdős-Rényi $\left(n, \frac{c}{n}\right)$,

$$\mu_{G_n}(\{0\}) \xrightarrow[n\to\infty]{} \lambda^* + e^{-c\lambda^*} + c\lambda^* e^{-c\lambda^*} - 1,$$

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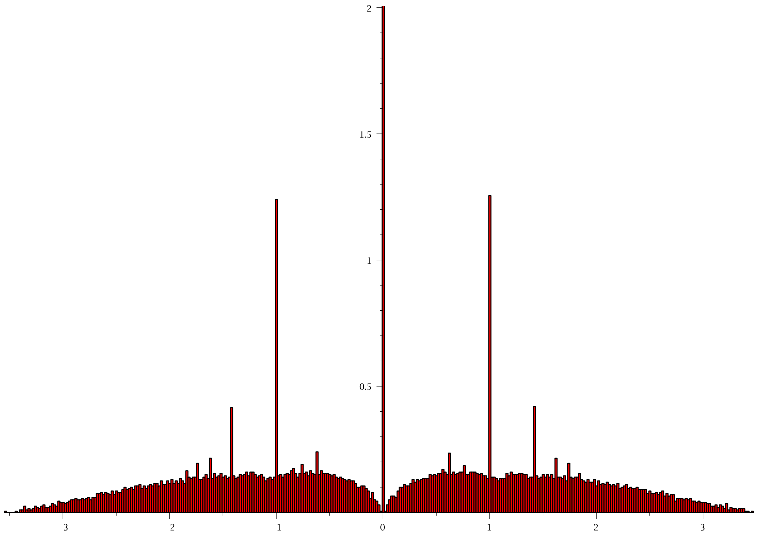
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▷ all tree eigenvalues are atoms of $\mu_{\mathcal{L}}$ (e.g. $0, 1, \sqrt{3}, 2\cos\frac{2\pi}{5}, \dots$)

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In particular, $\mu_{\mathcal{L}}(\{\lambda\}) = 0$ unless λ is a **totally real algebraic integer** (= root of some real-rooted monic integer polynomial).

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Corollary: many graph limits have the set of totally real algebraic integers as atomic support. This includes all *Galton-Watson trees* with $\text{supp}(\nu) = \mathbb{N}$, as well as the *Infinite Skeleton Tree*.

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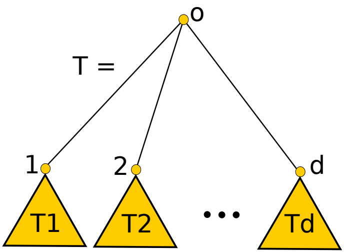

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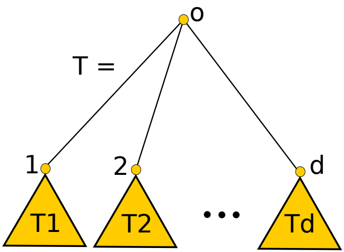

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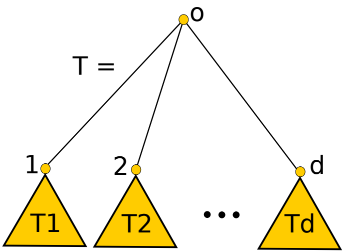

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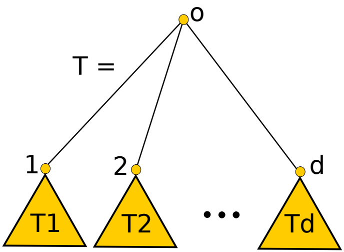

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▷ $\lambda \neq 0$ is a tree eigenvalue $\iff$ 1 can be generated from 0 by repeated applications of $(x_1, \ldots, x_d) \mapsto \frac{1}{\lambda^2}\sum_i \frac{1}{1-x_i}$ $(d \in \mathbb{N})$.

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Conclusion: λ is an eigenvalue of $T =$ 

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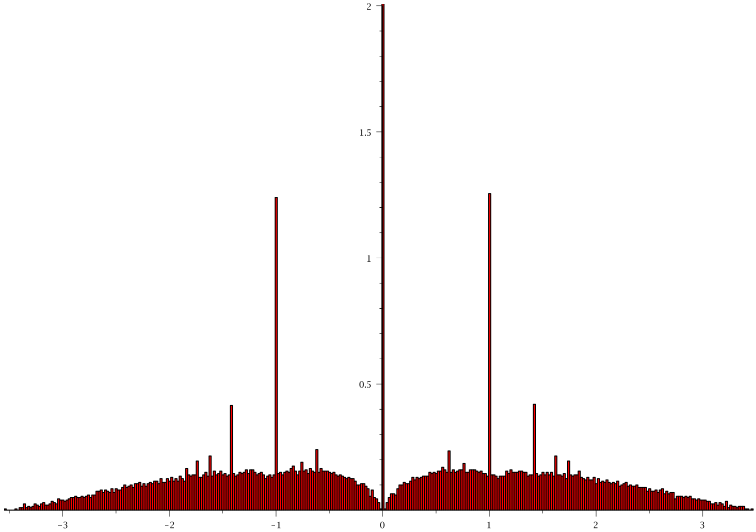

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