Entropy of Eigenfunctions on Quantum Graphs

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Distribution of eigenfunctions: Quantum ergodicity

(M, g) compact Riemannian manifold:

$$-\Delta_g \psi_n = k_n^2 \psi_n \ , \qquad \|\psi_n\|_{L^2} = 1$$

Shnirelman (74), Zelditch (87), Colin de Verdiere (85), Quantum Ergodicity: ergodic geodesic flow, then almost all eigenfunctions equidistribute for $k_n \to \infty$:

$$\lim_{j\to\infty} \int_M a(x)|\psi_{n_j}(x)|^2 \, \mathrm{d}\nu = \int_M a(x) \, \mathrm{d}\nu \ ,$$

along a subsequence n_j of density 1, $\mathrm{d}\nu$ Riemannian measure.

- density 1: $\lim_{N\to\infty} \frac{|\{n_j \leq N\}|}{N} = 1$
- valid for $\langle \psi_{n_j}, \mathrm{Op}[a]\psi_{n_j} \rangle$ with $a \in C^\infty(S^*M)$

Quantum Unique Ergodicity?

Quantum Unique Ergodicity: Does

$$\lim_{n\to\infty}\int_M a(x)|\psi_n(x)|^2\,\mathrm{d}\nu = \int_M a(x)\,\mathrm{d}\nu\ .$$

hold?

- Kurlberg Rudnick '00, Marklof Rudnik '00: **Yes** for Hecke eigenbasis of cat maps and for parabolic maps
- Faure, Nonnenmacher, De Bièvre '03; Chang, Krueger, RS, Troubetzkoy '08: **No** for cat maps and other quantised maps
- Lindenstrauss '06: **Yes** Quantum Unique Ergodicity holds for Hecke eigenbasis on arithmetic surfaces.
- Hassell '10: **No** for Stadium billiards
- Anantharaman et.al. '07: lower bounds on the entropy of quantum limits on manifolds of negative curvature.

Quantum Graphs

- $G = (V, E)$, finite undirected connected graph.
  V-vertices, E-edges, $E \ni e = [i, j]$, $i, j \in V$,
  bonds = oriented edges, $b = (i, j)$, then $\hat{b} := (j, i) \neq b$
- Length $\mathbf{L} \in \mathbb{R}_+^{|E|}$: assign to each edge a length $L_e > 0$, identify e with interval $[0, L_e]$.

$$L^2(G, \mathbf{L}) := \bigoplus_{e \in E} L^2([0, L_e]) \ , \quad H_s(G, \mathbf{L}) := \bigoplus_{e \in E} H_s((0, L_e)) \ .$$

- Laplace operator: $\Delta : H_2(G, \mathbf{L}) \to L^2(G, \mathbf{L})$,
  $f = (f_1, f_2, \cdots, f_{|E|}) \in H_2(G, \mathbf{L})$, then

$$\Delta f = (f_1'', f_2'', \cdots, f_{|E|}'') \ .$$

- need boundary conditions at vertices to define self-adjoint operator

S-matrix and Boundary conditions

describe boundary conditions on vertex i of degree d_i in terms of S-matrix $S^{(i)}$: unitary $d_i \times d_i$ matrix

- $[i,j]$, $j \sim i$, edges adjacent to i, oriented away from i:
- Solutions to $-\Delta f = k^2 f$:

$$f_{[i.j]}(x) = a_{(j,i)}\mathrm{e}^{-\mathrm{i}kx} + a_{(i,j)}\mathrm{e}^{\mathrm{i}kx}$$

- $\mathbf{a}_i^{in} := (a_{(j_1,i)}, \cdots, a_{(j_{d_i},i)})$, $\mathbf{a}_i^{out} := (a_{(i,j_1)}, \cdots, a_{(i,j_{d_i})})$.

$$\mathbf{a}_i^{out} = S^{(i)}(k)\mathbf{a}_i^{in}$$

- Boundary conditions classified by Kostrykin Schrader '99

Examples

- Neumann conditions: $f_e = f_{e'}$ for all e, e' meeting at i and $\sum f'_e = 0$.

$$S^{(i)}{}_{e,e'} = \frac{2}{d_i} - \delta_{e,e'}$$

  For large d_i backscattering dominates!

- Equi-transmitting conditions (Harrison, Smilansky, Winn 07):

$$|S^{(i)}{}_{e,e'}|^2 = \begin{cases} 0 & e = e' \\ \frac{1}{d_i - 1} & e \neq e' \end{cases}$$

  No backscattering!

- non-Robin boundary conditions: S independent of k. Equivalent to $S^* = S$. Then $S = P_+ - P_-$ where $P_\pm$ orthogonal projections with $P_+ + P_- = I$, $P_+ P_- = 0$ and boundary conditions are

$$P_- \mathbf{f} = 0 \qquad P_+ \mathbf{f}' = 0$$

Bond S-matrix and quantisation conditions

- Quantum Graph: $(G, \mathbf{L}, \{S^{(i)}\}_{i \in V})$
- Bond S-matrix $\mathcal{U}(k) = (u_{b,b'})$: $2|E| \times 2|E|$ matrix defined by

$$u_{(i,j),(k,l)} = \delta_{jk} S^{(j)}_{(i,j),(j,l)} \mathrm{e}^{\mathrm{i}kL_{[i,j]}} \ , \quad \mathcal{U}(k) = \mathrm{e}^{\mathrm{i}k\mathbf{L}} S$$

- Quantisation conditions:

$$\mathcal{U}(k)\mathbf{a} = \mathbf{a} \ , \quad \mathbf{a} \in \mathbb{C}^{2|E|} \setminus \{0\} \ ,$$

  if and only if f defined by

$$f_{[i,j]} = a_{(i,j)} \mathrm{e}^{\mathrm{i}kx_{i,j}} + a_{(j,i)} \mathrm{e}^{\mathrm{i}kx_{j,i}}$$

  is eigenfunction.
- eigenvalues determined by secular equation

$$\det(\mathcal{U}(k) - I) = 0$$

Paths and classical dynamics

- Path of length $t \in \mathbb{N}$: $\gamma = (b_1, b_2, \cdots, b_{t-1}, b_t)$ where if $b_s = (i, j)$ and $b_{s+1} = (k, l)$ then $j = k$.
  - $\Gamma_t(b, b')$-set of paths connecting b and b' in t steps
  - $\Gamma'_t(b, b')$-set of paths without backtracking: $b_{s+1} \neq \hat{b}_s$.
- Set $L_\gamma = \sum_{b\in\gamma} L_b$, $s_\gamma = \prod_{s=1}^{t} S_{b_s, b_{s+1}}$, then

$$\mathcal{U}(k)^t = (u^{(t)}_{b,b'}) \quad u^{(t)}_{b,b'} = \sum_{\gamma\in\Gamma_t(b,b')} s_\gamma \mathrm{e}^{\mathrm{i}kL_\gamma} ,$$

if no backscattering: $u^{(t)}_{b,b'} = \sum_{\gamma\in\Gamma'_t(b,b')} s_\gamma \mathrm{e}^{\mathrm{i}kL_\gamma}$

Classical dynamics: Set $M = (m_{b,b'})$ with $m_{b,b'} := |u_{b,b'}|^2$. M is doubly stochastic and defines a Markov chain with

$$M^t \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{e}}{2|E|} \mathbf{e} + O_{G,\mathbf{x}}(\mathrm{e}^{-\gamma_G t})$$

for some $\gamma_G > 0$ and $\mathbf{e} = (1, 1, \cdots, 1)$.

Quantum Graphs: History

- introduced independently in different areas: Chemistry, Physics, Mathematics
- Quantum ergodicity on quantum graphs is open! Partial results:
  - Berkolaiko, Keating, Winn (04): No quantum ergodicity on star graphs
  - Berkolaiko, Keating, Smilanski (07): Quantum ergodicity for graphs related to interval maps.
  - Gnutzman, Keating, Piotet (10): quantum ergodicity under gap condition, non-rigorous.
  - Anantharaman, LeMasson (13): quantum ergodicty on d-regular combinatorial graphs.
  - Jakobson, Strohmaier, Safarov (13): quantum ergodicity with ray-splitting
  - Winn (14, in preparation): quantum ergodicity on d-regular quantum graphs which large girth.
  - Colin de Verdière (14): classification of quantum limits on finite graphs with Neumann bc: no quantum ergodicity

Entropy

Let $\mathbf{a} \in \mathbb{C}^N$ with $\|\mathbf{a}\| = 1$. Entropy:

$$S(\mathbf{a}) := \frac{1}{\ln N} \sum_{n=1}^{N} -|a_n|^2 \ln |a_n|^2$$

- $0 \leq S(\mathbf{a}) \leq 1$
- $S(\mathbf{a}) = 0$ iff $\mathbf{a} = \mathbf{e}_m = (\delta_{m,n})$ and $S(\mathbf{a}) = 1$ iff $\mathbf{a} = \frac{1}{\sqrt{N}}\mathbf{e}$
- if $\mathbf{a} = (a_n)$, $a_n = 0$ for $n \in K \subset \{1, 2, \cdots, N\}$ then

$$S(\mathbf{a}) \leq \frac{\ln(N - |K|)}{\ln N}$$

Entropy large $\longrightarrow$ $\mathbf{a}$ can't be concentrated on small set

Entropy is a measure for the distribution of $\mathbf{a}$

Entropic Uncertainty Principle

Maassen Uffink '88: Let $U = (u_{n,m}) \in \mathbb{C}^{N\times N}$ be unitary, then

$$S(\mathbf{a}) + S(U\mathbf{a}) \geq -\frac{\ln\left(\max_{n,m}|u_{n,m}|^2\right)}{\ln N}$$

$\sum_n |u_{n,m}|^2 = 1$: optimal case $|u_{n,m}|^2 = 1/N$, $S(\mathbf{a}) + S(U\mathbf{a}) \geq 1$
Example: Fourier transform $F = (f_{n,m})$, $f_{n,m} = \frac{1}{\sqrt{N}}\,\mathrm{e}^{2\pi\mathrm{i}\frac{nm}{N}}$

$$S(\mathbf{a}) + S(F\mathbf{a}) \geq 1$$

Application to eigenvectors: If $U\mathbf{a} = \mathbf{a}$ then

$$S(\mathbf{a}) \geq -\frac{1}{2\ln N}\ln\left(\max_{n,m}|u_{n,m}|^2\right)$$

and

$$S(\mathbf{a}) \geq -\frac{1}{2\ln N}\ln\left(\max_{n,m}|u^{(t)}_{n,m}|^2\right), \quad \text{where} \quad U^t = (u^{(t)}_{n,m})$$

Star Graphs, equi-transmitting

Theorem (Kameni, RS 13/14)

Let (G, E) be a star graph with equi-transmitting boundary conditions, then for any eigenfunction

$$S(\mathbf{a}) \geq \frac{1}{2}\frac{\ln(|E|-1)+2\ln 2}{\ln|E|+\ln 2} > \frac{1}{2}$$

- eigenfunctions: $f_e(x) = A_e \cos\big(k(x-L_e)\big)$,
  $S(\mathbf{A}) := \frac{1}{\ln|E|}\sum_{e=1}^{|E|} -|A_e|^2 \ln|A_e|^2$, $\|\mathbf{A}\| = 1$
- $\mathrm{e}^{\mathrm{i}k\mathbf{L}} S \mathrm{e}^{\mathrm{i}k\mathbf{L}} \mathbf{A} = \mathbf{A}$, $|S_{e,e'}|^2 = (1-\delta_{e,e'})\frac{1}{|E|-1}$,

$$S(\mathbf{A}) \geq \frac{1}{2}\frac{\ln(|E|-1)}{\ln|E|}$$

- $S(\mathbf{a}) = \frac{\ln|E|}{\ln(2|E|)} S(\mathbf{A}) + \frac{\ln 2}{\ln(2|E|)}$

Star Graphs, Neuman

Theorem (Kameni, RS 13/14)

Let (G, E) be a star graph with Neumann boundary conditions, $\mathbf{L}$ rationally independent, and $\mathbf{a}^{(n)}$ is the n'th eigenfunction, then the average entropy $\langle S \rangle := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S(\mathbf{a}^{(n)})$ satisfies

$$\langle S(\mathbf{a})\rangle = \frac{\alpha}{\ln|E|} + O(|E|\Delta L)$$

where $\Delta L = \max_e L_e - \min_e L_e$ and $\alpha = 1.2692\ldots$.

- proof based on Barra & Gaspard (99), further developed in Keating, Marklof and Winn (03), Colin de Verdière (14)
- quantisation condition $\det(I - \mathcal{U}(k)) = F(k\mathbf{L} \mod 2\pi)$, function on torus $\mathbb{T}^{|E|}$ evaluated on trajectory $k\mathbf{L} \mod 2\pi$, as are $\mathcal{U}(k)$ and eigenvectors $\mathbf{a}$.
- use Weyl's Theorem (unique ergodicity of $\phi^t(\mathbf{x}) = \mathbf{x} + t\mathbf{L} \mod 2\pi$) to transform energy average in average over torus $\mathbb{T}^{|E|}$.

Equi-transmitting versus Neumann, Star Graphs

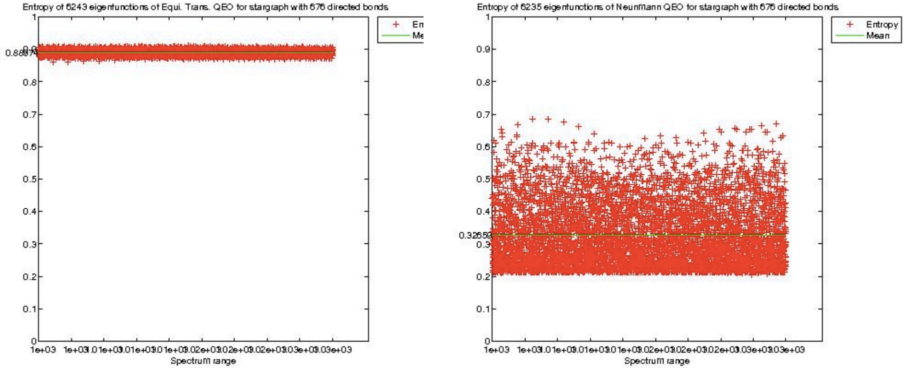

Figure: 6235 eigenfunctions on Star graph with 338 edges.
Left: Entropy of eigenfunctions with equi-transmitting boundary conditions.
Right: Entropy of eigenfunctions with Neumann boundary conditions.

Regular graphs

- G $d+1$ regular graph if every vertex has degree $d+1$
  - $2|E| = (d+1)|V|$
  - equi-transmitting boundary conditions: $|S_{e,e'}|^2 = \frac{1}{d}(1-\delta_{e,e'})$
- $G_N = (V_N, E_N)$, $N \in \mathbb{N}$, graphs with $\lim_{N\to\infty}|V_N| = \infty$
  - G_N **expander** if there exits $\gamma > 0$ such that

$$M_N^t \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{e}}{2|E_N|}\mathbf{e} + O(\mathrm{e}^{-\gamma t})$$

    Expansion rate uniform in N!
  - G_N has **large girth** if there exist a $\delta > 0$ such that the length T_N of the shortest cycle satisfies

$$T_N \geq 2\delta \ln(2|E_N|)$$

    - If b, b' have distance t less then $\delta \ln(2|E|)$ then there exist only one path of length t connecting them.
    - Any ball of radius less then $\delta \ln(2|E_N|)$ is a tree.

Regular graphs: Large girth

Theorem (Kameni, RS 13/14)

Let G be a $d+1$ regular graph with girth $T_G = 2R_G + 1$, then for equi-transmitting boundary conditions

$$S(\mathbf{a}) \geq \frac{1}{2}\frac{R_G \ln d}{\ln(2|E|)} \ .$$

Corollary

Assume G_N has large girth, $T_G = 2\delta \ln(2|E|)$, then $S(\mathbf{a}) \geq \frac{\delta \ln d}{2}$.

Main idea: for $t \leq R_G$, we have $|\Gamma'_t(b, b')| \leq 1$ hence for $t = R_G$

$$|u^{(t)}_{b,b'}|^2 = \Big| \sum_{\gamma \in \Gamma'_t(b,b')} s_\gamma e^{ikL_\gamma} \Big|^2 \leq |s_\gamma|^2 = \frac{1}{d^t}$$

Regular graphs: Large girth and expanding

t large: $\Gamma'_t(b,b')$ contains exponentially many elements, turn

$$u^{(t)}_{b,b'} = \sum_{\gamma\in\Gamma'_t(b,b')} s_\gamma \mathrm{e}^{\mathrm{i}kL_\gamma}$$

into a sum over random variables by making length $\mathbf{L}$ random.

Assumption: L_e independent and

- $\mathbb{P}(L_e \le \delta) = 0$ with $\delta > 0$ independent of e and G_N.
- there exits an $f(k) \in C(\mathbb{R})$ with $\lim_{k\to\pm\infty} f(k) = 0$ such that $|\mathbb{E}(\mathrm{e}^{\mathrm{i}kL_e})| \le f(k)$ independent of $e \in E_N$ and G_N.

Theorem (Kameni, RS 13/14)

$G_N = (V_N, E_N)$ expanding, large girth, random length and equi-transmitting. Then for any $\varepsilon > 0$ there exist a $k_0 > 0$ such that if $k \ge k_0$ and $\mathbf{a}$ is an eigenvector of $\mathcal{U}(k)$ we have

$$\mathbb{P}\left(S(\mathbf{a}) \ge \frac{1-\varepsilon}{2}\right) \ge 1 - \frac{4d}{|V_N|^\varepsilon}$$

Regular graphs: Large girth and expanding

Proof strategy:

- Chebychev's inequality: $|u^{(t)}_{b,b'}| \sim \sqrt{\mathbb{E}(|u^{(t)}_{b,b'}|^2)}$
- $\mathbb{E}(|u^{(t)}_{b,b'}|^2) = \sum_{\gamma,\gamma' \in \Gamma'_t(b,b')} s_\gamma s_{\gamma'} \mathbb{E}(\mathrm{e}^{\mathrm{i}k(L_\gamma - L_{\gamma'})})$
  large girth: if $\gamma \neq \gamma'$

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}k(L_\gamma - L_{\gamma'})}) \leq \left[f(k)\right]^{(2R_G)}$$

- $N_t(b,b') := |\Gamma'_t(b,b')|$, $|s_\gamma|^2 = d^{-t}$, then

$$\mathbb{E}(|u^{(t)}_{b,b'}|^2) \leq \frac{N_t(b,b')}{d^t}\left(1 + N_t(b,b')\left[f(k)\right]^{(2R_G)}\right)$$

- expander: there exist $\mu < 1$, independent of G_N, such that

$$\frac{N_t(b,b')}{d^t} \leq \frac{1}{2|E|} + \mu^t\ .$$

Equi-transmitting versus Neumann, Regular Graphs

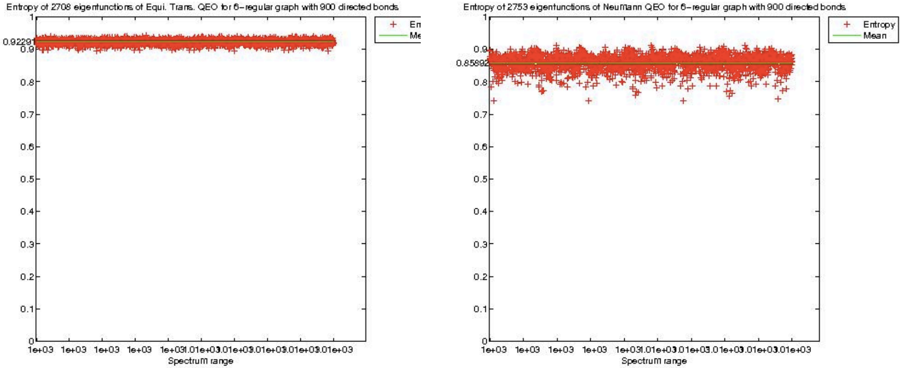

Figure: 2708 eigenfunctions on a 6-regular graph with 450 edges.
Left: Entropy of eigenfunctions with equi-transmitting boundary conditions.
Right: Entropy of eigenfunctions with Neumann boundary conditions.

Summary

- Entropy of eigenfunctions on graphs gives a measure for their localisation or delocalisation.
- We derive lower bounds on the entropy by using the Entropic Uncertainty Principle.
- Main assumptions are large girth and expansion, which allow to explore the Entropic Uncertainty Principle
- For regular graphs with large girth, expanding, and with random bond-length, we obtain a bound similar to the Anantharaman bound on manifolds of negative curvature.