Location and Weyl formula for the eigenvalues of some non self-adjoint operators

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Abstract We present a survey of some recent results concerning the location and the Weyl formula for the complex eigenvalues of two non self-adjoint operators. We study the eigenvalues of the generator G of the contraction semigroup e^{tG}, $t \geq 0$, related to the wave equation in an unbounded domain Ω with dissipative boundary conditions on $\partial\Omega$. Also one examines the interior transmission eigenvalues (ITE) in a bounded domain K obtaining a Weyl formula with remainder for the counting function $N(r)$ of complex (ITE). The analysis is based on a semi-classical approach.

1 Introduction

Let $P(x, D_x)$ be a second order differential operator with $C^\infty(K)$ real-valued coefficients in a bounded domain $K \subset \mathbb{R}^d$, $d \geq 2$, with C^∞ boundary ∂K. Consider a boundary problem

$$\begin{cases} P(x, D_x)u = f \text{ in } K, \\ B(x, D_x)u = g \text{ on } \partial K, \end{cases} \tag{1.1}$$

where $B(x, D_x)$ is a differential operator with order less or equal to 1 and the principal symbol $P(x, \xi)$ of $P(x, D_x)$ satisfies $p(x, \xi) \geq c_0|\xi|^2$, $c_0 > 0$. Assume that there exists $0 < \varphi < \pi$ such that the problem

$$\begin{cases} (P(x, D_x) - z)u = f \text{ in } K, \\ B(x, D_x)u = g \text{ on } \partial K. \end{cases} \tag{1.2}$$

is parameter-elliptic for every $z \in \Gamma_\psi = \{z : \arg z = \psi\}$, $0 < |\psi| \leq \varphi$. Then following a classical result of Agranovich-Vishik [1] we can find a closed operator A with

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domain $D(A) \subset H^2(K)$ related to the problem (1.1). Moreover, for every closed angle $Q = \{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta\} \subset \{z \in \mathbb{C} : |\arg z| < \varphi\}$ which does not contain $\mathbb{R}^+$ there exists $a_Q > 0$ such that the resolvent $(A - z)^{-1}$ exists for $z \in Q, |z| \geq a_Q$ and the operator A has a *discrete spectrum* in $\mathbb{C}$ with eigenvalues with finite multiplicities.

Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of A ordered as follows

$$0 \leq |\lambda_1| \leq ... \leq |\lambda_m| \leq ....$$

In general A is not a self-adjoint operator and the analysis of the asymptotics of the counting function

$$N(r) = \#\{|\lambda_j| \leq r\} \text{ as } r \to +\infty,$$

where every eigenvalues is counted with its multiplicity, is a difficult problem. In particular, it is quite complicated to obtain a Weyl formula for $N(r)$ with a remainder and many authors obtained results which yield only the leading term of the asymptotics. On the other hand, even for parameter-elliptic boundary problems the result in [1] says that in any domain $0 < \psi < |\arg z| < \varphi$ we can have only finite number eigenvalues but we could have a bigger eigenvalues-free domains. To obtain a better remainder in the Weyl formula for $N(r)$ we must obtain a eigenvalues-free region outside some *parabolic neighborhood* of the real axis.

On the other hand, in mathematical physics there are many problems which are not parameter-elliptic. Therefore, the results of [1] cannot be applied and the analysis of the eigenvalues-free regions must be studied by another approach.

For the spectrum of non self-adjoint operators we have three important problems:

(I) Prove the discreteness of the spectrum of A in some subset $U \subset \mathbb{C}$,

(II) Find eigenvalues-free domains in $\mathbb{C}$ having the form

$$|\operatorname{Im} z| \geq C_{\pm\delta}(|\operatorname{Re} z| + 1)^{\delta_\pm}, \pm\operatorname{Re} z \geq 0, 0 < \delta_\pm < 1,$$

(III) Establish a Weyl asymptotic with remainder for the counting function

$$N(r) = cr^d + \mathscr{O}(r^{d-\kappa}), r \to \infty, 0 < \kappa < 1.$$

In this survey we discuss mainly the problems (II) and (III) for two non self-adjoint operators related to the scattering theory. The problem (I) is easer to deal with and the analysis of (II) in many cases implies that $A - z$ is a Fredholm operator for z in a suitable region. We apply a new semi-classical approach for both problems (II) and (III). The analysis of (II) is reduced to the inversibility of a h-pseudo-differential operator, while for the asymptotic of $N(r)$ one exploits in a crucial way the existence of parabolic neighborhood of the real axis containing the (ITE). The purpose of this survey is to present the recent results in [21], [22], [14], [15], [13], [6], where the above problems are investigated by the same approach. We expect that our arguments can be applied to more general non self-adjoint operators covering the case of parameter-elliptic boundary problems (1.2).

2 Two spectral problems related to the scattering theory

I. Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of Ω is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x) u_t = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0,x) = f_1,\, u_t(0,x) = f_2 \end{cases} \tag{2.1}$$

with initial data $f = (f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathscr{H}$. Here $\nu(x)$ is the unit outward normal to $x \in \Gamma$ pointing into Ω and $\gamma(x) \geq 0$ is a C^∞ function on Γ. The solution of (2.1) is given by

$$(u, u_t) = V(t) f = e^{tG} f,\ t \geq 0,$$

where $V(t)$ is a contraction semi-group in $\mathscr{H}$ whose generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

has a domain $D(G)$ which is the closure in the graph norm of functions $(f_1, f_2) \in C^\infty_{(0)}(\mathbb{R}^n) \times C^\infty_{(0)}(\mathbb{R}^n)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ. The spectrum of G in $\mathrm{Re}\, z < 0$ is formed by isolated eigenvalues with finite multiplicity (see [11] for d odd and [13] for d even), while the continuous spectrum of G coincides with $\mathbf{i}\mathbb{R}$. Next, if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda \gamma f_1 = 0 \text{ on } \Gamma. \end{cases} \tag{2.2}$$

Thus if $\mathrm{Re}\,\lambda < 0$, $f \neq 0$, $(u(t,x), u_t(t,x)) = V(t) f = e^{\lambda t} f(x)$, then $u(t,x)$ is a solution of (2.1) with *exponentially decreasing global energy.* Such solutions are called asymptotically disappearing and they perturb the inverse scattering problems. Recently it was proved (see [5]) that if we have a least one eigenvalue λ of G with $\mathrm{Re}\,\lambda < 0$, then the wave operators $W_\pm$ related to the problem (2.1) and the Cauchy problem for the wave equation are not complete, that is $\mathrm{R}an\, W_- \neq \mathrm{R}an\, W_+$. Hence we cannot define the scattering operator S related to (2.1) by $S = W_+^{-1} \circ W_-$. We may define S by another evolution operator. For problems associated to unitary groups, the associated scattering operator $S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ satisfies the equality

$$S^{-1}(z) = S^*(\bar{z}),\ z \in \mathbb{C},$$

provided that $S(z)$ is invertible at z. This implies that $S(z)$ is invertible for $\mathrm{Im}\, z > 0$, since $S(z)$ and $S^*(z)$ are analytic for $\mathrm{Im}\, z < 0$ (see [10] for more details). For dissipative boundary problems the above relation is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \mathrm{Im}\, z_0 > 0$. In the case of odd dimensions d Lax and Phillips [11] proved that $\mathbf{i} z_0$ is an eigenvalue of G. Consequently, the analysis of the location of the eigenvalues of G is important for the inverse scattering problems.

The eigenvalues of G are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. A. Majda [12] proved that if $\sup_{x\in\Gamma}\gamma(x)<1$, then the eigenvalues of G lie in the region

$$E_1=\{z\in\mathbb{C}:\ |\operatorname{Re} z|\le C_1(|\operatorname{Im} z|^{3/4}+1),\ \operatorname{Re} z<0\},$$

while if $\sup_{x\in\Gamma}\gamma(x)\ge 1$, the eigenvalues of G lie in $E_1\cup E_2$, where

$$E_2=\{z\in\mathbb{C}:\ |\operatorname{Im} z|\le C_2(|\operatorname{Re} z|^{1/2}+1),\ \operatorname{Re} z<0\}.$$

The case $\gamma(x)=1$, $\forall x\in\Gamma$, is special since as it was mentioned by Majda [12] for some obstacles there are no eigenvalues of G. On the other hand, to our best knowledge we did not found a proof of this result in the literature. In the Appendix in [13], the case when $K=B_3=\{x\in\mathbb{R}^3:\ |x|\le 1\}$ is ball and $\gamma>0$ is a constant has been examined and it was proved that if $\gamma=1$, there are no eigenvalues of G. On the other hand, for $\gamma=const>1$ all eigenvalues of G are real and for $0<\gamma<1$ there are no real eigenvalues.

We will improve the above result of Majda and one examines two cases:

$$(A):\ 0<\gamma(x)<1,\ \forall x\in\Gamma.$$

$$(B):\ \gamma(x)>1,\ \forall x\in\Gamma.$$

II. We discuss another important spectral problem for inverse scattering leading to non self-adjoint operator. For simplicity we assume that d is odd. The inhomogeneous medium in K is characterized by a smooth function $n(x)>0$ in $\bar{K}$, called *contrast*. The scattering problem is related to an *incident wave* u_i which satisfies the equation $(\Delta+k^2)u_i=0$ in $\mathbb{R}^d$ and the *total wave* $u=u_i+u_s$ satisfies the transmission problem

$$\begin{cases}\Delta u+k^2u=0 \text{ in } \mathbb{R}^d\setminus\bar{K},\\ \Delta u+k^2n(x)u=0 \text{ in } K,\\ u^+=u^- \text{ on } \Gamma,\\ \Big(\frac{\partial u}{\partial\nu}\Big)^+=\Big(\frac{\partial u}{\partial\nu}\Big)^- \text{ on } \Gamma,\end{cases}\qquad (2.3)$$

where $f^\pm(x)=\lim_{\varepsilon\to 0}f(x\pm\varepsilon\nu(x))$ for $x\in\Gamma$. Here $k>0$ and the outgoing scattering wave u_s satisfies the outgoing Sommerfeld radiation condition

$$\lim_{r\to+\infty}r^{(1-d)/2}\Big(\frac{\partial u_s}{\partial r}-\mathbf{i}ku_s\Big)=0$$

uniformly with respect to $\theta=x/r\in\mathbb{S}^{d-1}$, $r=|x|$.

If the incident wave has the form $u_i=e^{\mathbf{i}k\langle x,\omega\rangle}$, $\omega\in\mathbb{S}^{d-1}$, then

$$u_s(r\theta,k)=e^{\mathbf{i}kr}r^{-(d-1)/2}\Big(a(k,\theta,\omega)+\mathscr{O}(\frac{1}{r})\Big),\ r\to+\infty.$$

The function $a(k,\theta,\omega)$ is called scattering amplitude and the *far-field operator* $F(k) : L^2(\mathbb{S}^{d-1}) \longrightarrow L^2(\mathbb{S}^{d-1})$ has the form

$$(F(k)f)(\theta) = \int_{\mathbb{S}^{d-1}} a(k,\theta,\omega)f(\omega)d\omega.$$

Notice also that the scattering operator has the representation

$$S(k) = I + \Big(\frac{\mathbf{i}k}{2\pi}\Big)^{(d-1)/2} F(k).$$

The inverse scattering problem of the reconstruction of K based on the linear sampling method of Colton and Kress (see [3]) breaks down for frequencies k such that $F(k)$ has a non trivial kernel or co-kernel. Assume that for some $k \in \mathbb{R}^+$ the kernel of $F(k)$ is not trivial and let $F(k)f = 0,\ f \neq 0$. We may consider an incident Herglotz wave

$$u_i(x) = \int_{\mathbb{S}^{d-1}} e^{\mathbf{i}k\langle x,\omega\rangle} f(\omega)d\omega.$$

Then one obtains a scattering wave $u_s = \mathscr{O}(\frac{1}{r^2})$ since the leading term

$$\int_{\mathbb{S}^{d-1}} a(k,\theta,\omega)f(\omega)d\omega = 0$$

vanishes. On the other hand, $(\Delta + k^2)u_s = 0$ in $\mathbb{R}^d \setminus \bar{K}$, so the Rellich theorem implies $u^s = 0$ in $\mathbb{R}^d \setminus \bar{K}$. Therefore the functions $u = u_i|_K \neq 0$ and $w = (u_i + u_s)|_K$ satisfy the following problem

$$\begin{cases} \Delta u + k^2 u = 0 \,\text{in}\, K, \\ \Delta w + k^2 n(x) w = 0 \,\text{in}\, K, \\ u = w,\ \partial_\nu u = \partial_\nu w \,\text{on}\, \Gamma \end{cases} \tag{2.4}$$

and $\lambda = k^2$ is called *interior transmission eigenvalue* (ITE). The inverse statement in general is not true and we may have complex (ITE).

We consider a more general setting. For $d \geq 2$, a complex number $\lambda \in \mathbb{C} \setminus \{0\}$, is called interior transmission eigenvalue (ITE) if the following problem has a non-trivial solution $(u_1,u_2) \neq 0$:

$$\begin{cases} (\nabla c_1(x) \nabla + \lambda n_1(x))\, u_1 = 0 \,\text{in}\, K, \\ (\nabla c_2(x) \nabla + \lambda n_2(x))\, u_2 = 0 \,\text{in}\, K, \\ u_1 = u_2,\ c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 \,\text{on}\, \Gamma, \end{cases} \tag{2.5}$$

where $c_j(x), n_j(x) \in C^\infty(\overline{K}),\ j = 1,2$ are strictly positive real-valued functions. For the analysis of (ITE) one imposes the condition

$$d(x) = c_1(x)n_1(x) - c_2(x)n_2(x) \neq 0, \quad \forall x \in \Gamma. \tag{2.6}$$

Partial cases: 1) isotropic case: $c_1(x) = c_2(x)$, $\forall x \in \Gamma$, $n_1(x) = 1, n_2(x) \neq 1$, $\forall x \in \Gamma$.
2) anisotropic case: $c_1(x) \neq c_2(x)$, $\forall x \in \Gamma$.

3 Dirichlet-to-Neumann map

The analysis of the eigenvalues-free domains is based on a semi-classical analysis. Let $0 < h \ll 1$ and let $P(h) = -h^2\Delta$. Introduce the sets

$$Z_1 = \{z \in \mathbb{C} : \text{Re } z = 1, h^{1/2-\varepsilon} \leq \text{Im } z \leq 1,\, 0 < \varepsilon \ll 1\},$$
$$Z_2 = \{z \in \mathbb{C} : \text{Re } z = -1, |\text{Im } z| \leq 1\},$$
$$Z_3 = \{z \in \mathbb{C} : |\text{Re } z| \leq 1, \text{Im } z = 1\}.$$

and consider for $z \in Z_1 \cup Z_2 \cup Z_3$ the semi-classical problem

$$\begin{cases} (P(h) - z)u = 0 \text{ in } \Omega,\, u \in H^2(\Omega), \\ u = f \text{ on } \Gamma, \end{cases} \tag{3.1}$$

We need to introduce some h-pseudo-differential operators on a manifold with boundary V. We say that $a(x, \xi; h) \in S^{k,m}_\delta(V)$ if the following conditions are satisfied:

(i) for $|\xi| \geq L \gg 1$ we have

$$|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha,\gamma,L}(1 + |\xi|)^{m-|\gamma|},\, \forall \alpha, \forall \gamma.$$

(ii) for $|\xi| \leq L$ we have

$$|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha,\gamma,L} h^{-k-\delta(|\alpha|+|\gamma|)},\, \forall \alpha, \forall \gamma.$$

For $a \in S^{k,m}_\delta(V)$, consider the operator

$$\Big(\text{Op}_h(a) f\Big)(x) = (2\pi h)^{-d+1} \int \int e^{\mathbf{i}\langle x-y,\xi\rangle/h} a(x, \xi; h) f(y) dy d\xi.$$

We have a calculus for the h-pseudo-differential operators with symbols in $S^{k,m}_\delta$ if $0 < \delta < 1/2$. In particular, if $a \in S^{0,1}_\delta$, $b \in S^{0,-1}_\delta$, one gets

$$\|\text{Op}_h(a)\text{Op}_h(b) - \text{Op}_h(ab)\|_{L^2} \leq C h^{1-2\delta}.$$

We refer to [7] and [21] for the calculus of h-pseudo-differential operators.

Let $D_\nu = -\mathbf{i}\partial_\nu$, and let γ_0 denote the trace on Γ. It is important to construct a semi-classical parametrix for the problem (3.1) in $Z_1 \cup Z_2 \cup Z_3$ and to find an approximation for the (exterior) *semi-classical* Dirichlet-to-Neumann map defined by

$$\mathscr{N}_{ext}(z,h) : H_h^s(\Gamma) \ni f \longrightarrow \gamma_0 h D_\nu u \in H_h^{s-1}(\Gamma). \tag{3.2}$$

Here for $s \in \mathbb{R}$, $H_h^s(\Gamma)$ is the semi-classical Sobolev space with norm $\|\langle hD\rangle^s u\|_{L^2(\Gamma)}$. Vodev [21] constructed a semi-classical parametrix $\tilde{u}$ where the equation in (3.1) is satisfied for $x \in K$. In fact the construction in [21] is made in a very small neighborhood of the boundary Γ and the *local* parametrix is a Fourier integral operator with complex phase function. By using the resolvent $(-h^2\Delta_D - z)^{-1}$ of the Dirichlet Laplacian in Ω, one may modify the proof in [21] to obtain a parametrix in Ω (see [13] for more details).

To describe the local parametrix, consider *normal geodesic coordinates* (x_1, x') in a neighborhood of a fixed point $x_0 \in \Gamma$, where $x_1 = \mathrm{dist}\,(x,\Gamma)$. Then locally the boundary Γ is given by $x_1 = 0$. Let $\psi(x') \in C_0^\infty(\Gamma)$ be a cut-off function with support in a small neighborhood of $x_0 \in \Gamma$ and $\psi(x') = 1$ in another neighborhood of x_0. Then $-\frac{h^2}{c(x)}\nabla c(x)\nabla - z\frac{n(x)}{c(x)}$ in these coordinates has the form

$$\mathscr{P}(z,h) = h^2 D_{x_1}^2 + r(x, hD_{x'}) + q(x, hD_x) + h^2\tilde{q}(x) - zm(x).$$

with

$$D_{x_1} = -\mathbf{i}\partial_{x_1},\, D_{x'} = -\mathbf{i}\partial_{x'},\, m(x) = \frac{n(x)}{c(x)},\, r(x,\xi') = \langle R(x)\xi', \xi'\rangle,\, q(x,\xi) = \langle q(x), \xi\rangle.$$

Here $R(x)$ is a symmetric $(d-1)\times(d-1)$ matrix with smooth real-valued enters and $r(0,x',\xi') = r_0(x',\xi')$ is the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$ on Γ. Let

$$\rho = \sqrt{zm(x) - r_0(x',\xi')} \in C^\infty(T^*(\Gamma))$$

be the root of the equation $\rho^2 + r_0(x',\xi') - zm(x) = 0$ with $\mathrm{Im}\,\rho > 0$. Let $\phi(\sigma) \in C^\infty(\mathbb{R})$ be cut-off function such that $\phi(\sigma) = 1$ for $|\sigma| \le 1$, $\phi(\sigma) = 0$ for $|\sigma| \ge 2$. In [21] for small $\delta_1 > 0$ and for x close to the boundary it was constructed a parametrix

$$\begin{cases} \tilde{u}_\psi(x) = (2\pi h)^{-d+1} \int\int e^{\frac{\mathbf{i}}{h}\varphi(x,y',\xi',z)} \phi\left(\frac{x_1}{\delta_1}\right) \\ \times \phi\left(\frac{x_1}{\delta_1\rho_1}\right) a(x,\xi',z;h) f(y') dy' d\xi', \\ \tilde{u}_\psi|_{x_1=0} = \psi f, \end{cases} \tag{3.3}$$

where $0 < \delta_1 < 1$ is small enough and $\rho_1 = 1$ if $z \in Z_2 \cup Z_3$, $\rho_1 = |\rho|^3$ if $z \in Z_1$. The phase $\varphi(x,y',\xi',z)$ is complex-valued and we have

$$\varphi|_{x_1=0} = -\langle x'-y', \xi'\rangle,\; \partial_{x_1}\varphi|_{x_1=0} = \rho,\; \mathrm{Im}\,\varphi \ge x_1\,\mathrm{Im}\,\rho/2,$$

while $a\big|_{x_1=0} = \psi(x')$. Next, $a = \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} x_1^k h^j a_{k,j}(x',\xi',z)$,

$$\varphi = -\langle x'-y', \xi'\rangle + \sum_{k=1}^{N-1} x_1^k \varphi_k(x',\xi',z),\; \varphi_1 = \rho,$$

$N \gg 1$ being a large integer. The phase φ and the amplitude a are determined so that

$$e^{-\frac{\mathbf{i}\varphi}{h}}\mathscr{P}(z,h)e^{\frac{\mathbf{i}\varphi}{h}}a = x_1^N A_N(x,\xi',z;h) + h^N B_N(x,\xi',z;h),$$

where A_N, B_N are smooth functions and their behavior for $|\xi'| \to \infty$ is related to negative powers of $|\rho|$. For example,

$$|\partial_{x_1}^k \partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} A_N(\phi(\delta_0 r_0(x',\xi')))| \leq C_{k,\alpha,\beta}|\rho|^{2-3N-3k-2|\alpha|-2|\beta|}.$$

Moreover, for $x_1 > 0$ the parametrix $\tilde{u}_\psi$ has a decay $\mathscr{O}\Big(e^{-x_1\frac{|\operatorname{Im} z|}{2|\rho|h}}\Big)$ and for $x_1 \geq |\rho|^3/\delta$ we get an estimate $\mathscr{O}\Big(e^{-C\frac{|\rho|^2|\operatorname{Im} z|}{h}}\Big)$.

Consider the (interior) semi-classical Dirichlet-to-Neumann map $\mathscr{N}_{int}(z,h)f = \gamma_0\partial_\nu u$, related to the problem

$$\begin{cases} (-\frac{h^2}{n(x)}\nabla c(x)\nabla - z)u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma, \end{cases} \tag{3.4}$$

where $n(x) > 0$, $c(x) > 0$ are C^∞ functions on Γ. Then we have the following

Proposition 1 ([21]). *Given* $0 < \varepsilon \ll 1$, *there exists* $0 < h_0(\varepsilon) \ll 1$ *such that for* $z \in Z_1$ *and* $0 < h \leq h_0(\varepsilon)$ *we have*

$$\|\mathscr{N}_{int}(z,h)f - \mathrm{Op}_h(\rho + hb)f\|_{H^1_s(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}}\|f\|_{L^2(\Gamma)}, \tag{3.5}$$

where $b \in S_0^{0,0}(\Gamma)$ *does not depend on* z, h *and the function* $n(x)$. *Moreover, for* $z \in Z_2 \cup Z_3$ *the above estimate holds with* $|\operatorname{Im} z|$ *replaced by* 1.

With some modifications of the proof the same result remains true for unbounded domains $\mathbb{R}^d \setminus \bar{K}$ and obtain the estimate (3.5) for the semi-classical Dirichlet-to-Neumann operator $\mathscr{N}_{ext}(z,h)$ related to the problem (3.1) with $n(x) = c(x) = 1$. (see [13]).

4 Location of the eigenvalues of G

Let $u = (u_1, u_2) \neq 0$ be an eigenfunction of G with eigenvalue λ, $\operatorname{Re}\lambda < 0$, and let $f = u_1|_\Gamma$. Then from (2.2) we deduce $(-\Delta + \lambda^2)u_1 = 0$ and $\partial_\nu u_1 - \lambda\gamma u_1 = 0$ on Γ. Setting

$$\lambda = \frac{\mathbf{i}\sqrt{z}}{h},\ 0 < h \ll 1,$$

for $z \in Z_1 \cup Z_2 \cup Z_3$, one obtains the problem

$$\begin{cases}(-h^2\Delta - z)u_1 = 0 \text{ in } \Omega,\\ \mathscr{N}_{ext}(z,h)f - \sqrt{z}\gamma f = 0 \text{ on } \Gamma.\end{cases}$$

Consider the case (A) and notice that there exists $\varepsilon_0 > 0$ such tat

$$0 < \varepsilon_0 \leq \gamma(x) \leq 1 - \varepsilon_0,\ \forall x \in \Gamma.$$

We will discuss the case $z \in Z_1$, the case $z \in Z_2 \cup Z_3$ is more simple. According to Proposition 1 for $\mathscr{N}_{ext}(z,h)$, for $z \in Z_1$, $1 \geq \operatorname{Im} z \geq h^\delta$, $\delta = 1/2 - \varepsilon$, we have

$$\|\mathrm{Op}_h(\rho)f - \gamma\sqrt{z}f\|_{L^2(\Gamma)} \leq C\frac{h}{\sqrt{|\operatorname{Im} z|}}\|f\|_{L^2(\Gamma)}, \tag{4.1}$$

while for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\operatorname{Im} z|$ replaced by 1. Consider the symbol

$$c(x',\xi',z) = \rho(x',\xi',z) - \gamma\sqrt{z} = \frac{(1-\gamma^2)z - r_0(x',\xi')}{\rho(x',\xi',z) + \gamma\sqrt{z}}.$$

We will show that $c(x',\xi',z)$ is elliptic in a suitable class.

Clearly, c is elliptic for $|\xi'|$ large enough. So it remains to examine the behavior of c for $|\xi'| \leq C_0$ and for these values of ξ' we have $|\rho + \gamma\sqrt{z}| \leq C_1$. Introduce the set

$$\mathscr{F} = \{(x',\xi') : |1 - r_0(x',\xi')| \leq \frac{\varepsilon_0^2}{2}\}.$$

Then $\operatorname{Re}\Big((1-\gamma^2)z - r_0\Big) = 1 - r_0 - \gamma^2 \leq -\frac{\varepsilon_0^2}{2}$. If $(x',\xi') \notin \mathscr{F}$, we get

$$\operatorname{Im}\Big((1-\gamma^2)z - r_0\Big) = (1-\gamma^2)\operatorname{Im} z \geq (1-\gamma^2)h^\delta \geq \varepsilon_1 h^\delta,\ \varepsilon_1 > 0.$$

Consequently, the symbol c is elliptic and

$$\operatorname{Im}(\rho + \gamma\sqrt{z}) = \operatorname{Im}\rho + \gamma \operatorname{Im}\sqrt{z} \geq Ch^\delta.$$

Thus, for bounded $|\xi'|$ we have $|c| \geq C_3 h^\delta, C_3 > 0$, while for large $|\xi'|$ we have $|c| \sim |\xi'|$. Introduce the function

$$\chi(x',\xi') = \phi(\delta_0 r_0(x',\xi')),\ 0 < \delta_0 \leq 1/2$$

and define $\mathscr{M}_1 := Z_1 \times \operatorname{supp}\chi$, $\mathscr{M}_2 := (Z_1 \times \operatorname{supp}(1-\chi)) \cup ((Z_2 \cup Z_3) \times T^*\Gamma)$. Set $\langle\xi'\rangle = (1 + |\xi'|)^{1/2}$. It is easy to see that for $(z,x',\xi') \in \mathscr{M}_1$, $\operatorname{Im} z \neq 0$, we have

$$\left|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho\right| \leq C_{\alpha,\beta}|\operatorname{Im} z|^{1/2-|\alpha|-|\beta|},\ |\alpha| + |\beta| \geq 1, \tag{4.2}$$

$|\rho| \leq C$, while for $(z,x',\xi') \in \mathscr{M}_2$ we have

$$\left|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\rho\right| \leq C_{\alpha,\beta}\langle\xi'\rangle^{1-|\beta|}. \tag{4.3}$$

Thus, we conclude that $c = (\rho - \gamma\sqrt{z}) \in S_{\delta}^{0,1}$. A similar analysis shows that $|\operatorname{Im} z|c^{-1} \in S_{\delta}^{0,-1}$, while for $z \in Z_2 \cup Z_3$ we get $c^{-1} \in S_{\delta}^{0,-1}$. Therefore

$$\|\mathrm{Op}_h(c^{-1})g\|_{L^2(\Gamma)} \leq C|\operatorname{Im} z|^{-1}\|g\|_{L^2(\Gamma)}$$

and we deduce

$$\|\mathrm{Op}_h(c^{-1})\mathrm{Op}_h(c)f\|_{L^2(\Gamma)} \leq C_1\frac{h}{|\operatorname{Im} z|^{3/2}}\|f\|_{L^2(\Gamma)}.$$

A more fine analysis (see [13]) shows that

$$\|\mathrm{Op}_h(c^{-1})\mathrm{Op}_h(c)f - f\|_{L^2(\Gamma)} \leq C_2\frac{h}{|\operatorname{Im} z|^2}\|f\|_{L^2(\Gamma)}.$$

Consequently, one concludes that

$$\|f\|_{L^2(\Gamma)} \leq C_3\left(h^{1-2\delta} + h^{1-\frac{3}{2}\delta}\right)\|f\|_{L^2(\Gamma)}. \tag{4.4}$$

Since $\delta = 1/2 - \varepsilon$, $0 < \varepsilon \ll 1$, from (4.4) it follows $f = 0$ for $0 < h \leq h_0(\varepsilon)$ small enough. Since $-h^2\Delta$ with Dirichlet boundary conditions does not have eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, one gets $u_1 = 0$. Going back to the eigenvalues and using the scaling, one obtains that in the case (A) the eigenvalues of G lie in the region

$$\Lambda_{\varepsilon} = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_{\varepsilon}(|\operatorname{Im} z|^{\frac{1}{2}+\varepsilon} + 1), \operatorname{Re} z < 0\}.$$

In the case (B) the above analysis works only for $z \in Z_1 \cup Z_3$. Indeed for $z \in Z_1$ we have

$$\operatorname{Re}((1-\gamma^2) - r_0) \leq (1-\gamma^2) < -\eta_0 < 0$$

and again $c \in S_{\delta}^{0,1}, c^{-1} \in S_{\delta}^{0,-1}$. Thus for $z \in Z_1 \cup Z_3$ we obtain that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ must lie in Λ_{ε}. For $z \in Z_2$ the argument, exploited in the case (A), breaks down since for $\operatorname{Re} z = -1$, $\operatorname{Im} z = 0$ the symbol

$$1 + r_0(x', \xi') - \gamma(x')$$

is not elliptic and it may vanish for some (x'_0, ξ'_0).

Let $z = -1 + \mathbf{i}\operatorname{Im} z \in Z_2$. For such z we have an better approximation $T(z,h)$ of the operator $\mathcal{N}_{ext}(z,h)$ (see [21], [13]) for which we have

$$\|\mathcal{N}_{ext}(z,h)f - T(z,h)f\|_{H^1(\Gamma)} \leq C_N h^{-s_d+N}\|f\|_{L^2(\Gamma)}, \forall N \in \mathbb{N}, \tag{4.5}$$

with $s_d > 0$ depending only on the dimension d. Therefore, if f is related to the trace of an eigenfunction of G, from the equality $\mathcal{N}_{ext}(z,h)f - \gamma\sqrt{z}f = 0$ on Γ we obtain

$$|\operatorname{Re}(T(z,h)f-\gamma\sqrt{z}f,f)_{L^2(\Gamma)}|\leq C_N h^{-s_d+N}\|f\|_{L^2(\Gamma)}.$$

Next, by applying Taylor formula, we write

$$\begin{aligned}\operatorname{Re}\Big((T(z,h)-\gamma\sqrt{z})f,f\Big)_{L^2(\Gamma)}&=\operatorname{Re}\Big((T(-1,h)-\mathbf{i}\gamma)f,f\Big)_{L^2(\Gamma)}\\&\quad-\operatorname{Im}z\operatorname{Im}\Big(\Big[\frac{\partial T}{\partial z}(z_t,h)-\gamma\frac{1}{2\sqrt{z_t}}\Big]f,f\Big)_{L^2(\Gamma)}\end{aligned}\tag{4.6}$$

with $z_t=-1+\mathbf{i}t\operatorname{Im}z\in Z_2$, $0<t<1$. We may replace in (4.6) the operator $\frac{\partial T}{\partial z}(z_t,h)$ by the operator $\operatorname{Op}_h(\frac{d\rho}{dz}(z_t,h))$ modulo $\mathscr{O}(h)\|f\|^2_{L^2(\Gamma)}$ term and a sharp analysis shows that

$$\operatorname{Im}\Big(\Big(\operatorname{Op}_h(\frac{d\rho}{dz}(z_t,h))-\gamma\frac{1}{2\sqrt{z_t}}\Big)f,f\Big)_{L^2(L^2(\Gamma))}\geq\alpha_0\|f\|^2_{L^2(\Gamma)},\ \alpha_0>0.$$

We refer to [13] for the details of this argument. Combining (4.5) and (4.6), one estimates $|\operatorname{Im}z|$ and for small h and every $N\in\mathbb{N}$, we obtain that the eigenvalues $\lambda=\frac{\mathbf{i}\sqrt{z}}{h}$ of G with $z\in Z_2$ must lie in the region

$$\mathscr{R}_N=\{z\in\mathbb{C}:\ |\operatorname{Im}z|\leq C_N(|\operatorname{Re}z|+1)^{-N},\ \operatorname{Re}z<0\}.$$

Finally, we have the following

Theorem 1 ([13]). *In the case (A) for every ε, $0<\varepsilon\ll1$, the eigenvalues of G lie in the region Λ_ε. In the case (B) for every ε, $0<\varepsilon\ll1$, and every $N\in\mathbb{N}$ the eigenvalues of G lie in the region $\Lambda_\varepsilon\cup\mathscr{R}_N$.*

For *strictly convex* obstacles K we have a more precise result concerning the operator $\mathscr{N}_{out}(z,h)$ based on the construction of a semi-classical parametrix for the problem (3.1) when $\operatorname{Re}z=1$ and $h^{1/2-\varepsilon}\geq\operatorname{Im}z\geq h^{2/3}$ (see [22], [13]) or $0<\operatorname{Im}z\leq h^{2/3}$ (see [20]). This makes possible to improve the above result in the case (B) and to obtain the following

Theorem 2 ([13]). *In the case (B) for every $N\in\mathbb{N}$ outside the region $\mathscr{R}_N$ we have only finite number eigenvalues of the generator G.*

Moreover, we have the following

Proposition 2 ([5]). *Assume that d is odd. Then the operator G has no a sequence of eigenvalues λ_j, $\operatorname{Re}\lambda_j<0$ such that $\lim_{j\to\infty}\lambda_j=\mathbf{i}z_0$, $z_0\in\mathbb{R}$.*

It is world noting that the Dirichlet-to-Neumann map can be used to establish the discreetness of the spectrum of G in $\{z\in\mathbb{C}:\operatorname{Re}z<0\}$. We follow below the argument of [13]. For $\operatorname{Re}\lambda<0$ introduce the map

$$\mathscr{N}(\lambda):H^s(\Gamma)\ni f\longrightarrow\partial_\nu u|_\Gamma\in H^{s-1}(\Gamma),$$

where u is the solution of the problem

$$\begin{cases} (\Delta - \lambda^2)u = 0 \text{ in } \Omega,\, u \in H^2(\Omega), \\ u = f \text{ on } \Gamma. \end{cases} \tag{4.7}$$

It is well known that $\mathscr{N}(\lambda)$ is a meromorphic function in $\mathbb{C}$ for d odd and in the logarithmic covering of $\mathbb{C}$ for d even and the poles of $\mathscr{N}(\lambda)$ in $\mathbb{C} \setminus \{0\}$ coincide with the resonances of the Dirichlet problem for the Laplacian (see for instance, [20]). On the other hand, $u \in H^2(\Omega)$ implies that u is $\lambda-$incoming in the sense of Lax and Phillips (see Chapter IV in [10]). Notice that the definition of outgoing/incoming solutions in [20] is different from that in [10] and the resonances in [20] lie in $\text{Im}\, z < 0$, while in [10] they are in the half-plan $\text{Im}\, z > 0$. Consequently, $\mathscr{N}(\lambda)$ is analytic for $\text{Re}\, \lambda < 0$. The same is true for the Neumann problem for the $\Delta - \lambda^2$, hence $\mathscr{N}^{-1}(\lambda)$ is also analytic for $\text{Re}\, \lambda < 0$ and the poles of $\mathscr{N}^{-1}(\lambda)$ are the resonances of the Neumann problem ([19]). Therefore, the boundary condition in (2.2) may be written as follows

$$\mathscr{N}(\lambda)\Big(I - \lambda \mathscr{N}^{-1}(\lambda)\gamma\Big) f_1 = 0,\, \text{Re}\, \lambda < 0,\, x \in \Gamma.$$

The operator $\mathscr{N}(\lambda) : L^2(\Gamma) \longrightarrow H^1(\Gamma)$ is compact and Theorem 1 guarantees that there are points λ_0, $\text{Re}\, \lambda_0 < 0$, for which $(I - \lambda_0 \mathscr{N}^{-1}(\lambda_0)\gamma)$ is invertible. Applying the analytic Fredholm theorem, we conclude that the spectrum of G in $\{z \in \mathbb{C} : \text{Re}\, \lambda < 0\}$ is formed by isolated eigenvalues with finite multiplicities.

We finish this section by a trace formula involving the operator

$$C(\lambda) := \mathscr{N}(\lambda) - \lambda\gamma = \mathscr{N}(\lambda)\Big(I - \lambda \mathscr{N}^{-1}(\lambda)\gamma\Big),$$

which is an analytic operator-valued function in $\{z \in \mathbb{C} :\ \text{Re}\, z < 0\}$, while $C(\lambda)^{-1}$ is meromorphic in the same domain. We wish to find a formula for the trace

$$\text{tr}\, \frac{1}{2\pi i} \int_\delta (\lambda - G)^{-1} d\lambda, \tag{4.8}$$

where $\omega \subset \{\text{Re}\, z < 0\}$ has as a boundary the curve δ and $(G - \lambda)^{-1}$ is analytic on δ. We know that $(G - \lambda)^{-1}$ is meromorphic in ω and if λ_0 is a pole of $(G - \lambda)^{-1}$, then the multiplicity of the eigenvalue λ_0 of G is given by

$$\text{mult}\,(\lambda_0) = \text{rank} \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varepsilon_0} (\lambda - G)^{-1} d\lambda,$$

with $\varepsilon_0 > 0$ small enough. Therefore, (4.8) is equal to the number of the eigenvalues of G in ω contented with their multiplicities.

Let $(u, w) = (G - \lambda)^{-1}(f, g)$. Then $w = \lambda u + f$ and setting $q = u|_\Gamma$, one gets

$$u = R_D(\lambda)(g + \lambda f) + K(\lambda)q.$$

Here $R_D(\lambda) = (\Delta_D - \lambda^2)^{-1}$ is the resolvent of the operator Δ_D with Dirichlet boundary conditions and $K(\lambda)$ satisfies

$$\begin{cases}(\Delta - \lambda^2)K(\lambda) = 0 \text{ in } \Omega, \\ K(\lambda) = Id \text{ on } \Gamma.\end{cases}$$

The boundary condition on Γ yields

$$\partial_\nu[R_D(\lambda)(g+\lambda f) + K(\lambda)q] - \gamma\lambda[R_D(\lambda)(g+\lambda f) + K(\lambda)q] - \gamma f = 0, \, x \in \Gamma$$

and the term $\gamma\lambda[R_D(\lambda)(g+\lambda f)$ vanishes. Since $\mathscr{N}(\lambda) = \partial_\nu K(\lambda)|_\Gamma$ is the Dirichlet-to-Neumann map, assuming that $C^{-1}(\lambda)$ is invertible, we deduce

$$q = C^{-1}(\lambda)\Big([\partial_\nu R_D(\lambda)(g+\lambda f)] - \gamma f\Big).$$

Therefore

$$u = \Big[\lambda R_D(\lambda) + K(\lambda)C^{-1}(\lambda)\lambda\partial_\nu R_D(\lambda) - \gamma\Big]f + Xg,$$

$$w = Yf + \Big[\lambda R_D(\lambda) + \lambda K(\lambda)C^{-1}(\lambda)\partial_\nu R_D(\lambda)\Big]g,$$

where the operators X and Y are not important for the calculus of the trace. Thus we are going to study the integral

$$\operatorname{tr}\int_\delta \Big(2\lambda K(\lambda)C^{-1}(\lambda)\partial_\nu R_D(\lambda) - C^{-1}(\lambda)\gamma\Big)d\lambda.$$

For the first term we apply the cyclicity of the trace and the fact that

$$\frac{\partial\mathscr{N}}{\partial\lambda}(\lambda) = \partial_\nu\frac{\partial K}{\partial\lambda}(\lambda) = 2\lambda\partial_\nu R_D(\lambda)K(\lambda).$$

Finally, we obtain the following

Proposition 3 ([13]). *Let $\delta \subset \{z \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ be a closed positively oriented curve and let ω be the domain bounded by δ. Assume that $C^{-1}(\lambda)$ is meromorphic in ω without poles on δ . Then*

$$\operatorname{tr}\frac{1}{2\pi i}\int_\delta (\lambda - G)^{-1}d\lambda = \operatorname{tr}\frac{1}{2\pi i}\int_\delta C^{-1}(\lambda)\frac{\partial C}{\partial\lambda}(\lambda)d\lambda. \tag{4.9}$$

The idea to write the right-hand side of (4.9) as the trace of an integral involving the product of a meromorphic function $T^{-1}(\lambda)$ and its derivative $\frac{dT}{d\lambda}(\lambda)$ going back to [19], [4] (see also Proposition 3 in the next section). We expect that in the case (B) Proposition 3 combined with the techniques in [19] will imply a Weyl formula for the eigenvalues of G lying in $\mathscr{R}_N$.

We conjecture that for N large enough and $\gamma(x) > 1, \forall x \in \Gamma$, the counting function

$$N(r) = \#\{\lambda_j \in \sigma_p(G) : |\lambda_j| \le r, \, \lambda_j \in \mathscr{R}_N\}$$

has the asymptotic

$$N(r) = (2\pi)^{-d+1}\omega_{d-1}\Big(\int_{\Gamma}(\gamma^2(y') - 1)^{(d-1)/2} dy'\Big) r^{d-1} + \mathscr{O}_{\gamma}(r^{d-2}),\ r \geq r_0(\gamma), \tag{4.10}$$

where $\omega_{d-1} = \mathrm{vol}\,\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}$. For strictly convex obstacles and $\gamma(x) > 1$ this will imply a Weyl asymptotics of all eigenvalues of G. Notice that for ball B_3 we have the following

Proposition 4 ([13]). *For $\gamma \equiv const > 1$ and $K = B_3$ all eigenvalues λ_j of G are real and they lie in the interval $(-\infty, -\frac{1}{\gamma-1}]$. Moreover, there is an infinite number of real eigenvalues of G.*

Hence in this case we must study the asymptotic of $N(r)$ for $r \geq -\frac{1}{\gamma-1} = r_0(\gamma)$. Moreover, following the analysis in [13], we may prove that (4.10) holds for $K = B_3$ and constant γ.

By a similar argument we may study the eigenvalues of the generator G of the contraction semigroup associated to Maxwell's equations with dissipative boundary conditions

$$\begin{aligned} &\partial_t E = \mathrm{curl}\, B,\ \partial_t B = -\mathrm{curl}\, E \text{ in } \mathbb{R}_t^+ \times \Omega, \\ &E_{tan} - \gamma(x)(\nu(x) \wedge B_{tan}) = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ &E(0,x) = e_0(x),\ B(0,x) = b_0(x). \end{aligned} \tag{4.11}$$

The solution of the problem (4.11) is given by a contraction semigroup

$$(E,B) = V(t)f = e^{tG_b} f,\ t \geq 0,$$

where the generator G_b has domain $D(G_b)$ that is the closure in the graph norm of functions $u = (v,w) \in (C^{\infty}_{(0)}(\mathbb{R}^3))^3 \times (C^{\infty}_{(0)}(\mathbb{R}^3))^3$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ. Here $u_{tan} = u - \langle u, \nu\rangle \nu$. For Maxwell's equations for $0 < \gamma(x) < 1$ and $\gamma(x) > 1$ we have the same location of eigenvalues of G_b. This location has been examined in [6] by a semi-classical analysis of a h-pseudo-differential system on the boundary Γ. We have the following

Theorem 3 ([6]). *Assume that for all $x \in \Gamma$, $\gamma(x) \neq 1$. Then for every $0 < \varepsilon \ll 1$ and every $N \in \mathbb{N}$ there are constants $C_\varepsilon > 0$ and $C_N > 0$ such that the eigenvalues of G_b lie in the region $\Lambda_\varepsilon \cup \mathscr{R}_N$, where*

$$\Lambda_\varepsilon = \{z \in \mathbb{C} :\ |\mathrm{Re}\, z| \leq C_\varepsilon(|\mathrm{Im}\, z|^{1/2+\varepsilon} + 1),\ \mathrm{Re}\, z < 0\},$$

$$\mathscr{R}_N = \{z \in \mathbb{C} :\ |\mathrm{Im}\, z| \leq C_N(|\mathrm{Re}\, z| + 1)^{-N},\ \mathrm{Re}\, z < 0\}.$$

It is interesting to notice that for Maxwell's equation if $\gamma(x) \equiv 1, \forall x \in \Gamma$, and $K = B_3$ is the unit ball in $\mathbb{R}^3$, then G_b has no eigenvalues (see [6] for other results concerning the case $\gamma = const$ and B_3).

5 Location and Weyl formula for the (ITE)

To examine the location of the (ITE), set $\lambda = \frac{z}{h^2}$, $z \in Z_1 \cup Z_2 \cup Z_3$. If λ is an (ITE) with eigenfunction (u,w), consider $u|_\Gamma = w|_\Gamma = f$. Introduce the Dirichlet-to-Neumann operators $\mathscr{N}_j = \mathscr{N}_j(z,h)$, $j = 1,2$ related to

$$\mathscr{P}_j(z,h) = -\frac{h^2}{n_j(x)}\nabla c_j(x)\nabla - z\frac{c_j(x)}{n_j(x)},\ j = 1,2.$$

The boundary condition in the problem (2.5) implies

$$c_1\mathscr{N}_1(z,h)f - c_2\mathscr{N}_2(z,h)f = 0.$$

As in the Section 3, one introduces normal geodesic coordinates (x_1,x') and define

$$\rho_j = \sqrt{z\frac{n_j(x)}{c_j(x')} - r_0(x',\xi')},\ j = 1,2$$

with $\mathrm{Im}\,\rho_j > 0$. Applying Proposition 1 for the operators $\mathscr{N}_j(x,h)$, we deduce

$$\|c_1\mathrm{Op}_h(\rho_1)f - c_2\mathrm{Op}_h(\rho_2)f\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\mathrm{Im}\,z|}}\|f\|_{L^2(\Gamma)}.$$

Below we discuss only the case $c_1(x) = c_2(x) \equiv 1$, $\forall x \in \Gamma$. Then we have a better estimate

$$\|\mathrm{Op}_h(\rho_1)f - \mathrm{Op}_h(\rho_2)f\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\mathrm{Im}\,z|}}\|f\|_{L^2(\Gamma)} \tag{5.1}$$

and we must invert the operator $\mathrm{Op}_h(\rho_1) - \mathrm{Op}_h(\rho_2)$. Writing

$$\rho_1 - \rho_1 = \frac{z(n_1(x') - n_2(x'))}{\rho_1 + \rho_2},$$

it is easy to see that $\rho_1 - \rho_2$ is elliptic and $(\rho_1 - \rho_2)^{-1} \in S_\delta^{0,-1}$ for $z \in Z_1$, while $(\rho_1 - \rho_2)^{-1} \in S^{0,-1}$ for $z \in Z_2 \cup Z_3$. For $\delta = 1/2 - \varepsilon < 1/2$ we may use the calculus of h-pseudo-differential operators and (5.1) implies, as in Section 4, $f = 0$. The latter yields $u = w = 0$. Returning to the eigenvalues $\lambda = \frac{z}{h^2}$, we get that the (ITE) lie in the domain Λ_+ defined below. The analysis of the general case when $c_j(x)$ are not equal to 1 is more complicated and we refer to [21] for the details. Thus we have the following

Theorem 4 ([21]). *Assume* (2.6) *fulfilled together with the condition*

$$c_1(x) = c_2(x),\ \partial_\nu c_1(x) = \partial_\nu c_2(x),\ \forall x \in \Gamma.$$

Then for every $0 < \varepsilon \ll 1$ *the (ITE) lie the region*

$$\Lambda_{+,\varepsilon} := \{z \in \mathbb{C} : \text{Re}\,\lambda \geq 0, |\text{Im}\,\lambda| \leq C_\varepsilon(\text{Re}\,\lambda + 1)^{3/4+\varepsilon}\}$$

and there are only a finite number (ITE) *with* Re $\lambda < 0$. *If* $c_1(x) \neq c_2(x), \forall x \in \Gamma$, *the* (ITE) *lie in*

$$\Lambda'_{+,\varepsilon} := \{z \in \mathbb{C} : \text{Re}\,\lambda \geq 0, |\text{Im}\,\lambda| \leq C_\varepsilon(\text{Re}\,\lambda + 1)^{4/5+\varepsilon}\}.$$

If $(c_1(x) - c_2(x))d(x) > 0, \forall x \in \Gamma$, *we have only a finite number* (ITE) *with* Re $\lambda < 0$. *Moreover, if we assume that* $(c_1(x) - c_2(x))d(x) < 0$, $\forall x \in \Gamma$, *then for* Re $\lambda \geq 0$ *the* (ITE) *are in* Λ_+, *while for* Re $\lambda < 0$ *and every* $N \geq 1$ *there exists* $C_N > 0$ *such that* (ITE) *lie in*

$$\mathscr{R}_N = \{\lambda \in \mathbb{C} : |\text{Im}\,\lambda| \leq C_N(|\text{Re}\,\lambda| + 1)^{-N}, \text{Re}\,\lambda \leq 0\}.$$

A weaker result in a partial case $n_1(x) \equiv 1, n_2(x) > 1$ in K with eigenvalues-free region

$$\{z \in \mathbb{C} : \text{Re}\,\lambda \geq 0, |\text{Im}\,\lambda| \geq C(\text{Re}\,\lambda + 1)^{24/25}\}$$

has been obtained in [8].

For strictly convex obstacles one may construct a parametrix for the problem (3.4) and Re$z = 1$, $h^{1/2-\varepsilon} \geq \text{Im}\,z \geq h^{1-\varepsilon}$ by using more complicated construction and exploiting the properties of the Airy function Ai(z) (see [22] for more details). This leads to the following improvement of Theorem 4.

Theorem 5 ([22]). *Assume* K *strictly convex, the condition* (2.6) *satisfied and* $c_1(x) = c_2(x), \partial_\nu c_1(x) = \partial_\nu c_2(x)$, $x \in \Gamma$. *Then for every* $\varepsilon > 0$ *the* (ITE) *lie in the region*

$$M_{+,\varepsilon} := \{z \in \mathbb{C} : \text{Re}\,\lambda \geq 0, |\text{Im}\,\lambda| \leq C_\varepsilon(\text{Re}\,\lambda + 1)^{1/2+\varepsilon}\}$$

and there are only a finite number (ITE) *with* Re $\lambda < 0$.

This results is almost optimal, since for the unit ball in $\mathbb{R}^d$ we have the following

Theorem 6 ([15]). *Let* $K = \{x \in \mathbb{R}^d : |x| \leq 1\}$, $d \geq 2$. *Suppose that the functions* c_j, n_j, $j = 1, 2$, *are constants everywhere in* K, $c_1 = c_2$, *and the condition* (2.6) *is satisfied. Then, there are no* (ITE) *in the region* $\mathscr{M}_{+,0}$

The case $d = 1$ and $K = \{x \in \mathbb{R} : |x| \leq 1\}$ has been previously examined in [18] and [16].

Now we pass to the Weyl formula for the counting function $N(r)$ of the (ITE) and introduce the coefficients

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_K \Big(\frac{n_j(x)}{c_j(x)}\Big)^{d/2} dx,\ j = 1, 2,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

In the anisotropic case $c_1(x) = 1, n_1(x) = 1, c_2(x) \neq 1, c_2(x)n_2(x) \neq 1, \forall x \in \bar{K}$, the asymptotics

$$N(r) \sim (\tau_1 + \tau_2) r^d,\, r \to +\infty. \tag{5.2}$$

has been obtained by Lakshatanov and Vainberg [9] under some additional assumptions which guarantee that the boundary problem is *parameter-elliptic.*

By the results of Agranovich and Vishik [1] for the closed operator $\mathscr{A}$ related to (2.5) outside every angle $D_\alpha = \{z \in \mathbb{C} :\, |\arg z| \leq \alpha\}$, we have only a finite number of (ITE) and the following estimate holds

$$\|(z - \mathscr{A})^{-1}\| \leq C_\alpha |z|^{-1},\, z \notin D_\alpha,\, |z| \gg 1.$$

The authors applied directly a result of Boimanov-Kostjuchenko [2] leading to (5.2).

The isotropic case $c_1(x) = c_2(x) = 1$, $\forall x \in \bar{K}$, $n_1(x) = 1$, $n_2(x) \neq 1$, $\forall x \in \Gamma$, is more difficult since the corresponding operator $\mathscr{A}$ has domain

$$D(\mathscr{A}) = \{(u,w) \in L^2(K) \times L^2(K) :\, \Delta u \in L^2(K),\, \Delta v \in L^2(K),$$

$$u - w = 0, \partial_\nu (u - w) = 0 \,\text{on}\, \Gamma\}.$$

Thus $D(A)$ is not included in $H^2(K)$, and the problem is not parameter-elliptic. In this case Robbiano [17] obtained (5.2) by establishing the asymptotics

$$\sum_j \frac{1}{|\lambda_j|^p + t} = \alpha t^{-1+\frac{d}{2p}} + o(t^{-1+\frac{d}{2p}}),\, t \to +\infty.$$

where $p \in \mathbb{N}$ is sufficiently large. An application of the Tauberian theorem of Hardy-Littlewood yields the result. By this argument one obtains a very week estimate for the remainder which can be estimated by the principal term divided by a logarithmic factor. To get better results, it is important to take into account *parabolic eigenvalues-free regions* and to apply different techniques which are not based on Tauberian theorems.

Theorem 7 ([14]). *Under the condition* (2.6)*, assume that there are no* (ITE) *in the region*

$$\{\lambda \in \mathbb{C} :\, |\mathrm{Im}\,\lambda| \geq C(|\mathrm{Re}\,\lambda| + 1)^{1-\frac{\kappa}{2}}\},\, C > 0, 0 < \kappa \leq 1. \tag{5.3}$$

Then for every $0 < \varepsilon \ll 1$ *we have the asymptotics*

$$N(r) = (\tau_1 + \tau_2) r^d + \mathscr{O}_\varepsilon(r^{d-\kappa+\varepsilon}),\, r \to +\infty. \tag{5.4}$$

- According to Theorem 4, for arbitrary obstacles and $c_1(x) = c_2(x)$, $\partial_\nu c_1(x) = \partial_\nu c_2(x), \forall x \in \Gamma$, we can take $\kappa = \frac{1}{2} - \varepsilon$ and we obtain a remainder $\mathscr{O}_\varepsilon(r^{d-1/2+\varepsilon})$.
- Taking into account Theorem 5, for strictly convex obstacles we choose $\kappa = 1 - \varepsilon$, $\forall \varepsilon$. Consequently, we have in this case a remainder $O_\varepsilon(r^{d-1+\varepsilon})$.
- The optimal result should be to have a eigenvalues-free region with $\kappa = 1$ as it was proved in [15], [18], [16] for the case when K is a ball and the functions c_j, n_j are constants. However, even with $\kappa = 1$, to obtain an optimal remainder $\mathscr{O}(r^{d-1})$ some extra work is needed and this is an interesting open problem.

The proof of Theorem 7 is long and technical. After a semi-classical scaling, the idea is to reduce the analysis of $N(r)$ to the trace of an integral involving the product of a meromorphic function $T^{-1}(\lambda)$ and its derivative $\frac{dT}{d\lambda}(\lambda)$ similar to Proposition 3. Set $Z = \{z \in \mathbb{C};\ \frac{1}{2} \leq |\operatorname{Re} z| \leq 3,\ |\operatorname{Im} z| \leq 1\}$ and consider for $z \in Z$ and $0 < h \ll 1$ the operator

$$hT(z/h^2) := c_1\mathcal{N}_1(z,h) - c_2\mathcal{N}_2(z,h),$$

where the DN-maps $N_j(z,h)$ are defined in the beginning of this section.

Let $G_D^{(j)}$, $j = 1,2$, be the Dirichlet self-adjoint realization of the operator $L_j := -n_j^{-1}\nabla c_j \nabla$ in the space $H_j = L^2(K, n_j(x)dx)$. Set $\mathscr{H} = H_1 \oplus H_2$ and let $R(\lambda)$ be the resolvent of the transmission boundary problem. We omit in the notation $j = 1,2$ and consider the operators

$$\mathcal{N}(z,h)Op_h(1-\chi)f = \tilde{\mathcal{N}}(z,h)f - \gamma_0 D_\nu (h^2 G_D - z)^{-1}\frac{c}{n}\mathrm{Op}_h(p)f,$$

$$F(z,h) = \mathcal{N}(z,h) - \tilde{\mathcal{N}}(z,h) = \mathcal{N}(z,h)\mathrm{Op}_h(\chi) - \gamma_0 D_\nu (h^2 G_D - z)^{-1}\frac{c}{n}\mathrm{Op}_h(p),$$

where $\chi(x',\xi') = \Phi(\delta_0 r_0(x',\xi'))$ with $\Phi(\sigma) = 1$ for $|\sigma| \leq 1$ and $\Phi(\sigma) = 0$ for $|\sigma| \geq 2$, while $0 < \delta_0 \ll 1$ is small enough. Here $\tilde{\mathcal{N}}(z,h)$ is the parametrix of the DN operator $\mathcal{N}(z,h)\mathrm{Op}_h(1-\chi)$ in the domain where $r_0(x',\xi') > \frac{1}{\delta_0}$ and p is some symbol having behavior $\mathscr{O}(h^N)$ with all its derivatives. The number N will be taken large enough and it depends only on the parametrix construction.

The operator $F(z,h)$ is meromorphic with values in the space of trace class operators and we denote by $\mu_j(F(z,h))$ its characteristic eigenvalues.

Lemma 1. *If z/h^2 does not belong to spec G_D, then for every integer $0 \leq m \leq N/4$ we have*

$$\mu_j(F(z,h)) \leq \frac{C}{\delta(z,h)}\left(hj^{1/(d-1)}\right)^{-2m},\ \forall j \in \mathbb{N},$$

where $\delta(z,h) := \min\{1, \operatorname{dist}\{z, \operatorname{spec} h^2 G_D\}\} > 0$ and $C > 0$ depends on m and N but is independent of z,h,j.

Let

$$T(\lambda) := \gamma_0 c_1 D_\nu K_1(\lambda) - \gamma_0 c_2 D_\nu K_2(\lambda),$$

where $K_j(\lambda)f = u$, and u is the solution of the problem

$$\begin{cases} \Big(L_j - \lambda\Big)u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma. \end{cases}.$$

Proposition 5. *Assume that $T(\lambda)^{-1}$ is a meromorphic function with residues of finite rank. Let $\delta \subset \mathbb{C}$ be a simple closed positively oriented curve which avoids the eigenvalues of $G_D^{(j)}$, $j = 1,2$, as well as the poles of $T(\lambda)^{-1}$. Then we have the identity*

$$-\mathrm{tr}_{\mathscr{H}}\,(2\pi i)^{-1}\int_\delta R(\lambda)d\lambda = \sum_{j=1}^{2}\mathrm{tr}_{H_j}\,(2\pi i)^{-1}\int_\delta(\lambda - G_D^{(j)})^{-1}d\lambda$$
$$-\mathrm{tr}_{L^2(\Gamma)}\,(2\pi i)^{-1}\int_\delta T(\lambda)^{-1}\frac{dT(\lambda)}{d\lambda}d\lambda. \tag{5.5}$$

Let us mention that if $R(\lambda)$ is an operator-valued meromorphic function with residues of finite rank, the multiplicity of a pole $\lambda_k \in \mathbb{C}$ of $R(\lambda)$ is defined by

$$\mathrm{mult}\,(\lambda_k) = -\mathrm{rank}\,(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\varepsilon} R(\lambda)d\lambda,\ 0<\varepsilon\ll 1.$$

On the other hand, the rank of the operator above is equal to the trace of this operator and on the left-hand side of (5.5) we have the sum of the mutiplicities of the (ITE) lying in the domain $\omega_\delta \subset \mathbb{C}$ bounded by δ. Clearly, the terms with $(\lambda - G_D^{(j)})^{-1}$ yield the sum of eigenvalues of $G_D^{(j)}$ in ω_δ counted with their multiplicities.

It is possible to construct invertible, bounded operator $E(z,h): H_h^s(\Gamma) \to H_h^{s+1}(\Gamma)$ with bounded inverse $E(z,h)^{-1}: H_h^s(\Gamma) \to H_h^{s-1}(\Gamma)$, $\forall s \in \mathbb{R}$, so that

$$hT(z/h^2) = E^{-1}(z,h)(I+\mathscr{K}(z,h)),$$
$$(hT(z/h^2))^{-1} = (I+\mathscr{K}(z,h))^{-1}E(z,h)$$

with a trace class operator

$$\mathscr{K}(z,h) = E(z,h)(c_1F_1(z,h) - c_2F_2(z,h)) + \mathscr{L}(z,h).$$

Moreover, the operators $E(z,h), E^{-1}(z,h)$, are holomorphic with respect to z in Z while $\mathscr{K}(z,h)$ is meromorphic operator-valued function in this region. Then

$$\mathrm{tr}\int_\delta T^{-1}(z/h^2)\frac{d}{dz}T(z/h^2)dz = \mathrm{tr}\int_\delta (I+\mathscr{K}(z,h))^{-1}\frac{d}{dz}\mathscr{K}(z,h)dz.$$

Set $g_h(z) := \det(I+\mathscr{K}(z,h))$ and denote by $M_\delta(h)$ the number of the poles $\{\lambda_k\}$ of $R(\lambda)$ such that $h^2\lambda_k$ are in ω_δ. Similarly, we denote by $M_\delta^{(j)}(h)$ the number of the eigenvalues ν_k of $G_D^{(j)}$ such that $h^2\nu_k \in \omega_\delta$. Then using the well-known formula

$$\mathrm{tr}\left[(I+\mathscr{K}(x,h))^{-1}\frac{\partial\mathscr{K}(z,h)}{\partial z}\right] = \frac{\partial}{\partial z}\log\det(I+\mathscr{K}(z,h)),$$

we get from (5.5) the following

Lemma 2. *Let $\delta \subset Z$ be closed positively oriented curve which avoid the eigenvalues of $h^2G_D^{(j)}$, $j=1,2$ as well as the poles of $T(z/h^2)^{-1}$. Then we have*

$$M_\delta(h) = M_\delta^{(1)}(h) + M_\delta^{(2)}(h) + \frac{1}{2\pi i}\int_\delta \frac{d}{dz}\log g_h(z)dz. \tag{5.6}$$

The leading term in (5.4) is obtained from the $M_\delta^{(1)}(h) + M_\delta^{(2)}(h)$ after a scaling. The crucial point is to examine the asymptotic of the integral involving $\log g_h(z)$. The details of this analysis are given in [14].

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