Acta Math., 166 (1991), 69–120

Effective Bezout identities in $\mathbf{Q}[z_1, \ldots, z_n]$

by

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1. Introduction

Let $p_1, \ldots, p_m \in \mathbf{Z}[z_1, \ldots, z_n] = \mathbf{Z}[z]$ without common zeros in $\mathbf{C}^n$. Hilbert's Nullstellensatz ensures that there is $\mathfrak{d} \in \mathbf{Z}^+$ and polynomials $q_1, \ldots, q_m \in \mathbf{Z}[z]$ such that for every $z \in \mathbf{C}^n$

$$\mathfrak{d} = p_1(z)\, q_1(z) + \ldots + p_m(z)\, q_m(z). \tag{1.1}$$

The explicit resolution of the Bezout equation (1.1) consists in giving an algorithm to find such polynomials $q_1, \ldots, q_m$. One such algorithm is due to G. Hermann [18] and Seidenberg [33]; another one, very effective, has been developed by Buchberger [12]. Masser–Wüstholz [28] used Hermann's method to estimate the degree and the size of the polynomials q_j, and the size of $\mathfrak{d}$. Denote by $h(P)$ the logarithmic size of a polynomial $P \in \mathbf{Z}[z]$, i.e., $h(P)$=the logarithm of the modulus of the coefficient of P of largest absolute value. They showed that using the Hermann algorithm one could find $q_1, \ldots, q_m$ satisfying:

$$\max(\deg q_j) \leq 2(2D)^{2^{n-1}}, \quad D = \max(\deg p_j) \tag{1.2}$$

$$\max(\log |\mathfrak{d}|, h(q_j)) \leq (8D)^{4\times 2^{n-1}-1} \cdot (h + 8D \log 8D), \quad h = \max h(p_j). \tag{1.3}$$

Recently, using a combination of methods from elimination theory and several complex variables, Brownawell [10] has obtained an essentially sharp bound for the degrees of polynomials q_j satisfying (1.1):

$$\max(\deg q_j) \leq \mu n D^{\mu} + \mu D, \quad \mu = \inf\{n, m\}. \tag{1.4}$$

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[1] This research has been supported in part by NSF Grant DMS-8703072 and by the AFOSR-URI Grant 870073.

Later on, Kollár [22] has succeeded in obtaining an even sharper bound using only algebraic methods:

$$\max(\deg q_j) \leqslant D^{\mu}, \tag{1.5}$$

with μ as in (1.4). To be completely correct, the inequalities (1.4) of Brownawell and (1.5) of Kollár are slightly more precise, we refer to the respective papers for the details. Later on we will state the precise version of the Nullstellensatz from [22, Corollary 1.7].

To be able to compare the nature of the algorithmic approach in [12] and the construction from [10], a word is necessary about Brownawell's polynomials q_j. (The polynomials in (1.5) are obtained by a non-constructive argument.) First one proves that there exist $q_j^* \in \mathbf{C}[z]$ satisfying the equation (1.1) with $\mathfrak{d}=1$, with degrees bounded as in (1.4). These q_j^* are obtained as integrals over the whole space $\mathbf{C}^n$ of some conveniently constructed kernels. In some sense we whould say the q_j^* are given by explicit formulas, but these formulas do not constitute an algorithm. One also obtains an upper bound for the absolute value of the coefficients of the q_j^*. This follows from the effective bounds for the constant c_1 appearing in the Lojasiewicz' type inequality [10], [30]

$$\left(\sum_{j=1}^{m} |p_j(z)|^2\right)^{1/2} \geqslant c_1(1+\|z\|)^{1-(n-1)D^n}. \tag{1.6}$$

Since the p_j have integral coefficients, the existence of q_j^* implies the existence of $\mathfrak{d} \in \mathbf{Z}^+$, $q_j \in \mathbf{Z}[z]$ satisfying (1.1) and (1.4); this is simply linear algebra. Namely, the equation (1.1) (with $\mathfrak{d}=1$) can be written as a system of linear equations with integral coefficients for the unknown rational coefficients of the q_j, once the degree of them has been estimated. Therefore, we might as well apply this principle with the estimates (1.5). One could then ask what is the size of $\mathfrak{d} \in \mathbf{Z}^+$ and of the polynomials $q_j \in \mathbf{Z}[z]$ obtained by solving this system of equations. Setting $\delta = D^{\mu}$, and using a lemma of Masser–Wüstholz ([28], Lemma 1, section 4) one obtains the estimate

$$\max(\log \mathfrak{d}, h(q_j)) \leqslant m\binom{n+\delta}{\delta}\left\{h+\log m+\log\binom{n+\delta}{\delta}\right\}. \tag{1.7}$$

For $m \geqslant n$ the order of magnitude of the right hand side of (1.7) is essentially

$$\frac{me^nD^{n^2}}{n^{n-(1/2)}}(h+\log m+n^2\log D). \tag{1.8}$$

Note that in special cases where a better estimate than (1.4) is possible, then this same Lefschetz' principle provides a better bound in (1.7). Such is the case studied by Macaulay [23], [25], when the polynomials $p_1, \dots, p_m$ have no common points at infinity. Then one can find q_j satisfying the estimate

$$\deg q_j \leq n(D-1). \tag{1.9}$$

The corresponding estimate for a and $h(q_j)$ is essentially

$$\max(\log \mathfrak{d}, h(q_j)) \leq mn^n D^n (h + \log m + n \log n + n \log D). \tag{1.10}$$

As soon as there is even a single common point at ∞ for $p_1, \dots, p_m$, the estimate (1.9) is false. This is precisely the situation for the example of Masser–Philippon in [10]

$$p_1 = z_1^D, \quad p_2 = z_1 - z_2^D, \quad \dots, \quad p_{n-1} = z_{n-2} - z_{n-1}^D, \quad p_n = 1 - z_{n-1} z_n^{D-1}, \tag{1.11}$$

for which the best estimate possible for $\deg q_j$ is $D^n - D^{n-1}$. This example shows that (1.5) is practically best possible (cf. [22] for an optimal version).

One of the objectives of this paper is to obtain a better bound than (1.8) for the size of $\mathfrak{d}$ and the q_j. The idea is to use that the choice of q_j is not unique and that by losing a little bit in the estimate of the degrees of q_j, $\varkappa_1 n^2 D^n$ instead of D^n, the size estimate is basically (1.8) where D^{n^2} is replaced by $D^{\varkappa_2 n}$ ($\varkappa_1, \varkappa_2$ absoslute constants), see Theorem 5.1 below.

Our method also depends on complex function theory, except that we have succeeded in obtaining by this method a solution q_j, $\mathfrak{d}$ lying directly in $\mathbf{Z}[z]$, $\mathbf{Z}$ respectively. $\mathbf{Z}$ can be replaced by the ring of integers of any number field. The formulas we introduce can also be used to study the question of finding a division formula in $\mathbf{C}[z]$ as we have done elsewhere [7].

The interest of sharp estimates for the degree and size of the polynomials q_j appearing in the Nullstellensatz lies in applications to Transcendental Number Theory and Computational Geometry. For the last application, it would seem that the algorithms of Buchberger type can be modified to take into account estimates of degree and size (see [13]). Purely algebraic methods appear to be able to improve bounds obtained by analytic methods as well as give insight into the algorithmic questions. Such has been the case in the period between the two versions of this paper, and we have certainly profited from the work of Kollár [22], Ji, Kollár and B. Shiffman [21], and Philippon [32] that appeared between August 1987 and now. One particularly simple and tantalizing question which we would like to pose is finding the sharp estimate for the number of arithmetical operations needed to decide whether a system of n quadratic equations (with integral coefficients) in n variables has or does not have a solution in $\mathbf{C}^n$ (or in $\mathbf{R}^n$).

Apart from the intrinsic interest of the result obtained here, we would like to point out the power of the explicit integral representation formulas of the Henkin type, even when dealing with problems that are algebraic in nature. Another feature of this paper is the crucial role played by multidimensional residues, used as a tool in computations and not purely as an abstract concept, as they had been used essentially until now (see also [1] and [7]).

This paper was written while C. Berenstein was on a sabbatical leave supported by the General Research Board of the University of Maryland and A. Yger was a visiting professor in that institution. An announcement of the results herein appeared in [9].

We would like to thank Dale Brownawell, Patrice Philippon and Bernard Shiffman for many useful remarks.

§ 2. Residue currents

We incorporate in this section some results of Complex Analysis which form the basis for the rest of the paper. We start by fixing some notation that will be used throughout.

Let $f=(f_1,\dots,f_n)$ be a $\mathbf{C}^n$-valued function, $m\in\mathbf{N}^n$ a multi-index of length $|m|=m_1+\dots+m_n$. For an integer $p\in\mathbf{N}^*$ we let $\underline{p}=(p,\dots,p)$. Then we denote

$$f^m=f_1^{m_1}\dots f_n^{m_n},\quad F=f_1\dots f_n,\quad \|f\|=\left(\sum|f_j|^2\right)^{1/2}$$

$$\partial f=\partial f_1\wedge\dots\wedge\partial f_n=\bigwedge_{j=1}^n\partial f_j,\quad \partial f_j=\sum_{k=1}^n\frac{\partial f_j}{\partial z_k}dz_k$$

$$\bar\partial f=\bar\partial f_1\wedge\dots\wedge\bar\partial f_n=\bigwedge_{j=1}^n\bar\partial f_j,\quad \bar\partial f_j=\sum_{k=1}^n\frac{\partial f_j}{\partial \bar z_k}d\bar z_k \tag{2.1}$$

$$df=df_1\wedge\dots\wedge df_n,\quad df_j=\partial f_j+\bar\partial f_j,$$

where $\partial/\partial z_k$, $\partial/\partial\bar z_k$ are the standard first order complex derivative operators [17], [20], and the functions f_j are continuously differentiable. Note that $dz=dz_1\wedge\dots\wedge dz_n$ and $d\bar z=d\bar z_1\wedge\dots\wedge d\bar z_n$ are particular cases of (2.1). Also note $\bigwedge_{j=1}^n$ is always understood in increasing order.

If Q is a $(1,0)$ form, i.e. $Q(\zeta)=\sum_{j=1}^n Q_j(\zeta)\,d\zeta$, then $\bar\partial Q$ is a $(1,1)$ form, and there is no ambiguity in writing for $k \in \mathbf{N}$

$$(2.2) \qquad (\bar{\partial} Q)^k = \bar{\partial} Q \wedge \ldots \wedge \bar{\partial} Q \quad (k \text{ times})$$

since $(1, 1)$ forms commute for the wedge product ($(\bar{\partial} Q)^0 = 1$).

The space of differential forms of type (j, k) with smooth coefficients of compact support in $\mathbf{C}^n$ is denoted $\mathscr{D}_{j,k}$. $\varphi \in \mathscr{D}_{j,k}$ is called a test form. The dual space of $\mathscr{D}_{n-j,n-k}$, $\mathscr{D}'_{n-j,n-k}$, is called the space of currents of type (j, k). It can be identified to the space of differential forms of type (j, k) with coefficients in the space $\mathscr{D}'$ of distributions in $\mathbf{C}^n$ [24].

Given n entire holomorphic functions f_j defining a discrete variety $V = V(f)$, $V := \{z \in \mathbf{C}^n : f_1(z) = \ldots = f_n(z) = 0\}$, we can define the *Grothendieck residue current* $\bar{\partial}(1/f)$ as the current of type $(0, n)$ defined on test forms $\varphi \in \mathscr{D}_{n,0}$ by

$$(2.3) \qquad \left\langle \bar{\partial}\frac{1}{f}, \varphi \right\rangle = \lim_{\lambda \to 0} \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \lambda^n \int_{\mathbf{C}^n} |F|^{2(\lambda-1)} \bar{\partial} f \wedge \varphi,$$

where $F = f_1, \ldots, f_n$ and the meaning of the integral on the right hand side of (2.2) is the following. First, it is well defined as a holomorphic function of λ for $\operatorname{Re}\lambda > 1$. Then, the product $\lambda^n \int_{\mathbf{C}^n} |F|^{2(\lambda-1)} \bar{\partial} f \wedge \varphi$ can be analytically continued to the whole complex plane to become a meromorphic function of λ, which is holomorphic in a neighborhood of $\lambda = 0$. In fact, the limit in (2.3) is just the evaluation of this analytically continued function at $\lambda = 0$ (cf. [7]). This coincides with the usual definition of the Grothendieck residue current [15]. If we want to emphasize the components of f we will write

$$\bar{\partial}\frac{1}{f} = \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_n}.$$

In particular

$$\bar{\partial}\frac{1}{f^m} = \bar{\partial}\frac{1}{f_1^{m_1}} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_n^{m_n}}.$$

Note there is no contradiction between this notation and (2.1). If the holomorphic function f_j is such that $1/f_j$ is differentiable, it means that f_j has no zeros. Therefore the usual differential form $\bar{\partial}(1/f_j) = 0$, but the Grothendieck residue will also be zero since $V = \varnothing$. Furthermore, this observation holds in a local sense also, that is, if $\operatorname{supp}\varphi \cap V = \varnothing$ we have $\langle \bar{\partial}(1/f), \varphi \rangle = 0$.

If $f_1, \ldots, f_n$ are polynomials defining a discrete (hence finite) variety V and if h is a function which is C^∞ in a neighborhood of V we can define the action of $\bar{\partial}(1/f)$ on the form $h\,dz$ by

$$\left\langle \bar\partial \frac{1}{f}, h\,dz \right\rangle := \left\langle \bar\partial \frac{1}{f}, \varphi h\,dz \right\rangle,$$

where $\varphi \in \mathcal{D}$, $\varphi=1$ on a (small) neighborhood of V. When h is actually holomorphic in a neighborhood of V then

$$\left\langle \bar\partial \frac{1}{f}, h\,dz \right\rangle := \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^n} \int_{|f|=\varepsilon} h(z) \frac{dz}{f_1(z)\dots f_n(z)} = \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^n} \int_{|f|=\varepsilon} h \frac{dz}{F}, \tag{2.4}$$

where $\{|f|=\varepsilon\}$ is the smooth cycle $\{z\in \mathbf{C}^n: |f_j(z)|=\varepsilon, 1\leq j\leq n)\}$ defined (by Sard's theorem) for $0<\varepsilon$ outside a negligible set, and it is taken to be positively oriented (that is $d(\arg f_1)\wedge\dots\wedge d(\arg f_n)\geq 0$ on $|f|=\varepsilon$) (cf. [17], [37]). Furthermore, once $0<\varepsilon<<1$, the limit coincides with the integral over $\{|f|=\varepsilon\}$.

It follows from the fact that the current $\bar\partial(1/f)$ has support in V that for $\varphi\in\mathcal{D}$

$$\left\langle \bar\partial \frac{1}{f}, \varphi\,dz \right\rangle = \left(\sum_{\zeta\in V} \sum_{\alpha} c_{\alpha,\zeta}\, \delta_\zeta^{(\alpha)} \right)(\varphi), \tag{2.5}$$

where the interior sum takes place over multi-indices α, $|\alpha|\leq N$, $c_{\alpha,\zeta}\in\mathbf{C}$. In case the point $\zeta\in V$ is a simple zero then $c_{\alpha,\zeta}=0$ for $\alpha\neq 0$ and $c_{0,\zeta}=1/J(\zeta)$, $J(\zeta)$=the determinant Jacobian $\partial(f_1\dots f_n)/\partial(z_1\dots z_n)$ at $z=\zeta$. More generally, we have the identity ([14], § 1.9) for $\varphi\in\mathcal{D}$:

$$\left\langle J\bar\partial \frac{1}{f}, \varphi\,dz \right\rangle = \left(\sum_{\zeta\in V} m_\zeta \delta_\zeta \right)(\varphi) = \sum_{\zeta\in V} m_\zeta \varphi(\zeta) \tag{2.6}$$

where m_ζ is the multiplicity of ζ as a common zero of $f_1,\dots,f_n$. Here we use the fact that a current can be multiplied by a smooth function g by the rule $\langle g\bar\partial(1/f),\varphi\rangle := \langle\bar\partial(1/f), g\varphi\rangle$. Note this multiplication will also make sense if g is of class C^N in a neighborhood of V, N the integer from (2.5). We remark that in (2.5) the only derivatives that appear are with respect to the variable ζ and not $\bar\zeta$ (cf. [14], [7]).

The identity (2.6) allows us to write Cauchy's formula in terms of residues. Namely, let $\varphi\in C_0^1(\mathbf{C}^n)$ and consider the functions $f_j(\zeta)=\zeta_j-z_j$, $j=1,\dots,n$, for $z\in\mathbf{C}^n$ fixed. Then we have

$$\left\langle \bar\partial \frac{1}{\zeta - z}, \varphi(\zeta)\,d\zeta \right\rangle = \varphi(z).$$

In fact this is a particular case of (2.6), where $V=\{z\}$, $m_z=1$, $J\equiv 1$.

Another property that will play a role is that

$$f_j \bar{\partial} \frac{1}{f} = 0, \quad j = 1, \ldots, n. \tag{2.8}$$

Therefore, $\bar{\partial}(1/f)$ vanishes on the C_0^∞-submodule of $\mathcal{D}_{(n,0)}$ generated by $f_1, \ldots, f_n$.

The three properties (2.5) (conveniently modified), (2.6), and (2.8) hold also for entire functions f_j.

LEMMA 2.1. *Let K be a subfield of $\mathbf{C}$, $f_1, \ldots, f_n \in K[z]$ defining a discrete variety V, $g \in K[z]$. Then*

$$\sum_{\zeta \in V} m_\zeta g(\zeta) \in K.$$

Proof. By (2.6) we have

$$\sum_{\zeta \in V} m_\zeta g(\zeta) = \left\langle J \bar{\partial} \frac{1}{f}, g\, dz \right\rangle.$$

By elimination theory [36] there are polynomials $q_1, \ldots, q_n \in K[z]$, q_j a polynomial depending only on the jth variable such that

$$q_k = \sum_{j=1}^{n} h_{k,j} f_j, \quad h_{k,j} \in K[z]. \tag{2.9}$$

Let us denote $\Delta = \det(h_{k,j})_{k,j}$. The transformation law for the residue states that for any function g smooth in $\mathbf{C}^n$ one has:

$$\left\langle \bar{\partial} \frac{1}{f}, g\, dz \right\rangle = \left\langle \Delta \bar{\partial} \frac{1}{q}, g\, dz \right\rangle,$$

(cf. [7, Proposition 2.5]). In particular

$$\sum m_\zeta g(\zeta) = \left\langle \bar{\partial} \frac{1}{q}, \Delta J g\, dz \right\rangle.$$

To finish the proof, it is enough to show that for any monomial $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, we have $\langle \bar{\partial}(1/q), z^\alpha dz \rangle \in K$. To compute this value we can apply (2.4):

$$\left\langle \bar{\partial} \frac{1}{q}, z^\alpha dz \right\rangle = \lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^n} \int_{|q| = \varepsilon} z^\alpha \frac{dz}{q_1 \cdots q_n}$$

$$= \prod_{j=1}^{n} \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|q_j|=\varepsilon} z_j^{\alpha_j} \frac{dz_j}{q_j(z_j)}$$

$$= \prod_{j=1}^{n} \left(\sum_{q_j(\beta)=0} \operatorname{res}_{\beta} t^{\alpha_j}/q_j(t) \right),$$

where $\operatorname{res}_\beta h$ denotes the usual one variable residue of the function h at the point β. The easiest way to compute the inner sums is to recall that, for rational functions of one variable, the sum of the residues over all the poles plus the point at ∞ is zero. Therefore

$$\sum_{q_j(\beta)=0} \operatorname{res}_{\beta} t^{\alpha_j}/q_j(t) = -\operatorname{res}_{\infty} t^{\alpha_j}/q_j(t) = a_{-1},$$

where $t^{\alpha_j}/q_j(t) = a_l t^l + \ldots + a_0 + a_{-1}/t + a_{-2}/t^2 + \ldots$ in a neighborhood of ∞. The coefficients a_k are rational linear combinations of the coefficients of q_j. Hence each sum is in K. □

COROLLARY 2.2. *Let K be a number field of degree e, $f_1, \ldots, f_n, g$ as in Lemma* 2.1. *Let $\zeta_0 = (\zeta_1^0, \ldots, \zeta_n^0) \in V$, then $g(\zeta_0)$ is an algebraic number of degree $\leq e(\Sigma_{\zeta \in V} m_\zeta)$. If* $\max_j \deg f_j = D$ *then the degree of $g(\zeta_0) \leq eD^n$.*

Proof. Let $M = \Sigma_{\zeta \in V} m_\zeta$=total number of finite zeros of $f_1, \ldots, f_n$, and denote $\zeta_1, \ldots, \zeta_M$ these zeros, each repeated according to its multiplicity. Then the polynomial $\Pi_{j=1}^{M}(x - g(\zeta_j))$ has coefficients in K. In fact, the symmetric functions of $g(\zeta_j)$ can be written as rational combinations of the elementary symmetric functions (Newton sums) [36], i.e., as rational combinations of

$$\sum_{j=1}^{M} g(\zeta_j)^p = \sum_{\zeta \in V} m_\zeta (g(\zeta))^p \in K$$

by Lemma 2.1. The last statement follows from Bezout's theorem.

LEMMA 2.3. *Let $K, f_1, \ldots, f_n$ as in Lemma* 2.1. *Let $r \in K(z)$ without any poles on V, then $\langle \bar\partial(1/f), r\, dz \rangle \in K$.*

Proof. Let $q_1, \ldots, q_n$ be the same as in the proof of Lemma 2.1. Let $r = g/p$, g, p coprime polynomials in $K[z]$, $V(p, f_1, \ldots, f_n) = \varnothing$. The difficulty in carrying over the proof as in Lemma 2.1 consists in that p could vanish on some points of $V(q_1, \ldots, q_n) \setminus V$. (In the application of the transformation law for the residue one had to assume h was globally smooth, it would be enough to know it is smooth in a neighborhood of

$V(q_1, \dots, q_n)$ but if r has a pole there we cannot apply that formula.) We first show we can in fact assume this is not the case.

Let N be the integer defined by (2.5) and consider the polynomial

$$P = \lambda_0 p + \lambda_1 f_1^{N+1} + \dots + \lambda_n f_n^{N+1}. \tag{2.10}$$

By Lemma 1 from ([28], section 4), we can choose $\lambda_0, \dots, \lambda_n \in \mathbf{Z}$ such that P does not vanish on $V(q_1, \dots, q_n)$. In particular $\lambda_0 \neq 0$. Therefore we can set $\lambda_0 = 1$ and $\lambda_1, \dots, \lambda_n \in \mathbf{Q}$. From (2.5) it follows now that

$$\left\langle \bar\partial \frac{1}{f}, \frac{g}{p} dz \right\rangle = \left\langle \bar\partial \frac{1}{f}, \frac{g}{P} dz \right\rangle$$

since g/p and g/P coincide and have the same derivatives up to order N at each point of V.

Since we are now assuming that r has no poles on $V(q_1, \dots, q_n)$ we have, as in Lemma 2.1,

$$\left\langle \bar\partial \frac{1}{f}, r\, dz \right\rangle = \left\langle \bar\partial \frac{1}{q}, \Delta r\, dz \right\rangle.$$

This time Δr is a rational function, hence we cannot reduce ourselves to the case of monomials as in Lemma 2.1. To overcome this difficulty let us factorize each q_j in $K[t]$ into irreducible factors:

$$q_j = q_{j,1}^{n_1} \cdots q_{j,s}^{n_s}, \quad q_{j,k} \in K[t], \; s = s(j), \; n_k \in \mathbf{Z}^+. \tag{2.10}$$

From (2.4) we can take $0 < \varepsilon << 1$ so that $\Delta(z) r(z)$ is holomorphic in $\{|q_j| \leq \varepsilon, 1 \leq j \leq n\} = \{|q| \leq \varepsilon\}$ and

$$\left\langle \bar\partial \frac{1}{q}, \Delta r\, dz \right\rangle = \frac{1}{(2\pi i)^n} \int_{|q|=\varepsilon} \frac{\Delta(z)\, r(z)\, dz}{q_1(z_1) \cdots q_n(z_n)}.$$

This integral can be computed one variable at a time. Fixing z', $z' = (z_2, \dots, z_n)$, we have

$$\frac{1}{2\pi i} \int_{|q_1(z_1)| = \varepsilon} \frac{h(z_1, z')}{q_1(z_1)} dz_1 = \sum_{k=1}^{s(1)} \left(\sum_{q_{1,k}(a)=0} \operatorname*{res}_{z_1=a} \frac{h(z_1, z')/[q_1(z_1)/q_{1,k}^{n_k}(z_1)]}{(q_{1,k}(z_1))^{n_k}} \right). \tag{2.11}$$

Fix k, let $\nu = n_k$, $Q = q_{1,k}$, A = numerator in the interior sum of (2.11). The zeros of Q are all simple, let them be $\alpha_1, \dots, \alpha_\mu$. We can factorize $Q(t)$ as follows:

$$Q(t) = (t - \alpha_1)(Q'(\alpha_1) + \dots) = (t - \alpha_1) R_1(t),$$

$R_1(t)$ is a polynomial in t with coefficients in $K[\alpha_1]$. For a different root α_j we will have $Q(t)=(t-\alpha_j)R_j(t)$, where the coefficients of R_j are obtained by replacing α_1 to α_j everywhere in the computation of R_1. The function A is holomorphic at $t=\alpha_1,\dots,\alpha_\mu$, since the different irreducible factors of q_1 have no common zeros. Therefore $A(t)/Q(t)$ has a pole of order exactly ν at $t=\alpha_1$.

$$\operatorname*{res}_{t=\alpha_1}\frac{A(t)}{(Q(t))^\nu}=\frac{1}{(\nu-1)!}\frac{d^{\nu-1}}{dt^{\nu-1}}\frac{A(t)}{(R_1(t))^\nu}\bigg|_{t=\alpha_1}. \tag{2.12}$$

This expression is now a rational expression in α_1 (and z') with coefficients in K, such that the residue at $t=\alpha_j$ is obtained simply by replacing α_1 by α_j everywhere. Therefore

$$\sum_{q_{1,k}(\alpha)=0}\operatorname*{res}_{\alpha}\frac{A(t)}{(q_{1,k}(t))^{n_k}}$$

is a rational function in $K(z')$. Furthermore, we note that the portion of the denominator of $A(t)$ which depends on z' is $p(t,z')$. The expression (2.12) will have a common denominator which is $p(\alpha_1,z')^\nu$. Hence the inner sum of (2.11) has no poles for z' a zero of the product $q_2(z_2)\dots q_n(z_n)$. The same thing holds therefore for the expression (2.11). Now we can iterate the procedure and conclude that $\langle\bar\partial(1/q),\Delta r\,dz\rangle\in K$. Hence $\langle\bar\partial(1/f),r\,dz\rangle\in K$. □

Remark 2.4. Later on we will need a quantitative version of the fact that $\langle\bar\partial(1/f),r dz\rangle\in K$. For this purpose we will use the local character of the residue current $\bar\partial(1/f)$. That is by using a partition of unity $\{\varphi_\zeta\}$ we have

$$\langle\bar\partial(1/f),r\,dz\rangle=\Sigma_{\zeta\in V}\langle\bar\partial(1/f),\varphi_\zeta r\,dz\rangle,$$

$\varphi_\zeta\equiv1$ near ζ. We can further assume that ζ is the only zero of $V(q_1,\dots,q_n)$ lying in the support of φ_ζ and that r is holomorphic on $\operatorname{supp}\varphi_\zeta$. Therefore for each term of this sum we can apply the transformation law for residues without changing r at all, i.e.,

$$\left\langle\bar\partial\frac{1}{f},r\,dz\right\rangle=\sum_{\zeta\in V}\left\langle\bar\partial\frac{1}{q},\Delta r\varphi_\zeta\,dz\right\rangle=\left\langle\bar\partial\frac{1}{q},\Delta r\,dz\right\rangle_V, \tag{2.13}$$

where we have introduced the last notation to indicate it is only the points of V that count. Note there are many less points in V than in $V(q_1,\dots,q_n)$. In the first case one has at most D^n points, while in the second one might have as many as D^{n^2} points.

In Section 3 we will need the following result from [7]:

THEOREM 2.5 (cf. [7, Proposition 2.4]). *Let $f_1,\dots,f_n$ be n polynomials in $\mathbf{C}^n$ defining a discrete variety V, φ a test function, m an n-tuple of non-negative integers. Then the function defined for* $\operatorname{Re}\lambda$ *sufficiently large by*

$$\lambda \mapsto \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} n\lambda \int \frac{|F|^{2(n+|m|)\lambda}}{\|f\|^{2(n+|m|)}} \bar{f}^m\, \overline{\partial f} \wedge \varphi\, d\zeta \tag{2.14}$$

has an analytic continuation to the whole plane as a meromorphic fuction. Moreover, this continuation is holomorphic at $\lambda=0$ and its value at this point is given by

$$\frac{m!}{(n+|m|)!}\left\langle \bar{\partial}\frac{1}{f^{m+1}}, \varphi\, d\zeta \right\rangle, \tag{2.15}$$

where $m!=m_1!\dots m_n!$, $m+\underline{1}=(m_1+1,\dots,m_n+1)$.

§ 3. Division formulas

The division formula we obtain here generalizes our previous representation formulas for solutions of the algebraic Bezout equation. We had originally considered them from the point of view of deconvolution (cf. [3], [5], [6]). The same techniques can be applied to entire functions, but to simplify we will only consider the algebraic case [7].

Throughout this section we will assume we have M polynomials $p_1,\dots,p_M \in \mathbf{C}[z]$ such that

$$M \geqslant n, \tag{3.1}$$

and that the first n satisfy the following property:

$$\exists \varkappa>0,\ c>0 \text{ and } d>0 \text{ such that when } \|\zeta\|\geqslant\varkappa \text{ we have} \quad \left(\sum_{j=1}^{n}|p_j(\zeta)|^2\right)^{1/2} \geqslant c\|\zeta\|^d. \tag{3.2}$$

Since the first n polynomials play a special role, it is convenient to adopt the notation $f=(f_1,\dots,f_n)=(p_1,\dots,p_n)$, hence (3.2) can be written as $\|f(\zeta)\|\geqslant c\|\zeta\|^d$ and it implies that the variety $V=V(f)$ is discrete. We also let

$$\max_{1\leqslant j\leqslant n} (\deg f_j) = D. \tag{3.3}$$

For every polynomial p_j $(1\leqslant j\leqslant M)$ we can find polynomials $g_{j,k}$ of $2n$ variables, with degree $\leq \deg p_j$ in each variable, such that for every $z, \zeta \in \mathbf{C}^n$ we have

$$p_j(z)-p_j(\zeta)=\sum_{k=1}^{n} g_{j,k}(z,\zeta)(z_k-\zeta_k). \tag{3.4}$$

For instance, we can take

$$g_{j,k}(z,\zeta)=\frac{p_j(\zeta_1,\dots,\zeta_{k-1},z_k,\dots,z_n)-p_j(\zeta_1,\dots,\zeta_k,z_{k+1},\dots,z_n)}{z_k-\zeta_k}.$$

If $p_j \in \mathbf{Z}[z]$, $\deg p_j = D_j$, then $g_{j,k} \in \mathbf{Z}[z,\zeta]$, $\deg g_{j,k} \leq D_j$ and $h(g_{j,k}) \leq h(p_j)+2n\log(D_j+1)$.

THEOREM 3.1. *Assume* (3.1) *and* (3.2) *hold. Let P be a polynomial in $I(p_1,\dots,p_M)$ and let $u_1,\dots,u_M$ be any functions holomorphic in a neighborhood Ω of V such that*

$$P=u_1p_1+\dots+u_Mp_M \quad \text{in } \Omega. \tag{3.5}$$

Then for $q \in \mathbf{N}$ satisfying

$$dq \geq \deg P+(n-1)(2D-d)+1, \tag{3.6}$$

and, for any $z \in \mathbf{C}^n$ we have

$$P(z)=\sum_{|m|\leq q-n}\left\langle \bar\partial\frac{1}{f^{m+1}}, \sum_{j=1}^{M} u_j \begin{vmatrix} g_{1,1}(z,\cdot) & \dots & g_{n,1}(z,\cdot) & g_{j,1}(z,\cdot) \\ \vdots & & & \\ g_{1,n}(z,\cdot) & \dots & g_{n,n}(z,\cdot) & g_{j,n}(z,\cdot) \\ f_1(z)-f_1(\cdot) & \dots & f_n(z)-f_n(\cdot) & p_j(z) \end{vmatrix} d\zeta \right\rangle f^m(z) \tag{3.7}$$

where $m \in \mathbf{N}^n$, $m+\underline{1}=(m_1+1,m_2+1,\dots,m_n+1)$, and the dot in the determinant represents the variable ζ on which the residue current $\bar\partial(1/f^{m+1})$ acts.

Remark 3.2. (i) The only term in the sum (3.7) that a priori might not belong to $I(p_1,\dots,p_M)$ is that one corresponding to $m=(0,\dots,0)$. In that case the development of the determinants along the last row shows that either one has a multiple of $p_j(z)$ for some j, $1\leq j\leq M$, or a multiple of $f_j(\zeta)$ for some j, $1\leq j\leq n$. This last type of term vanishes since $\bar\partial(1/f)$ annihilates the ideal generated by the f_j. Therefore (3.7) has the form

$$P(z)=A_1(z)p_1(z)+\dots+A_m(z)p_M(z).$$

(ii) In the case $M=n+1$ and $V(p_1,\dots,p_M)=\varnothing$ this theorem improves upon Theorem 3 [6] and its applications in [3].

(iii) Note that the conditions (3.5) and $P \in I(p_1, \ldots, p_M)$ are equivalent by Cartan's Theorem B [20].

Example 3.3. Let $M=n+1$, $V(p_1, \ldots, p_{n+1})=\emptyset$, $p_j \in \mathbf{Z}[z]$. For $P=1$ we can take $u_1=\ldots=u_n=0$, $u_{n+1}=1/p_{n+1}$. In that case Lemma 2.3 implies that (3.7) gives a Bezout formula in $\mathbf{Q}[z]$, that is

$$1 = p_1(z)A_1(z)+\ldots+p_{n+1}(z)A_{n+1}(z)$$

with $A_j \in \mathbf{Q}[z]$. Note that the result remains true if $\mathbf{Q}$ is replaced by a number field K and $\mathbf{Z}$ by the field of integers $\mathcal{O}_K$ of K.

Proof of Theorem 3.1. The germ of the idea of this proof goes back to our papers on deconvolution [5], [6] except that here we have to deal inevitably with multiple zeros in V. In the recent past we have found that the best way to treat this question is through the principle of analytic continuation of the distributions $|f|^{2m\lambda}$ as functions of λ [7]. We also use the recent work of Andersson–Passare on integral representation formulas [2].

Let us fix once and for all $\vartheta \in \mathcal{D}(\Omega)$, $\vartheta=1$ in a neighborhood of V.

Let $\varrho>1$ so that $\Omega_\varrho=\{\zeta \in \mathbf{C}^n: \|\zeta\|<\varrho\} \supseteq \operatorname{supp}\vartheta \cup \{z\}$. Let $\chi \in \mathcal{D}(\Omega_\varrho)$ such that $\chi=1$ in a neighborhood of $\operatorname{supp}\vartheta \cup \{z\}$, $0 \leq \chi \leq 1$.

Consider the differential form $Q_0=Q_0(z, \zeta)$ given by

$$Q_0 := (1-\chi(\zeta)) \frac{\sum_{j=1}^n (\bar{\zeta}_j-\bar{z}_j)\, d\zeta_j}{\|\zeta-z\|^2}. \tag{3.8}$$

If ω is an open set such that $z \in \omega$ and $\chi=1$ on ω then Q_0 is C^∞ in $\omega \times \mathbf{C}^n$. Let

$$\Gamma_0(t) = (1+t)^N, \tag{3.9}$$

with N any integer $>n$.

For $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda > 1+1/n$, let $Q_1=Q_1(z, \zeta, \lambda)$ be the differential form (with the notation of (2.1)):

$$Q_1 := |F(\zeta)|^{2\lambda} \frac{\sum_{j=1}^n \bar{f}_j(\zeta)\, G_j}{\|f(\zeta)\|^2}, \tag{3.10}$$

where the differential forms $G_j=G_j(z, \zeta)$ are given by

$$G_j := \sum_{k=1}^n g_{j,k}\, d\zeta_k. \tag{3.11}$$

The coefficients of G_j are therefore polynomials in z and ζ. Q_1 is of class C^1 and a polynomial in z. If we let $\mathrm{Re}\,\lambda >> 1$, we can make Q_1 of class C^l for any l given. With q as in (3.6) let

$$\Gamma_1(t) = (1+t)^q. \tag{3.12}$$

Finally, define a third differential form $Q_2 = Q_2(z, \zeta)$ by

$$Q_2 := \vartheta(\zeta) \sum_{j=1}^{M} u_j(\zeta) G_j \tag{3.13}$$

and let

$$\Gamma_2(t) := t. \tag{3.14}$$

These three differential forms are of type $(1,0)$ in ζ, hence they can be associated to $\mathbf{C}^n$-valued functions, simply take the coefficient of $d\zeta_j$ as its jth component. Using their bilinear products with the vector valued function $z-\zeta$ we can construct three auxiliary functions Φ_j. We have

$$\Phi_0 := \langle Q_0(z,\zeta), z-\zeta \rangle = \frac{(1-\chi(\zeta))}{\|\zeta - z\|^2} \sum_{j=1}^{n} (\bar{\zeta}_j - \bar{z}_j)(z_j - \zeta_j) = \chi(\zeta) - 1, \tag{3.15}$$

$$\begin{aligned} \Phi_1 := \langle Q_1(z,\zeta,\lambda), z-\zeta \rangle &= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \bar{f}_j(\zeta) \langle G_j, z-\zeta \rangle \\ &= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \bar{f}_j(\zeta) \left(\sum_{k=1}^{n} g_{j,k}(z,\zeta)(z_j - \zeta_j) \right) \\ &= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \bar{f}_j(\zeta)(f_j(z) - f_j(\zeta)), \end{aligned} \tag{3.16}$$

by (3.3).

The last one is given by

$$\Phi_2 := \langle Q_2(z,\zeta), z-\zeta \rangle + P(\zeta) = \vartheta(\zeta) \sum_{j=1}^{M} u_j(\zeta)(p_j(z) - p_j(\zeta)) + P(\zeta). \tag{3.17}$$

Note that in a neighborhood of V we have $\Phi_2 = \Sigma_{j=1}^{M} u_j(\zeta) p_j(z)$.

As a function of ζ consider the product

$$\zeta \mapsto \varphi = \Gamma_0(\Phi_0)\Gamma_1(\Phi_1)\Gamma_2(\Phi_2), \tag{3.18}$$

for z fixed and λ fixed, $\operatorname{Re}\lambda \gg 1$, this is a C^{n+1} function of compact support since $\Gamma_0(\Phi_0) = \chi(\zeta)^N$. Furthermore

$$\varphi(z) = P(z). \tag{3.19}$$

We need one more piece of notation: for $0 \leqslant j \leqslant 2$, and α a non-negative integer denote

$$\Gamma_j^{(\alpha)} = \Gamma_j^{(\alpha)}(z, \zeta) := \frac{d^\alpha}{dt^\alpha} \Gamma_j(t) \bigg|_{t=\Phi_j(z,\zeta)}. \tag{3.20}$$

(Recall that Φ_1 depends also on λ.)

The following lemma will allow us to compute $P(z)$ with the help of Cauchy's formula (2.7) applied to φ (cf. [2]). Its proof will be postponed to the end of the proof of Theorem 3.1.

LEMMA 3.2. *With the above notation we have, for* $\operatorname{Re}\lambda \gg 1$,

$$\begin{aligned} P(z) = {} & \frac{1}{(2\pi i)^n} \int_{\Omega_\varrho} \Phi_2(z,\zeta) \sum_{\alpha_0+\alpha_1=n} \frac{\Gamma_0^{(\alpha_0)} \Gamma_1^{(\alpha_1)}}{\alpha_0!\,\alpha_1!} (\bar\partial_\zeta Q_0(z,\zeta))^{\alpha_0} \wedge (\bar\partial_\zeta Q_1(z,\zeta,\lambda))^{\alpha_1} \\ & + \frac{1}{(2\pi i)^n} \int_{\Omega_\varrho} \sum_{\alpha_0+\alpha_1=n-1} \frac{\Gamma_0^{(\alpha_0)} \Gamma_1^{(\alpha_1)}}{\alpha_0!\,\alpha_1!} (\bar\partial_\zeta Q_0(z,\zeta))^{\alpha_0} \wedge (\bar\partial_\zeta Q_1(z,\zeta,\lambda))^{\alpha_1} \wedge \bar\partial_\zeta Q_2(z,\zeta). \end{aligned} \tag{3.21}$$

The next step will be to study the analytic continuation of this formula as a function of λ. For that purpose, we compute explicitly $(\bar\partial_\zeta Q_1)^\alpha$, $1 \leqslant \alpha \leqslant n$, always for $\operatorname{Re}\lambda \gg 1$. To simplify we simply write $\bar\partial$ for $\bar\partial_\zeta$. Let us write first

$$A = \frac{\sum_{j=1}^n \bar f_j G_j}{\|f\|^2}, \quad Q_1 = |F|^{2\lambda} A.$$

Then

$$(\bar\partial Q_1)^k = |F|^{2\lambda k} (\bar\partial A)^k + \lambda k |F|^{2(k\lambda-1)} F\, \overline{\partial F} \wedge A \wedge (\bar\partial A)^{k-1}. \tag{3.22}$$

A is a C^∞ form off the variety V. The form $(\bar\partial Q_1)^n$ can be written in a slightly different way by denoting

$$Q_1 = \sum_{j=1}^n \psi_j G_j, \quad \psi_j = \frac{|F|^{2\lambda}}{\|f\|^2} \bar f_j.$$

Then

$$(\bar\partial Q_1)^n = \left(\sum_{j=1}^{n} \bar\partial\psi_j \wedge G_j\right)^n = (-1)^{(n-1)n/2} n! \bigwedge_{j=1}^{n} \bar\partial\psi_j \wedge \bigwedge_{j=1}^{n} G_j.$$

We have

$$\bar\partial\psi_j = \frac{|F|^{2\lambda}}{\|f\|^2}\overline{\partial f_j} + \bar f_j\left(\frac{\lambda|F|^{2(\lambda-1)}}{\|f\|^2} F\,\overline{\partial F} - \frac{|F|^{2\lambda}}{\|f\|^4}\bar\partial\|f\|^2\right).$$

Hence

$$\bigwedge_{j=1}^{n} \bar\partial\psi_j = \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\overline{\partial f} + \lambda\frac{|F|^{2(\lambda n-1)}}{\|f\|^{2n}} F \sum_{j=1}^{n} \bar f_j \bigwedge_{k<j} \overline{\partial f_k} \wedge \overline{\partial F} \wedge \bigwedge_{j<k} \overline{\partial f_k}$$
$$- \frac{|F|^{2\lambda n}}{\|f\|^{2(n+1)}} \sum_{j=1}^{n} \bar f_j \bigwedge_{k<j} \overline{\partial f_k} \wedge \bar\partial\|f\|^2 \wedge \bigwedge_{j<k} \overline{\partial f_k}.$$

Note that $\overline{\partial F} = \Sigma_{k=1}^{n} (\bar F/\bar f_k)\overline{\partial f_k}$ and $\bar\partial\|f\|^2 = \Sigma f_k\overline{\partial f_k}$. Since $\overline{\partial f_k}\wedge\overline{\partial f_k} = 0$ we have

$$\bigwedge_{j=1}^{n} \bar\partial\psi_j = \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\overline{\partial f} + n\lambda\frac{|F|^{2\lambda n}}{\|f\|^{2n}}\overline{\partial f} - \frac{|F|^{2\lambda n}}{\|f\|^{2(n+1)}}\|f\|^2\,\overline{\partial f}$$
$$= n\lambda\frac{|F|^{2\lambda n}}{\|f\|^{2n}}\overline{\partial f}. \tag{3.23}$$

$$(\bar\partial Q_1)^n = (-1)^{(n-1)n/2} n!\, n\lambda \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\overline{\partial f} \wedge \bigwedge_{j=1}^{n} G_j. \tag{3.24}$$

Following the principle we introduced in [5] we have to transform, using Stokes' theorem, some terms in (3.21) to make them more singular. In this case we apply this idea to the second term of (3.21), the term with $\alpha_0=0$, $\alpha_1=n-1$. Then

$$\bar\partial(\Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge Q_2) = \Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge\bar\partial Q_2$$
$$+\Gamma_0^{(1)}\Gamma_1^{(n-1)}\bar\partial\chi\wedge(\bar\partial Q_1)^{n-1}\wedge Q_2 + \Gamma_0^{(0)}\Gamma_1^{(n)}\bar\partial\Phi_1\wedge(\bar\partial Q_1)^{n-1}\wedge Q_2.$$

Recall that $\Gamma_0^{(0)}$ has compact support in Ω_ϱ and that $\bar\partial\chi\wedge Q_2=0$, since $\chi=1$ on $\operatorname{supp}\vartheta$. Therefore

$$\int_{\Omega_\varrho} \Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge\bar\partial Q_2 = -\int_{\Omega_\varrho} \Gamma_0^{(0)}\Gamma_1^{(n-1)}\bar\partial\Phi_1\wedge(\bar\partial Q_1)^{n-1}\wedge Q_2.$$

To simplify the computation of this last integral, let us introduce polynomials $\Delta_{j,l}$ $(1 \leq j \leq n, 1 \leq l \leq M)$, and Δ_0, by

$$G_1 \wedge \ldots \wedge \hat{G}_j \wedge \ldots \wedge G_n \wedge G_l = \Delta_{j,l}\, d\zeta, \tag{3.25}$$

and

$$G_1 \wedge \ldots \wedge G_n = \Delta_0\, d\zeta. \tag{3.26}$$

Note that $\Delta_{j,j} = (-1)^{n-j}\Delta_0$.

Now we can compute the integrand above as follows:

$$\begin{aligned}
-\bar\partial\Phi_1 \wedge (\bar\partial Q_1)^{n-1} \wedge Q_2 &= -\left(\sum_{j=1}^{n} (f_j(z)-f_j(\zeta))\,\bar\partial\psi_j\right) \wedge \left(\sum_{1}^{n} \bar\partial\psi_j \wedge G_j\right)^{n-1} \wedge Q_2 \\
&= (-1)^{(n-2)(n-1)/2}(n-1)!\left(\bigwedge_{j=1}^{n} \bar\partial\psi_j\right) \wedge \sum_{j=1}^{n} (-1)^j (f_j(z)-f_j(\zeta)) \left(\bigwedge_{\substack{k=1\\k\neq j}}^{n} G_k\right) \wedge Q_2 \\
&= (-1)^{(n-2)(n-1)/2} n!\,\lambda \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\,\overline{\partial f} \\
&\quad \wedge \left(\sum_{j=1}^{n}\sum_{l=1}^{M} (-1)^j (f_j(z)-f_j(\zeta))\,\Delta_{j,l}(z,\zeta)\,\vartheta(\zeta)\,u_l(\zeta)\right) d\zeta.
\end{aligned}$$

Recall that we have already computed in (3.24) the term with $\alpha_0=0$ in the first integral of (3.21). Let us write now (3.21) as a sum of the contributions from $\alpha_0=0$ in both integrals and the other terms put together:

$$\begin{aligned}
P(z) &= \frac{(-1)^{(n-1)n/2}}{(2\pi i)^n} n\lambda \int_{\Omega_\varrho} \Gamma_0^{(0)}\Gamma_1^{(n)} \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\,\overline{\partial f} \\
&\quad \wedge \left[\Phi_2\Delta_0 + (-1)^{n-1}\vartheta\left(\sum_{j,l} (-1)^j (f_j(z)-f_j)\,\Delta_{j,l}\,u_l\right)\right] d\zeta + R(\lambda, z).
\end{aligned} \tag{3.27}$$

Let us call $T(z,\zeta)$ the term between brackets in (3.27). Let us show that in the set where $\vartheta=1$ this term is exactly the determinant that appears in the final formula (3.7). First we observe that since on $\operatorname{supp}\vartheta$ we have $P(\zeta)=\sum_{j=1}^{M} u_j(\zeta)\,p_j(\zeta)$ then

$$\Phi_2(z,\zeta) = \sum_{j=1}^{M} u_j(\zeta)\,p_j(z).$$

Now we can expand the determinants in (3.7) by the last row and obtain $T(z, \zeta)$. This function is therefore holomorphic on ζ in a neighborhood of V.

To evaluate (3.27), we will use the fact that both terms are holomorphic functions of λ for $\operatorname{Re}\lambda >> 1$ and that they have analytic continuations to the whole plane as meromorphic functions. We will further see that they are both holomorphic at $\lambda=0$, hence $P(z)$ will appear as

$$\lim_{\lambda\to 0} R(\lambda, z)+\lim_{\lambda\to 0} \text{(of the first term in (3.27))}.$$

We proceed now to verify these statements for the first term of (3.27). We have

$$\begin{aligned}\Gamma_1^{(n)} &= \frac{q!}{(q-n)!}(\Phi_1+1)^{q-n} = \frac{q!}{(q-n)!}\left((1-|F|^{2\lambda})+\sum_{j=1}^{n}\psi_j(\zeta)f_j(z)\right)^{q-n}\\ &= \frac{q!}{(q-n)!}\sum_{k=0}^{q-n}\binom{q-n}{k}(1-|F|^{2\lambda})^{q-n-k}\frac{|F|^{2\lambda k}}{\|f\|^{2k}}\sum_{|m|=k}\frac{k!}{m!}\bar{f}^m(\zeta)f^m(z)\\ &= \frac{q!}{(q-n)!}\sum_{k=0}^{q-n}\binom{q-n}{k}\frac{|F|^{2\lambda k}}{\|f\|^{2k}}\left(\sum_{j=0}^{q-n-k}\binom{q-n-k}{j}(-1)^j|F|^{2\lambda j}\right)\left(\sum_{|m|=k}\frac{k!}{m!}\bar{f}^m(\zeta)f^m(z)\right).\end{aligned}$$

In order to apply Theorem 2.5, we fix a k, a multi-index m, $|m|=k$, and an index j in the expansion of $\Gamma_1^{(n)}$. The corresponding term in (3.27) is then, up to a factor $f^m(z)$,

$$\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}n\lambda\, c_{k,m,j}\int_{\Omega_\varphi}\frac{|F|^{2\lambda(j+k+n)}}{\|f\|^{2(n+|m|)}}\bar{f}^m\,\bar{\partial}\bar{f}\wedge(\chi^N T)\,d\zeta, \tag{3.28}$$

where

$$c_{k,m,j}=(-1)^j\frac{q!}{(q-n)!}\frac{k!}{m!}\binom{q-n}{k}\binom{q-n-k}{j}.$$

Replacing λ by

$$\left(\frac{n+|m|}{j+k+n}\right)\lambda,$$

we are in the situation of (2.14) up to the new constant

$$c'_{k,m,j}=\left(\frac{n+k}{n+k+j}\right)c_{k,m,j}.$$

(Note $\chi=1$ in a neighborhood of V, hence $\chi^N T$ is holomorphic there.) Therefore, by Theorem 2.5, the analytic continuation exists, it is holomorphic at $\lambda=0$ and its value at this point is

$$c'_{k,m,j}\frac{m!}{(n+|m|)!}\left\langle \bar\partial \frac{1}{f^{m+1}}, \chi^N T\, d\zeta \right\rangle. \tag{3.29}$$

Note that the value in (3.29) is independent of the choice of χ. We need to evaluate the constant obtained by adding over all values of j.

$$\begin{aligned} c_{k,m} &= \sum_{j=0}^{q-n-k} c'_{k,m,j} = \sum_{j=0}^{q-n-k}\left(\frac{n+k}{n+k+j}\right) c_{k,m,j} \\ &= \frac{q!}{(q-n)!}\frac{k!}{m!}\binom{q-n}{k}\sum_{j=0}^{q-n-k}(-1)^j\binom{q-n-k}{j}\frac{m+k}{n+k+j}. \end{aligned} \tag{3.30}$$

This sum can be computed in terms of the beta function. Namely,

$$\begin{aligned} \sum_{j=0}^{q-n-k}(-1)^j\binom{q-n-k}{j}\frac{1}{n+k+j} &= \int_0^1 (1-u)^{q-n-k}u^{n+k-1}\,du \\ &= B(n+k, q-n-k+1) = \frac{(n+k-1)!\,(q-n-k)!}{q!}. \end{aligned}$$

We find

$$\frac{m!}{(n+|m|)!}c_{k,m} = 1. \tag{3.31}$$

Therefore, the value at $\lambda=0$ of the first term in (3.27) is exactly the right hand side of (3.7). We stress once more that the value we obtained is independent of the choice of χ.

To end the proof we need to study the analytic continuation of $R(\lambda, z)$ and evaluate it at $\lambda=0$. We assume first that $\operatorname{Re}\lambda > 1+1/n$. In $R(\lambda, z)$ we have all terms (3.21) where $\alpha_0>0$. Introducing the auxiliary differential forms

$$S = \sum_{j=1}^{n}(\bar\zeta_j - \bar z_j)\,d\zeta_j, \quad \bar S = \sum_{j=1}^{n}(\zeta_j - z_j)\,d\bar\zeta_j$$

we have

$$\bar\partial Q_0 = -\frac{\bar\partial\chi\wedge S}{\|\zeta - z\|^2} + (1-\chi)\left(\frac{\sum_{j=1}^n d\bar\zeta_j\wedge d\zeta_j}{\|\zeta-z\|^2} - \frac{\bar S\wedge S}{\|\zeta-z\|^4}\right). \tag{3.32}$$

This shows that $\bar\partial Q_0$ is identically zero in a neighborhood of $\operatorname{supp}\vartheta\cup\{z\}$ by the conditions imposed on χ. Since there is a factor ϑ in Q_2, it follows that all the terms with $\alpha_0>0$ in the second integral of (3.21) are identically zero.

Consider now the term with $\alpha_0=n$ in the first integral. Let us rewrite

$$\Phi_1+1 = 1-|F|^{2\lambda}+|F|^{2\lambda}\sum_{j=1}^{n}\theta_j f_j(z) = 1-|F|^{2\lambda}+|F|^{2\lambda}B, \tag{3.33}$$

where $\theta_j=\theta_j(\zeta)=\bar{f}_j(\zeta)/\|f(\zeta)\|^2$. On the support of $\bar{\partial}Q_0$ we have that B is C^∞, since $\chi=1$ on a neighborhood of the singular points of $\|f(\zeta)\|^{-2}$, namely V. Since F is a polynomial, it follows (for instance by the Weierstrass' Preparation Theorem or Hironaka's Resolution of Singularities) that on the ball $\bar{\Omega}_\varrho$ we have that $|F|^{-\varepsilon}$ is integrable for some $\varepsilon>0$. Whence, the term with $\alpha_0=n$, which is given by

$$\frac{1}{(2\pi i)^n}\binom{N}{n}\int_{\Omega_\varrho}\Phi_2\chi^{N-n}(1+\Phi_1)^q(\bar{\partial}Q_0)^n \tag{3.34}$$

for $\mathrm{Re}\,\lambda>1+1/n$, and depends on λ only in the factor $(1+\Phi_1)^q$, is holomorphic for $\mathrm{Re}\,\lambda>-\varepsilon$. Its value at $\lambda=0$ is obtained simply by taking $\lambda=0$ in the expression of Φ_1. That is, the value at $\lambda=0$ of (3.34) is

$$\frac{1}{(2\pi i)^n}\binom{N}{n}\int_{\Omega_\varrho}\Phi_2\chi^{N-n}B^q(\bar{\partial}Q_0)^n. \tag{3.35}$$

We now have left the case $0<\alpha_0<n$, $\alpha_1=n-\alpha_0$, to consider. By (3.22) we have $(\bar{\partial}Q_1)^{\alpha_1}$ as the sum of two terms. We study first the one that does not contain the factor λ. As we have just shown, A is smooth on the support of $\bar{\partial}Q_0$ and the whole integral is holomorphic for $\lambda=0$. Its value, obtained by simply setting $\lambda=0$, is the following

$$\frac{1}{(2\pi i)^n}\binom{N}{\alpha_0}\binom{q}{\alpha_1}\int_{\Omega_\varrho}\Phi_2\chi^{N-\alpha_0}B^{q-\alpha_1}(\bar{\partial}Q_0)^{\alpha_0}\wedge(\bar{\partial}A)^{\alpha_1}. \tag{3.36}$$

The other term can be written as a linear combination of integrals of the form

$$\lambda\int_{\Omega_\varrho}|F|^{2(r\lambda-1)}F\,\overline{\partial F}\wedge C,$$

r an integer $\geqslant\alpha_1$, C a smooth form of compact support. By Theorem 1.3 [7], this function has an analytic continuation as a meromorphic function of λ, whose value at $\lambda=0$ is, up to multiplicative constants

$$\left\langle\bar{\partial}\frac{1}{F}, FC\right\rangle$$

which is the residue on the hypersurface $F=0$. Since F divides the test form FC, this residue is zero.

At this point we can summarize what we have just done by saying that $\lambda \mapsto R(\lambda, z)$ has an analytic continuation which is holomorphic at $\lambda=0$, and

$$(3.37)\quad R_0 = R(\lambda, z)|_{\lambda=0} = \frac{1}{(2\pi i)^n} \int_{\Omega_\varrho} \sum_{j=1}^{n} \binom{N}{n} \binom{q}{n-j} \Phi_2 \chi^{N-j} B^{q-(n-j)} (\bar\partial Q_0)^j \wedge (\bar\partial A)^{n-j}.$$

By now we are essentially in the same situation as in the new Andersson–Passare proof of the Andersson–Berndtsson integral representation formula (cf. formula (6), proof of Theorem 2, [2]). They show we can let χ tend to the characteristic function of Ω_ϱ and use the fact that for a smooth form φ, and r integral ≥ 1, one has

$$\int_{\Omega_\varrho} \bar\partial \chi^r \wedge \varphi \to -\int_{\partial\Omega_\varrho} \varphi.$$

Since $B=(\Phi_1+1)|_{\lambda=0}$ and $A=Q_1|_{\lambda=0}$, the formula (3.37) is just the boundary term in the Andersson–Berndtsson formula for the single pair (A, t^q) (cf. [2], [7]):

$$(3.38)\qquad R_0 = \frac{1}{(2\pi i)^n} \int_{\partial\Omega_\varrho} \sum_{j=0}^{n-1} \Phi_2 \frac{1}{j!} \binom{q}{n-1-j} B^{q+j+1-n} \frac{S \wedge (\bar\partial S)^j \wedge (\bar\partial A)^{n-1-j}}{\|\zeta - z\|^{2(j+1)}},$$

where $\bar\partial = \bar\partial_\zeta$.

The last step of the proof is to verify that the estimates on A, B that we can obtain from the hypotheses are enough to let $\varrho \to \infty$ in (3.38).

Since $\|f(\zeta)\| \geq c\|\zeta\|^d$ if $\|\zeta\| \geq \varkappa$ we have that for $\varrho > \varkappa$ the following two estimates hold:

$$|B| \leq \text{const.}\ \|\zeta\|^{-d},$$

$$|\text{largest coefficient of } (\bar\partial A)^{n-1-j}| \leq \text{const.} \|\zeta\|^{2(D-d-1)(n-1-j)}.$$

Furthermore $\Phi_2 = P$ on $\partial\Omega_\varrho$. If follows that the worst term in the sum corresponds to $j=0$. From this we conclude that, since

$$\deg P + (n-1)(2D-d) + 1 < dq$$

by (3.6), the integral in (3.38) tends to zero when $\varrho \to \infty$.

This concludes the proof of Theorem 3.1, except for the proof of Lemma 3.2. □

Proof of Lemma 3.2. The defining properties (3.18) and (3.19) show that φ is a C^{n+1} function of compact support in Ω_ϱ which for a fixed z satisfies $\varphi(z)=P(z)$. Cauchy's formula (2.7) states that

$$\left\langle \bar\partial \frac{1}{\zeta-z}, \varphi(\zeta)\, d\zeta \right\rangle = \varphi(z) = P(z). \tag{3.39}$$

The proof of this lemma consists in evaluating the residue in the left hand side of (3.39) using the particular form of φ. It simplifies the computation of this residue to consider the slightly more general form of φ:

$$\varphi(\zeta) = \Gamma(\zeta, \langle Q(z,\zeta), z-\zeta\rangle), \tag{3.40}$$

where Γ is an entire function of $n+\nu$ variables (ζ, t), $Q=(Q_1, \dots, Q_\nu)$ a vector of $(1,0)$-differential forms in ζ, of class C^{n+1}, $\langle Q, z-\zeta\rangle := (\langle Q_1, z-\zeta\rangle, \dots, \langle Q_\nu, z-\zeta\rangle)$. For a multi-index α of ν components, we write, as above,

$$\Gamma^{(\alpha)} := D_1^{\alpha_1} \dots D_\nu^{\alpha_\nu} \Gamma := \left.\frac{\partial^\alpha}{\partial t^\alpha} \Gamma \right|_{t=\langle Q(z,\zeta), z-\zeta\rangle} \tag{3.41}$$

Let $c_n=(-1)^{(n-1)n/2}/(2\pi i)^n$. From (2.3) we see that

$$\left\langle \bar\partial \frac{1}{\zeta-z}, \varphi(\zeta)\, d\zeta \right\rangle = \lim_{\mu\to 0} c_n \mu^n \int \left| \prod_{j=1}^n (\zeta_j - z_j) \right|^{2(\mu-1)} \varphi(\zeta)\, d\bar\zeta \wedge d\zeta. \tag{3.42}$$

We compute the analytic continuation of this integral which is originally defined for $\operatorname{Re}\mu>0$.

One can easily verify that:

$$\begin{aligned} d&\left(\frac{\mu^{n-1}}{(\zeta_1-z_1)} |\zeta_1-z_1|^{2\mu} \left| \prod_{j=2}^n (\zeta_j-z_j) \right|^{2(\mu-1)} \varphi(\zeta)\, d\bar\zeta_2 \wedge \dots \wedge d\bar\zeta_n \wedge d\zeta \right) \\ &= \mu^n \left| \prod_{j=1}^n (\zeta_j-z_j) \right|^{2(\mu-1)} \varphi(\zeta)\, d\bar\zeta \wedge d\zeta \\ &\quad + (-1)^{n-1} \frac{\mu^{n-1}}{\zeta_1-z_1} |\zeta_1-z_1|^{2\mu} \left| \prod_{j=2}^n (\zeta_j-z_j) \right|^{2(\mu-1)} d\bar\zeta_2 \wedge \dots \wedge d\bar\zeta_n \wedge \bar\partial\varphi \wedge d\zeta. \end{aligned}$$

Here $d, \bar\partial$ are only computed with respect to ζ. Since the first term is the exact differential of a form of compact support, we have by Stokes' Theorem:

$$
(3.43)\quad
\begin{aligned}
&\int \mu^n \left| \prod_{j=1}^{n} (\zeta_j - z_j) \right|^{2(\mu-1)} \varphi(\zeta)\, d\bar{\zeta} \wedge d\zeta \\
&= (-1)^n \int \frac{\mu^{n-1}}{\zeta_1 - z_1} |\zeta_1 - z_1|^{2\mu} \left| \prod_{j=2}^{n} (\zeta_j - z_j) \right|^{2(\mu-1)} d\bar{\zeta}_2 \wedge \dots \wedge d\bar{\zeta}_n \wedge \bar{\partial}\varphi \wedge d\zeta.
\end{aligned}
$$

From (3.40) we have

$$
\bar{\partial}\varphi = \sum_{k=1}^{\nu} D_k \Gamma \left(\sum_{j=1}^{n} (z_j - \zeta_j)\, \bar{\partial} Q_{k,j}(z, \zeta) \right)
$$

where we recall $Q_k = \Sigma_{j=1}^{n} Q_{k,j}\, d\zeta_j$. Let us rewrite $\bar{\partial}\varphi$ as follows

$$
(3.44)\quad \bar{\partial}\varphi = -(\zeta_1 - z_1) \sum_{k=1}^{\nu} D_k \Gamma\, \bar{\partial} Q_{k,1} + R_1.
$$

The analytic continuation of the two separate terms obtained by replacing (3.44) into (3.43) exists by Theorem 1.3 [7]. The second one is a sum of integrals of the form: $(1 \leq k \leq \nu,\ 2 \leq i \leq n)$.

$$
\mu^{n-1} \int D_k \Gamma \frac{|\zeta_1 - z_1|^{2\mu}}{\zeta_1 - z_1} (z_i - \zeta_i) \left| \prod_{j=2}^{n} (\zeta_j - z_j) \right|^{2(\mu-1)} d\bar{\zeta}_2 \wedge \dots \wedge d\bar{\zeta}_n \wedge \bar{\partial} Q_{k,i} \wedge d\zeta.
$$

Since the two distributions

$$
\frac{|\zeta_1 - z_1|^{2\mu}}{\zeta_1 - z_1}, \quad \mu^{n-1} \left| \prod_{j=2}^{n} (\zeta_j - z_j) \right|^{2(\mu-1)}
$$

depend on different variables, their analytic continuations as distribution-valued meromorphic functions can be multiplied (this is just their tensor product). The first one is holomorphic for $\mu=0$, the second one leads to the residue current $\bar{\partial}(1/(\zeta_2 - z_2)) \wedge \dots \wedge \bar{\partial}(1/(\zeta_n - z_n))$. But the remaining differential form is in the ideal generated by the functions defining this current. Therefore the value of this product at $\mu=0$ is null.

We can therefore forget R_1 and consider only

$$
(3.45)\quad (-1)^{n-1} \mu^{n-1} \int |\zeta_1 - z_1|^{2\mu} \left| \prod_{j=2}^{n} (\zeta_j - z_j) \right|^{2(\mu-1)} d\bar{\zeta}_2 \wedge \dots \wedge d\bar{\zeta}_n \wedge \left(\sum_{k=1}^{\nu} D_k \Gamma\, \bar{\partial} Q_{k,1} \right) \wedge d\zeta.
$$

In ([7], Proof of Theorem 1.3), we have shown, in a much more general situation, not only that the analytic continuation of (3.45) is holomorphic at $\mu=0$, but its value is exactly the same as the one obtained from

$$(3.46)\qquad (-1)^{n-1}\mu^{n-1}\int \left|\prod_{j=2}^{n}(\zeta_j-z_j)\right|^{2(\mu-1)} d\bar{\zeta}_2\wedge\ldots\wedge d\bar{\zeta}_n\wedge\left(\sum_{k=1}^{\nu} D_k\Gamma\,\bar{\partial}Q_{k,1}\right)\wedge d\zeta.$$

(This also follows from the above remark on the product of the distributions of separate variables.) It is clear now what the general procedure is, the only point to verify is that the factor $(z_1-\zeta_1)$ does not reappear when we apply Stokes' theorem. For this, it is enough to compute $\Sigma_{k=1}^{\nu}(\bar{\partial}D_k\Gamma\wedge\bar{\partial}Q_{k,1})$.

$$\sum_{k=1}^{\nu}\bar{\partial}D_k\Gamma\,\bar{\partial}Q_{k,1}=\sum_{k=1}^{\nu}\sum_{j=1}^{\nu}D_jD_k\Gamma\left(\sum_{i=1}^{n}(z_i-\zeta_i)\,\bar{\partial}Q_{j,i}\right)\wedge\bar{\partial}Q_{k,1}.$$

The term $(z_1-\zeta_1)$ is the coefficient of $\Sigma_{j,k=1}^{\nu}D_jD_k\Gamma\,\bar{\partial}Q_{j,1}\wedge\bar{\partial}Q_{k,1}$ which is 0 by the anticommutativity of the wedge product.

After iterating this procedure a total of n times, and some algebra, one obtains, $(\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\nu),\ \alpha!=\alpha_1!\ldots\alpha_\nu!)$

$$\left\langle\bar{\partial}\frac{1}{\zeta-z},\varphi(\zeta)\,d\zeta\right\rangle=\frac{1}{(2\pi i)^n}\int\sum_{|\alpha|=n}\frac{1}{\alpha!}\Gamma^{(\alpha)}(\bar{\partial}Q_1)^{\alpha_1}\wedge\ldots\wedge(\bar{\partial}Q_\nu)^{\alpha_\nu}.$$

Note that $\bar{\partial}Q_j$ are (1, 1) forms which absorb the $d\zeta$ term from (3.44). For a detailed version of this algebraic computation see ([2], Proof of Theorem 1). The statement of the lemma follows from the explicit form of Γ in this case, we just use that $D_3^2\Gamma=0$. □

§4. On the Noether's Normalization theorem

In this section we reconsider the classical Noether's Normalization theorem [38]. Before we do that we need to recall some well known facts about the heights of polynomials in $\mathbf{Z}[z]$.

For a polynomial $p(z)=\Sigma_{|\alpha|\leq d}c_\alpha z^\alpha\in\mathbf{Z}[z]$, we let

$$(4.1)\qquad H(p)=\max_\alpha|c_\alpha|,\quad h(p)=\log H(p),$$

$h(p)$ is called the (logarithmic) height of p. Some easy properties of the height follows.

Let $C'_d=\binom{n+d-1}{n-1}$=number of monomials in n variables of degree exactly d and $C_d=\binom{n+d}{n}$=dimension of the vector space of polynomials in n variables of degree at most d. We have $C'_d\leq(1+d)^{n-1}$ and $C_d\leq(1+d)^n$.

Let $p, q \in \mathbf{Z}[z]$, $\deg p \leqslant d$, then

$$H(pq) \leqslant C_d H(p) H(q). \tag{4.2}$$

If one changes coordinates by $z=Aw$, A an invertible matrix with integral coefficients, and defines $q \in \mathbf{Z}[w]$ by $q(w)=p(Aw)$, then $\deg p = \deg q$ and

$$H(q) \leqslant C'_d (n\|A\|)^d H(p), \tag{4.3}$$

where $\|A\| = \max |a_{i,j}|$, $A=(a_{i,j})$.

PROPOSITION 4.1. *Let $p_1, \ldots, p_M \in \mathbf{Z}[z_1, \ldots, z_n]$ defining a variety V in $\mathbf{C}^n$. Assume $\dim V = k$, $0 \leqslant k \leqslant n-1$ (for the sake of simplicity, we take here $k=0$ to mean that V is either empty or discrete). Let $d = \max_{1\leqslant j \leqslant M} \deg p_j$ and $\mathfrak{h} = \max_{1 \leqslant j \leqslant M} h(p_j)$. One can find an invertible $n \times n$ matrix $A=(a_{i,j})$ with integral coefficents such that*

(i) $\|A\| \leqslant \varkappa d^{3+n(n-1)/2}$;

(ii) *After the change of coordinates $z=Aw$, $q_j(w)=p_j(Aw)$, let $\mathfrak{J}$ be the ideal generated by the q_j in $\mathbf{Z}[w]$. There are $n-k$ polynomials $Q_i \in \mathfrak{J}$ such that*

$$\begin{cases} Q_1(w) = q_{1,0} w_1^{d_1} + w_1^{d_1-1} q_{1,1}(w_2, \ldots, w_n) + \ldots \\ Q_2(w) = q_{2,0} w_2^{d_2} + w_2^{d_2-1} q_{2,1}(w_3, \ldots, w_n) + \ldots \\ \vdots \\ Q_{n-k}(w) = q_{n-k,0} w_{n-k}^{d_{n-k}} + w_{n-k}^{d_{n-k}-1} q_{n-k,1}(w_{n-k+1}, \ldots, w_n) + \ldots \end{cases} \tag{4.4}$$

with

$$d_1 = \deg Q_1 \leqslant d, \quad d_i = \deg(Q_1) \leqslant \varkappa d^{i+1} \quad (i \geqslant 2) \tag{4.5}$$

and

$$h(Q_i) \leqslant \varkappa d^{i+1} (\mathfrak{h} + d \log d), \tag{4.6}$$

where $\varkappa = \varkappa(n)$ is an effective constant that depends only on n.

This proposition is the usual Noether's Normalization theorem with good estimates on the degrees and heights of the polynomials Q_j and on $\|A\|$ (better than the ones obtained using Elimination theory).

Remarks. (1) This proposition still holds when the polynomials $p_j \in \mathbf{C}[z]$, except that in this case we obtain only (i) and the estimates for the degrees d_i.

(2) From now on we will keep the notation $\varkappa$ for any effective constant depending only on the number of variables n, even if the value of the constant changes from occurrence to occurrence. Whenever it is convenient, we will also assume $\varkappa$ to be a positive integer.

Before starting the proof we must recall what is *a ring with a size* $(\mathfrak{R}, \mathfrak{t})$ [32]. $\mathfrak{R}$ is a commutative Noetherian ring with identity, $\mathrm{Pol}(\mathfrak{R})$ is the algebra of polynomials in infinitely many variables with coefficients in $\mathfrak{R}$, $\mathfrak{R}^*$ the set of invertible elements of $\mathfrak{R}$. Then the (logarithmic) size $\mathfrak{t}$ is a map

$$\mathfrak{t}: \mathrm{Pol}(\mathfrak{R}) \to \{-\infty\} \cup \mathbf{R}_+$$

such that:

(1) $\mathfrak{t}(0)=-\infty$, $\mathfrak{t}(u)=0$ if $u \in \mathfrak{R}^*$.

(2) $\mathfrak{t}(fg)=\mathfrak{t}(f)+\mathfrak{t}(g)$ for every $f, g \in \mathrm{Pol}(\mathfrak{R})$.

(3) There are constants $c_1 \geq 1$, $c_2 \geq 0$ so that if we denote $\bar{\mathfrak{t}}(f) := \mathfrak{t}(f) + c_2 \log(m+1) \deg(f)$, where m is the number of variables appearing in f, then

$$\mathfrak{t}(f_1 + \ldots + f_k) \leq c_1 \max\{\bar{\mathfrak{t}}(f_1), \ldots, \bar{\mathfrak{t}}(f_k)\} + c_2 \log k.$$

(4) There is a constant $c_3 \geq 1$ such that if $f = \Sigma f_\beta x^\beta$, then

$$\max \mathfrak{t}(f_\beta) \leq c_3 \bar{\mathfrak{t}}(f).$$

The simplest example of such a ring is $\mathfrak{R} = \mathbf{C}[z_1, \ldots, z_m]$ with $\mathfrak{t}(f) = d^\circ f$=total degree of f as a polynomial in the z, x variables. In this case $c_1=1$, $c_2=0$, and $c_3=1$.

LEMMA 4.2. [32, Theorem 5]. *Let $(\mathfrak{R}, \mathfrak{t})$ be a ring with a size, $\mathfrak{R}$ being a regular ring, $\mathfrak{k}$ its quotient field and $\bar{\mathfrak{k}}$ the algebraic closure of $\mathfrak{k}$. Let $P_1, \ldots, P_s$ in $\mathfrak{R}[x_1, \ldots, x_m]$ have degree less than δ, $\delta \geq 1$, and size $\mathfrak{t}(P_i) \leq H$. If the polynomials $P_1, \ldots, P_s$ have no common zeros in $\bar{\mathfrak{k}}^m$, there is an element $b \in \mathfrak{R}$ such that b is in the ideal generated by $P_1, \ldots, P_s$ in $\mathfrak{R}[x_1, \ldots, x_m]$ with size estimated by*

$$\mathfrak{t}(b) \leq c_4(m)\, d^\mu (1 + H\mu),$$

where $\mu = \min\{s, m+1\}$ *and* $c_4(m) = (3c^{m+1}(8mc+1))^{m+2}$, *where* $c = \max(c_1, c_2, c_3)$.

For the proof of Proposition 4.1 we use the following lemma.

LEMMA 4.3. *Given a family of polynomials $F_1, \ldots, F_r \in \mathbf{Z}[T_1, \ldots, T_l][X_1, \ldots, X_v] = \mathbf{Z}[T][X]$ without common zeros in the algebraic closure of the quotient field of $\mathbf{Z}[T]$.*

Assume further that for every j, $1 \leq j \leq r$, they satisfy

(i) *if $d^\circ(F_j)$=degree of F_j as a polynomial in all the variables T, X then $d^\circ(F_j) \leq \mathfrak{D}$,*

(ii) *if $h(F_j)$=(logarithmic) height of F_j as a polynomial in the variables T, X, then $h(F_j) \leq \mathfrak{H}$.*

There exists a polynomial $b \in \mathbf{Z}[T]$ in the ideal generated by the F_j in $\mathbf{Z}[T][X]$ such that

$$\deg b = \deg_T b \leq 4c(\nu)\mu\mathfrak{D}^{\mu+1} \tag{4.7}$$

where $c(\nu)=(3^{\nu+2}(24\nu+1))^{\nu+2}$, $\mu=\min\{r, \nu+1\}$, and

$$h(b) \leq 10c(\nu)\mu\mathfrak{D}^{\mu+1}(\mathfrak{H}+(\nu+1)\log(\mathfrak{D}+1)). \tag{4.8}$$

Proof. We fix a constant $C>0$, to be chosen later, and define a function $\mathrm{t}: \mathrm{Pol}(\mathbf{Z}[T]) \to \{-\infty\} \cup \mathbf{R}_+$ by $\mathrm{t}(0)=-\infty$ and, if $P \in \mathrm{Pol}(\mathbf{Z}[T]) \setminus \{0\}$,

$$\mathrm{t}(p) = C\deg_T P + \int_0^1 \dots \int_0^1 \log\left|P(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_l}, e^{i2\pi\xi_1}, \dots, e^{i2\pi\xi_\nu})\right| d\theta_1 \dots d\xi_\nu, \tag{4.9}$$

where ν denotes the number of variables of P as a polynomial with coefficients in $\mathbf{Z}[T]$.

We claim that t is a size for the ring $\mathbf{Z}[T]$ with constants c_1, c_2, c_3 independent of C. First observe that properties (1) and (2) of the definition above are immediate from (4.9).

Let us write $P=\Sigma_\beta P_\beta(T)X^\beta=\Sigma_{\alpha,\beta}a_{\alpha,\beta}T^\alpha X^\beta$. Introduce the Mahler measures

$$M(P) = \exp\left(\int_0^1 \dots \int_0^1 \log\left|P(e^{i2\pi\theta_1}, \dots, e^{i2\pi\xi_\nu})\right| d\theta_1 \dots d\xi_\nu\right),$$

$$M(P_\beta) = \exp\left(\int_0^1 \dots \int_0^1 \log\left|P_\beta(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_l})\right| d\theta_1 \dots d\theta_l\right),$$

Mahler's inequality [26] as rewritten by Philippon [31, Lemma 1.13] states that, if $d_0=d^\circ P$, $d_1=\deg_T P$, $d_2=\deg_X P$,

$$M(P_\beta) \leq \frac{d_0!}{\beta!(d_0-|\beta|)!}M(P) \leq \frac{d_0!}{\beta!(d_0-d_2)!}M(P)$$

and

$$|a_{\alpha,\beta}| \leq \frac{d_1!}{\alpha!(d_1-|\alpha|)!}M(P_\beta) \leq \binom{d_0}{d_2}\frac{d_2!}{\beta!}\frac{d_1!}{\alpha!}M(P).$$

Hence

$$\sum_{\alpha,\beta} |a_{\alpha,\beta}| \leq 2^{d_0}(\nu+1)^{d_2}(l+1)^{d_1} M(P). \tag{4.10}$$

It is clear that

$$M(P) \leq \sum_{\alpha,\beta} |a_{\alpha,\beta}|. \tag{4.11}$$

It follows also from [26] that $M(P) \geq 1$, $M(P_a) \geq 1$. (This inequality depends on the fact that the polynomial has integral coefficients.)

We can now proceed to verify properties (3) and (4) of the definition of size: Let $R_1, \dots, R_k \in \mathrm{Pol}(\mathbf{Z}[T])$, write $R_j = \Sigma_{\alpha,\beta} a^{(j)}_{\alpha,\beta} T^\alpha X^\beta$. (The number of X variables might change from polynomial to polynomial.) Then, by (4.11)

$$M(R_1 + \dots + R_k) \leq \sum_{j,\alpha,\beta} |a^{(j)}_{\alpha,\beta}| \leq k \max_j \sum_{\alpha,\beta} |a^{(j)}_{\alpha,\beta}|.$$

Suppose, to simplify, that the maximum is achieved for $j=1$, let ν be the number of variables X of R_1. Then we can apply (4.10) (using that $d_1 \leq d_1 + d_2$, $d_1 = \deg_T R_1$, $d_2 = \deg_X R_1$)

$$\sum_{\alpha,\beta} |a^{(j)}_{\alpha,\beta}| \leq (2(\nu+1))^{d_2}(2(l+1))^{d_1} M(R_1).$$

Hence

$$\begin{aligned} \mathrm{t}(R_1 + \dots + R_k) &\leq (C+2l+2) \max_i (\deg_T R_i) + \log k + 2\log(\nu+1)\deg_X R_1 + \log M(R_1) \\ &\leq \left(1 + \frac{2l+2}{C}\right) \max_i (C \deg_T R_i) + \bar{\mathrm{t}}(R_1) + \log k, \end{aligned}$$

where

$$\bar{\mathrm{t}}(P) = \mathrm{t}(P) + 2\log(\nu+1)\deg_X P.$$

Since $M(P) \geq 1$, $\max_i (C \deg_T R_i) \leq \max_i \mathrm{t}(R_i) \leq \max_i \bar{\mathrm{t}}(R_i)$. Let us assume that $C \geq 2l+2$, then

$$\mathrm{t}(R_1 + \dots + R_k) \leq 3 \max_i \bar{\mathrm{t}}(R_i) + 2\log k.$$

This proves condition (3) with $c_1=3$ and $c_2=2$, which are independent of C (as long as $C \geqslant 2l+2$). Using (4.10) and (4.11) we obtain

$$\max_\beta \mathfrak{t}(P_\beta) \leqslant 2\tilde{\mathfrak{t}}(P),$$

so that $c_3=2$ in condition (4). The claim is therefore true.

To continue the proof of Lemma 4.3, we apply Lemma 4.2 with $\mathfrak{R}=\mathbf{Z}[T]$, $\mathfrak{t}$ as above, $C=\max(2l+2, \mathfrak{H}, (\nu+1)\log(\mathfrak{D}+1))$, to the given polynomials $F_1, \ldots, F_r \in \mathfrak{R}[X]$. Therefore, there is an element $b \in \mathbf{Z}[T]$ with size estimate

$$\mathfrak{t}(b) \leqslant c(\nu)\Big(\max_j \deg_X F_j\Big)^\mu \Big(1+\mu \times \max_j \mathfrak{t}(F_j)\Big),$$

where $\mu=\min\{\nu+1, r\}$, and $c(\nu)=(3^{\nu+2}(24\nu+1))^{\nu+2}$. From here we can obtain an estimate of the degree of b and of the height of its coefficients. Namely,

$$C \deg b \leqslant c(\nu)\, \mathfrak{D}^\mu (1+\nu(C\mathfrak{D}+\mathfrak{H}+(\nu+1)\log(\mathfrak{D}+1))).$$

Dividing by C we conclude that

$$\deg b \leqslant 4c(\nu)\mu \mathfrak{D}^{\mu+1},$$

as required. For the estimate of the height of the coefficients of b, we have

$$\begin{aligned} h(b) &\leqslant (\log 2)\deg b + \log M(b) \leqslant (\log 2)\deg b + \mathfrak{t}(b) \\ &\leqslant ((\log 2)+C)\, 4c(\nu)\mu\mathfrak{D}^{\mu+1} \\ &\leqslant 5Cc(\nu)\mu\mathfrak{D}^{\mu+1} \\ &\leqslant 10c(\nu)\mu\mathfrak{D}^{\mu+1}(\mathfrak{H}+(\nu+1)\log(\mathfrak{D}+1)). \end{aligned}$$

This concludes the proof of the lemma. □

Remark. The point of this lemma is that the estimate of the degree of b given in [32, Theorem 5] is much worse than (4.7), since it was also dependent on $\mathfrak{H}$.

Proof of Proposition 4.1. We can assume that none of the polynomials $p_1, \ldots, p_M$ is a constant, if it is zero we eliminate it from the list, if it is a non-zero constant the result is trivial. Let $d_1=\deg p_1$, and p_1° be the leading homogeneous term of p_1. By [27, Theorem 1] there is a point $a_1=(a_{11}, \ldots, a_{1n}) \in \mathbf{Z}^n$ such that $|a_{i,j}| \leqslant nd_1+1$ and $p_1^\circ(a_1) \neq 0$. Clearly $a_1 \neq 0$. We can choose $n-1$ elements of the canonical basis of $\mathbf{C}^n$ so that the $n \times n$ matrix A_1 with first column a_1, completed by them, is invertible. We now make the change of variable $z=A_1\zeta$, obtaining polynomials $F_j(\zeta)=p_j(A_1\zeta)$, $j=1,\ldots,M$. The first one will be

$$F_1(\zeta)=p_1^\circ(a_1)\,\zeta_1^{d_1}+\text{lower degree terms.}$$

If $k=n-1$, we take $A=A_1$, $Q_1=F_1$, and we will be done. We assume therefore that $k<n-1$. Consider now the polynomials F_j as polynomials in $\mathfrak{R}[\zeta_1]$, $\mathfrak{R}=\mathbf{Z}[\zeta_2,\ldots,\zeta_n]$. These polynomials $F_1,\ldots,F_M$ have no common zeros on $\bar{\mathfrak{k}}^{n-1}$, $\mathfrak{k}$ the quotient field of $\mathfrak{R}$, because of the assumption that the dimension $k<n-1$. Moreover, $\max_{1\leqslant j\leqslant n}\deg F_j=d$, as before, and their heights can be bounded using (4.3):

$$\begin{aligned}\max_{1\leqslant j\leqslant n}h(F_j)&\leqslant \log C_d'+d\log(n\|A_1\|)+\log\max_{1\leqslant j\leqslant n}h(p_j)\\&\leqslant \varkappa(n)\,(d\log d+\mathfrak{h}),\end{aligned}\tag{4.12}$$

where $\varkappa(n)$ is an effective constant.

We can now apply Lemma 4.3. We find $b_2\in\mathfrak{R}$, i.e., $b_2\in\mathbf{Z}[\zeta_2,\ldots,\zeta_n]$, in the ideal generated by $F_1,\ldots,F_M$ in $\mathbf{Z}[\zeta_1,\ldots,\zeta_n]=\mathfrak{R}[\zeta_1]$, such that

$$d_2=\deg b_2\leqslant 8c(1)\,d^3$$

$$\begin{aligned}h(b_2)&\leqslant 20c(1)\,d^3(\varkappa(n)\,(d\log d+\mathfrak{h})+2\log(d+1))\\&\leqslant \varkappa d^3(\mathfrak{h}+d\log d)\end{aligned}$$

for a new value of the constant $\varkappa$.

By the same [27], Theorem 1] we find $a_2=(0,a_{2,1},\ldots,a_{2,n})\in\mathbf{Z}^n$, $|a_{2,j}|\leqslant(n-1)\,d_2+1$, $b_2^\circ(a_2)\neq 0$. Complete the pair $e_1=(1,0,\ldots,0)$, a_2, to a basis of $\mathbf{C}^n$ using the elements of the canonical basis, so that the matrix A_2 with these columns is invertible. We change variables again with $z=A_2A_1\eta$, and we obtain two polynomials in the corresponding ideals of $\mathbf{Z}[\eta]$ of the form

$$G_1(\eta)=g_{1,0}\,\eta_1^{d_1}+(\text{terms of degree }\leqslant d_1)$$

$$G_2(\eta)=G_2(\eta')=g_{2,0}\,\eta_2^{d_2}+(\text{terms of degree }\leqslant d_2),$$

with $\eta'=(\eta_2,\ldots,\eta_n)$. Their heights can be estimated by $\varkappa d^3(h+d\log d)$, an estimate that also holds for all the polynomials $p_j(A_2A_1\eta)$. It is clear now what the inductive procedure is. Proposition 4.1 is therefore correct. □

Remark 4.4. In case we have L non-trivial finite families of polynomials in n variables, with corresponding ideals I_j and varieties V_j, $\dim V_j = k_j$, one can proceed as in Proposition 4.1 simultaneously for all the families. Namely, led d be a common bound for the degrees of all these polynomials. Then there is an invertible $n \times n$ matrix A with integral coefficients satisfying

$$\|A\| \leq \varkappa L^n d^{3+n(n-1)/2} \tag{4.13}$$

such that, after the change of coordinates $z = Aw$, we can find, for every j, polynomials $Q_{j,1}, \ldots, Q_{j,n-k_j}$ in the corresponding ideals $\mathfrak{J}_j$ of $\mathbf{Z}[w]$, of the form given in part (ii) of Proposition 4.1. The bounds for their degrees still remain (4.5) and the bounds for their heights are

$$h(Q_{j,i}) \leq \varkappa d^{i+1}(\mathfrak{h} + d\log(Ld)). \tag{4.14}$$

PROPOSITION 4.5. *Let $p_1, \ldots, p_M \in \mathbf{Z}[z_1, \ldots, z_n]$ be as in Proposition 4.1, $d \geq 3$. There is a linear change of coordinates $z = Aw$, A an invertible matrix with integral coefficients, $\|A\| \leq \varkappa d^{3+n(n-1)/2}$, and strictly positive constants ε, K such that if $q_j(w) = p_j(Aw)$, then*

$$\mathfrak{S} := \{w \in \mathbf{C}^n : \log \max_{1 \leq j \leq M} |q_j(w)| < \log \varepsilon - d^\mu(\log(1 + \|w\|^2))\}$$
$$\subseteq \mathfrak{C} := \{w \in \mathbf{C}^n : |w_1| + \ldots + |w_{n-k}| \leq K(1 + |w_{n-k+1}| + \ldots + |w_n|)\}$$

$\mu = \min\{M, n\}$. *Moreover, we have*

$$K \leq \exp[\varkappa d^{n-k+1}(\mathfrak{h} + d\log d)]. \tag{4.15}$$

Proof. Let A be the matrix A given by Proposition 4.1, Q_j the corresponding polynomials. Let $V' = \{w \in \mathbf{C}^n : q_1(w) = \ldots = q_M(w) = 0\}$. If $w \in V'$ then $Q_j(w) = 0$ for $j = 1, \ldots, n-k$. In particular, the equation

$$0 = Q_{n-k}(w) = q_{n-k,0}\, w_{n-k}^{d_{n-k}} + w_{n-k}^{d_{n-k}-1}\, q_{n-k,1}(w_{n-k+1}, \ldots, w_n) \ldots$$

implies that

$$|w_{n-k}| \leq K_1(1 + |w_{n-k+1}| + \ldots + |w_n|),$$

by a well known estimate on the location of the zeros of a polynomial of one variable. Namely, all the roots s_i of an algebraic equation of degree δ in a single variable

$$a_0 s^\delta + a_1 s^{\delta-1} + \ldots + a_\delta = 0$$

lie in the disk

$$|s_i| \leqslant \max_j |\delta(a_j/a_0)|^{1/j}. \tag{4.16}$$

Using that $\deg q_{n-k,i} \leqslant i$ we obtain from (4.5), (4.6) and (4.16) that

$$\begin{aligned} K_1 &\leqslant \varkappa d^{n-k+1} \exp[\varkappa d^{n-k+1}(\mathfrak{h}+d\log d)] \\ &\leqslant \exp[\varkappa d^{n-k+1}(\mathfrak{h}+d\log d)]. \end{aligned}$$

Iterating this process we find that

$$V' \subseteq \{w \in \mathbf{C}^n : |w_1| + \ldots + |w_{n-k}| \leqslant K'(1+|w_{n-k+1}|+\ldots+|w_n|)\}$$

for some $K'>0$ with same type of estimate (4.15). To conclude the proof we only need to show that whenever all the q_j are small at a point w, this point is close to a point V'. More precisely, let $d(w, V') = \min\{1, \operatorname{dist}(w, V')\}$, where $\operatorname{dist}(w, V')$ denotes the Euclidean distance from the point w to the variety V'. From the result in [21] one concludes that there is a positive constant $A>0$ such that

$$\log \max_{1\leqslant j\leqslant M} |q_j(w)| \geqslant -A + d^{\mu} \log(d(w, V')/(1+\|w\|^2))$$

(A is not an absolute constant). Choosing $\varepsilon>0$ so that $A+\log\varepsilon \leqslant 0$, every $w \in \mathfrak{S}$ satisfies

$$d(w, V') \leqslant 1.$$

It is now clear that by changing the constant K' slightly one obtains the inclusion $\mathfrak{S} \subseteq \mathfrak{C}$ we were looking for.

Remark 4.6. As in Remark 4.4, we see that given L finite families of polynomials, there is a change of coordinates $z=Aw$ and constants $\varepsilon>0, K>0$ such that for the jth family $q_{j,1}, \ldots, q_{j,M_j}$ (after change of coordinates)

$$\begin{aligned} \mathfrak{S}_j &:= \{w \in \mathbf{C}^n : \log \max_{1\leqslant j\leqslant M_j} |q_{j,i}(w)| < \log\varepsilon - d^{\mu_j} \log(1+\|w\|^2)\} \\ &\subseteq \mathfrak{C}_j = \{w \in \mathbf{C}^n : |w_1| + \ldots + |w_{n-k_j}| \leqslant K(1+|w_{n-k_j+1}|+\ldots+|w_n|)\} \end{aligned}$$

where $\mu_j = \min\{n, M_j\}$, $k_j = \dim V_j$, V_j the zero variety of the jth family $p_{j,1}, \ldots, p_{j,M_j}$. The matrix A has the estimates given in Remark 4.4. The constant K has the estimate

$$K \leqslant \exp[\varkappa d^{n-k'+1}(\mathfrak{h}+d\log(Ld))], \tag{4.17}$$

with $k'=\min\{k_j, 1\leqslant j\leqslant L\}$.

Remark 4.7. From the remark following the statement of Proposition 4.1, we can now conclude that Proposition 4.5 is still true when we consider polynomials p_j with complex coefficients, with the obvious exception that we do not have the bounds (4.15) for the constant K.

§ 5. Effective bounds for the size of the coefficients in the Bezout identity

In this section we will study the Bezout equation for polynomials in $\mathbf{Z}[z]=\mathbf{Z}[z_1, \dots, z_n]$. We remind the reader that for us $n\geqslant 2$, the case $n=1$ being well known as a consequence of the Euclidean division algorithm.

Using the division formula (3.7) we will prove

THEOREM 5.1. *Let $p_1, \dots, p_N \in \mathbf{Z}[z]$ without common zeros in $\mathbf{C}^n$, $\deg p_j \leqslant D$, $D\geqslant 3$, $h(p_j)\leqslant h$. There is an integer $\mathfrak{d} \in \mathbf{Z}^+$, polynomials $q_1, \dots, q_N \in \mathbf{Z}[z]$ such that*

$$p_1 q_1 + \dots + p_N q_N = \mathfrak{d},$$

satisfying the estimates:

$$\deg q_j \leqslant n(2n+1) D^n,$$

$$h(q_j) \leqslant \varkappa(n) D^{8n+3}(h+\log N + D\log D) \tag{5.2}$$

$$\operatorname{Log} \mathfrak{d} \leqslant \varkappa(n) D^{8n+3}(h+\log N + D\log D), \tag{5.3}$$

where $\varkappa(n)$ is an effective constant which can be computed explicitly following step by step the proof below.

Remark. We remind the reader that all constants are effective but, if they are not explicitly mentioned to be absolute constants, they will be denoted by the same letter $\varkappa$ and they will depend only on the dimension n. We assume moreover that $\varkappa$ is an integer whenever necessary. We keep track of the dependency on N, h and D, the values from the statement of Theorem 5.1. Once and for all, we assume $n\geqslant 2$ and $D\geqslant 2$.

We start by some preparatory considerations that will allow us to construct auxiliary polynomials $f_1, \ldots, f_{n+1}$ in the ideal $\mathfrak{I}$ generated by $p_1, \ldots, p_N$ in $\mathbf{Z}[z]$, for which the hypotheses from Theorem 3.1 will be satisfied.

As a first step, we adapt the proof of Lemma 2 in [28, section 4] to obtain the following

LEMMA 5.2. *There are integers $\lambda_{j,k}$, $1 \leq j \leq n$, $1 \leq k \leq N$ such that the polynomials*

$$g_j = \sum_{k=1}^{N} \lambda_{j,k} p_k, \tag{5.4}$$

have the property that for any non-empty subset $J \subseteq \{1, \ldots, n\}$ the variety

$$V_J = \{z \in \mathbf{C} : g_j = 0 \text{ for } j \in J\}$$

is either empty or of pure dimension $n - \#(J)$. Moreover, the $\lambda_{j,k}$ can be chosen so that

$$|\lambda_{j,k}| \leq (D+1)^{n-1}. \tag{5.5}$$

Proof. We start by taking $g_1 = p_1$. Let $\pi_1, \ldots, \pi_r$ be the distinct irreducible polynomials in the factorization of p_1 in $\mathbf{C}[z]$, then $r \leq D$. Since the original collection $p_1, \ldots, p_N$ have no common zeros, for any l, $1 \leq l \leq r$, not all the p_k are divisible by π_l. By Lemma 1 [28, Section 4] there are $\lambda_{2,k} \in \mathbf{Z}$, $|\lambda_{2,k}| \leq D$ such that if

$$g_2 = \sum_{k=2}^{N} \lambda_{2,k} p_k,$$

then the ideal (g_1, g_2) is either $\mathbf{C}[z]$ or a proper ideal of rank 2 and degree $\leq D^2$.

We will show now how to construct g_3, the general case is handled by induction. There are two cases which have to be dealt with separately. If (g_1, g_2) is $\mathbf{C}[z]$ then we consider the irreducible factors $\nu_1, \ldots, \nu_s$ of $g_1 g_2$. The previous argument allows us to construct g_3 in this case, so that both (g_1, g_3) and (g_2, g_3) are either $\mathbf{C}[z]$ or proper ideals of rank 2 and degree $\leq D^2$. Since $s \leq \deg(g_1 g_2) \leq 2D$, the size of the coefficients is at most $2D$. The most interesting case occurs when $(g_1, g_2) \neq \mathbf{C}[z]$. This ideal is unmixed. We consider all the ideals $\mathscr{I}_1, \ldots, \mathscr{I}_t$ in the primary decomposition of (g_1, g_2). We have now $t+s$ ideals $(\nu_1), \ldots, (\nu_s), \mathscr{I}_1, \ldots, \mathscr{I}_t$, and we know that $t+s \leq D^2 + 2D \leq (D+1)^2$ (cf. [27], p. 85). By the same Lemma 1 in [28, Section 4] we can find $\lambda_{3,k} \in \mathbf{Z}$, $|\lambda_{3,k}| \leq (D+1)^2$ such that if

$$g_3 = \sum_{k=3}^{N} \lambda_{3,k} p_k,$$

then (g_1, g_2, g_3) is either $\mathbf{C}[z]$ or a proper ideal of rank 3 and degree $\leq D^3$, and the ideals (g_1, g_3) and (g_2, g_3) are either $\mathbf{C}[z]$ or proper ideals of rank 2 and degree $\leq D^2$.

To construct g_{j+1} we have to consider the primary ideals corresponding to all proper ideals of the form $(g_{i_1}, \ldots, g_{i_l})$ with $i_1, \ldots, i_l \in \{1, \ldots, j\}$. The total number of these primary ideals is at most $D^j + jD^{j-1} + \ldots + jD \leq (D+1)^j$. The rest of the argument is the same as above. □

Another little lemma from linear algebra will prove useful.

LEMMA 5.3. *Given an integer $C \geq 1$ there are n linear forms $L_j \in \mathbf{Z}[w]$ such that*

(a) $H(L_j) \leq \varkappa C^{n-1}$ $(1 \leq j \leq n)$

(*with $\varkappa$ as usual an effective constant depending only on n*) *and*

(b) *there is a strictly positive constant γ (depending on n and C) such that for every k, $1 \leq k \leq n$, for every $J \subseteq \{1, \ldots, n\}$, $\#(J) = k$, we have*

$$\sum_{j \in J} |L_j(w)| \geq \gamma \|w\| \tag{5.6}$$

whenever

$$|w_1| + \ldots + |w_{n-k}| \leq C(|w_{n-k+1}| + \ldots + |w_n|). \tag{5.7}$$

Proof. We note that for $k = n$, the condition (b) is exactly the condition that $L_1, \ldots, L_n$ be linearly independent.

Let B be any $n \times n$ matrix with integral coefficients such that every minor of $B = (\beta_{ij})$ is different from zero. From [27, Theorem 1] one can obtain an explicit estimate of $\|B\|$ depending only on n. Let us denote by Δ the maximum absolute of any minor of B. It is clear that $\Delta \leq n! \|B\|^n$. Let $M = nC\Delta + 1$ and define, for $1 \leq i \leq n$,

$$L_i(w) = \beta_{i,1} w_1 + \beta_{i,2} M w_2 + \ldots + \beta_{i,n} M^{n-1} w_n. \tag{5.8}$$

The estimate (a) being obvious, we need to show (5.6) for an arbitrary k. For $k = n$, it is clear since the determinant of the coefficients of the L_i is just $M^{n(n-1)/2} \times \det B \neq 0$. To see the idea, consider the case $k = 1$. We have

$$\begin{aligned} |L_i(w)| &\geq M^{n-1}|w_n| - M^{n-2}\Delta(|w_1| + \ldots + |w_{n-1}|) \\ &\geq (M^{n-1} - M^{n-2}\Delta C)\,|w_n| \geq M^{n-2}|w_n| \geq \frac{\varkappa}{C} M^{n-2}\|w\|, \end{aligned}$$

in the cone given by the inequality (5.10) for $k = 1$.

For the general case we consider the set $J=\{1,\dots,k\}$ to simplify the notation. Consider the system of equations

$$(5.9)\qquad \begin{cases} \beta_{1,n-k+1}M^{n-k}w_{n-k+1}+\dots+\beta_{1,n}M^{n-1}w_n = L_1(w)-\sum_{j=1}^{n-k}\beta_{1,j}M^{j-1}w_j,\\ \vdots\\ \beta_{k,n-k+1}M^{n-k}w_{n-k+1}+\dots+\beta_{k,n}M^{n-1}w_n = L_k(z)-\sum_{j=1}^{n-k}\beta_{n-k,j}M^{j-1}w_j. \end{cases}$$

Eliminating any of the variables $w_{n-k+1},\dots,w_n$ by Cramer's rule, we obtain for $n-k+1\leqslant j\leqslant n$:

$$(5.10)\qquad \tilde{\Delta}M^{j-1}w_j=\sum_{i=1}^{k}\alpha_{i,j}L_i(w)+\sum_{i=1}^{n-k}\gamma_{i,j}M^{i-1}w_i,$$

where $\tilde{\Delta}$ denotes a certain $(n-k)\times(n-k)$ minor of B and $\alpha_{i,j}$, $\gamma_{i,j}$ are certain other minors of B. In the cone defined by (5.7), the identity (5.10) leads to the inequality

$$\begin{aligned}\Delta\sum_{i=1}^{k}|L_i(w)|\geqslant\left|\sum_{i=1}^{k}\alpha_{i,j}L_i(w)\right| &\geqslant |\tilde{\Delta}|M^{j-1}|w_j|-M^{n-k-1}\Delta\left(\sum_{i=1}^{n-k}|w_i|\right)\\ &\geqslant M^{n-k}|w_j|-M^{n-k-1}\Delta C(|w_{n-k+1}|+\dots+|w_n|).\end{aligned}$$

Adding the inequalities for $j=n-k+1,\dots,n$, we obtain

$$\begin{aligned}k\Delta\sum_{i=1}^{k}|L_i(w)| &\geqslant M^{n-k-1}\left(\sum_{j=n-k+1}^{n}|w_i|\right)(M-k\Delta C)\\ &\geqslant M^{n-k-1}\left(\sum_{j=n-k+1}^{n}|w_i|\right)\\ &\geqslant\frac{\varkappa}{C}M^{n-k-1}\|w\|.\end{aligned}$$

This is an inequality of the form (5.6), concluding the proof of the lemma. □

We are finally ready to start the proof.

Proof of Theorem 5.1. The first step is the construction of auxiliary functions $f_1,\dots,f_n\in\mathfrak{I}$ satisfying (3.2).

Let $g_1,\dots,g_n$ be given by Lemma 5.2. It follows immediately from the statement of that lemma that

(5.11) $$\deg g_j \leqslant D,$$

(5.12) $$h(g_j) \leqslant \varkappa(h+D+\log N).$$

If $J \subseteq \{1, \ldots, n\}$, $\#(J)=k$, $1 \leqslant k \leqslant n-1$, then the family $\mathfrak{G}_J$ of polynomials $(g_j)_{j \in J}$ either defines a complete intersection variety of dimension exactly $n-k \geqslant 1$, or is such that the ideal $(g_j)_{j \in J}$ is $\mathbf{C}[z]$. By Remark 4.6 there is a change of coordinates $z=Aw$ and constants $\varepsilon>0$, $K>0$ such that

$$\mathfrak{S}_J := \{w \colon \|w\| \geqslant 1,\ \log \max_{j \in J} |g_j(Aw)| \leqslant \log \varepsilon - D^n \log(1+\|w\|^2)\}$$

is contained in the cone

$$\mathfrak{C}_k := \{w \in \mathbf{C}^n \colon |w_1|+\ldots+|w_k| \leqslant K(|w_{k+1}|+\ldots+|w_n|)\}.$$

The total number of such families is 2^n-2, hence from (4.17) we obtain

(5.13) $$K \leqslant \exp[\varkappa D^n(h+\log N+D \log D)].$$

We apply Lemma 5.3 to obtain n linear forms $L_j \in \mathbf{Z}[w]$, with heights estimated by $\varkappa K^{n-1}$.

Let $\mathfrak{p}=\mathcal{W}D^n$, where $\mathcal{W}$ is a positive integer such that $\mathcal{W} \geqslant 2$, and consider the function

(5.14) $$\varphi_j(w) = (L_j(w))^{\mathfrak{p}} g_j(Aw) \quad (1 \leqslant j \leqslant n).$$

We claim that for some constant $\delta>0$ (δ depends on $K, N, D, \varepsilon, \mathcal{W}$) and $\|w\|>>1$ we have

(5.15) $$\left(\sum_{j=1}^{n} |\varphi_j(w)|^2\right)^{1/2} \geqslant \delta \|w\|^{(\mathcal{W}-1)D^n}.$$

(The value of δ plays no role whatsoever in the proof of Theorem 5.1 below.) Namely, by [22, Proposition 1.10] there are two positive constants ε_1, ϱ_1 (they depend on the polynomials $g_j, \ldots, g_n$) such that:

(5.16) For $\|w\| \geqslant \varrho_1$, we have $\log \max_{1 \leqslant j \leqslant n} |g_j(Aw)| \geqslant \log \varepsilon_1 - D^n \log(1+\|w\|)$.

Taking ϱ_2 sufficiently large, for $\|w\| \geqslant \varrho_2 \geqslant \varrho_1$ we have

$$\log \varepsilon_1 - D^n \log(1+\|w\|) \geqslant \log \varepsilon - D^n \log(1+\|w\|).$$

It follows from (5.14) that the set $\{w \in \mathbf{C}^n: \|w\| \geq \varrho_2\}$ can be written as the disjoint union of the sets

$$\mathfrak{T}_J: \{w \in \mathbf{C}^n: \|w\| \geq \varrho_2,\ \log|g_j(Aw)| \leq \log \varepsilon - D^n \log(1+\|w\|) \text{ if } j \in J$$
$$\text{and } \log|g_j(Aw)| > \log \varepsilon - D^n \log(1+\|w\|) \text{ if } j \notin J\},$$

where J is any subset of $\{1, \ldots, n\}$, $1 \leq \#(J) \leq n-1$. Any point of $\mathfrak{T}_J$ is contained in $\mathfrak{S}_J$, and a posteriori in $\mathfrak{C}_k$, $k = \#(J)$. By the definition of the L_j

$$\sum_{j \notin J} |L_j(w)| \geq \gamma \|w\| \quad \text{if} \quad w \in \mathfrak{T}_J. \tag{5.17}$$

Hence, for some $j_0 \notin J$, $|L_{j_0}(w)| \geq (\gamma/n)\|w\|$, so that

$$|\varphi_{j_0}(w)| = |L_{j_0}(w)|^{\mathfrak{p}}\, |g_{j_0}(Aw)| \geq (\gamma/n)^{\mathfrak{p}} \|w\|^{\mathfrak{p}}\, \| \varepsilon(1+\|w\|)^{-D^n} \geq \delta \|w\|^{(\mathscr{W}-1)D^n}$$

proving (5.15).

We define now

$$f_j(z) := ((\det A)\, L_j(A^{-1}z))^{\mathfrak{p}}\, g_j(z). \tag{5.18}$$

The linear forms $L_j(w)$ found in Lemma 5.3 have their heights bounded by

$$H(L_j) \leq \varkappa K^{n-1} = \exp[\varkappa D^n(h + \log N + D \log D)],$$

after an eventual change of constant $\varkappa$ which depends only on n. Therefore, the height of the corresponding linear forms $\Lambda_j(z) = (\det A)\, L_j(A^{-1}z)$, in the original variables, can be estimated by

$$H(\Lambda_j) = \exp[\varkappa D^n(h + \log N + D \log D)], \tag{5.19}$$

using that $\|(\det A)A^{-1}\| \leq n!\, \|A\|^n \leq \varkappa D^{n^3}$ and the formula (4.3). With this notation, the functions f_j defined by (5.18) are given by

$$f_j(z) = (\Lambda_j(z))^{\mathfrak{p}} g_j(z), \tag{5.20}$$

where

$$\mathfrak{p} = \mathscr{W} D^n. \tag{5.21}$$

We know therefore that for some constants $\gamma > 0$ and $\varrho > 0$ we have, for $\|z\| \geq \varrho$,

$$\|f(z)\| \geq \gamma \|z\|^{(\mathscr{W}-1)D^n}. \tag{5.22}$$

with $f=(f_1, \ldots, f_n)$. Moreover, the number $\mathfrak{N}$ of common zeros of the f_j, without counting multiplicities, is at most

$$\prod_{j=1}^{n} (1+\deg g_j) \leq \varkappa D^n, \tag{5.23}$$

as shown by the classical Bezout estimate. It is convenient to introduce the auxiliary polynomials

$$\Phi_j(z) := \Lambda_j(z)\, g_j(z). \tag{5.24}$$

Then

$$\max_j \deg \Phi_j \geq D+1, \tag{5.25}$$

and

$$\max_j h(\Phi_j) \leq \varkappa D^n(h+\log N+D\log D). \tag{5.26}$$

One last auxiliary polynomial f_{n+1} is obtained as a linear combination

$$f_{n+1} = \lambda_1 p_1 + \ldots + \lambda_N p_N,$$

$\lambda_j \in \mathbf{Z}$, $|\lambda_j| \leq \mathfrak{N} \leq \varkappa D^n$, in such a way that

$$\{z \in \mathbf{C}^n : f_1(z) = \ldots = f_{n+1}(z) = 0\} = \{z \in \mathbf{C}^n : \Phi_1(z) = \ldots = \Phi_n(z) = f_{n+1}(z) = 0\} = \varnothing.$$

The existence of such f_{n+1} is given by Lemma 2 in [28, section 4]. We have

$$h(f_{n+1}) \leq h+\log N+n\log D+\varkappa. \tag{5.27}$$

The sequence $f_1, \ldots, f_{n+1}$ fits exactly in the situation of Example 3.3 with $d=(\mathcal{W}-1)D^n$ and $\mathcal{W}D^n+D$ instead of D. Since $n\geq 2$, $D\geq 3$, as soon as $\mathcal{W}\geq 2n$, we get

$$n\mathcal{W}-n > (n-1)\mathcal{W}+n-1+\frac{2}{D^{n-1}},$$

so that the condition (3.6) is fulfilled for $P\equiv 1$, $q=n$, $(\mathcal{W}-1)D^n$ instead of d, and $\mathcal{W}D^n+D$ instead of D. We shall from now on choose $\mathcal{W}=2n$.

It follows that there are polynomials $A_j \in \mathbf{Q}[z]$ satisfying

$$\sum_{j=1}^{n+1} A_j f_j = 1. \tag{5.29}$$

They are explicitly obtained from the formula

$$\left\langle \bar{\partial}\frac{1}{f}, \frac{1}{f_{n+1}} \begin{vmatrix} g_{1,1} & \cdots & g_{n,1} & g_{n+1,1} \\ \vdots & & & \\ g_{1,n} & \cdots & g_{n,n} & g_{n+1,n} \\ f_1(z) & \cdots & f_n(z) & f_{n+1}(z) \end{vmatrix} d\zeta \right\rangle = 1, \tag{5.30}$$

where the $g_{j,k}$ are given by the formula following (3.4). It remains to estimate the degrees of the A_j, find a common denominator $\mathfrak{d} \in \mathbf{Z}^+$ of their coefficients, and obtain a good bound for the coefficients of the polynomials $\mathfrak{d}A_j$, which are now in $\mathbf{Z}[z]$.

It is immediate that

$$\deg A_j \leq n(2n+1)D^n.$$

Rewriting (5.29) in terms of the original polynomials p_j and clearing denominators we have

$$\sum_{j=1}^{N} q_j p_j = \mathfrak{d}$$

with $q_j \in \mathbf{Z}[z]$,

$$\deg q_j \leq n(2n+1)D^n.$$

Before proceeding to the estimate of the common denominator $\mathfrak{d}$, we need to recall a few definitions from Algebraic Number Theory. Given an algebraic number α one denotes

$$|\bar{\alpha}| = \max\{|\alpha'|: \alpha' \text{ conjugate of } \alpha \text{ over } \mathbf{Q}\}$$

$$s(\alpha) = \max\{\log \operatorname{den}(\alpha), \log|\bar{\alpha}|\},$$

where $\operatorname{den}(\alpha)$=denominator of α=smallest integer $d>0$ such that $d\alpha$ is an algebraic integer.

Finally, let $p \in \mathbf{Z}[z_1, \ldots, z_n]$, $\alpha_1, \ldots, \alpha_n$ algebraic numbers, and $\beta = p(\alpha_1, \ldots, \alpha_n)$. To estimate $\operatorname{den}(\beta)$ and $s(\beta)$, let $r_j \geq \deg_{z_j} p$=degree of p with respect to the variable z_j. Then

$$\operatorname{den}(\beta) \text{ is a divisor of } \prod_{j=1}^{n} (\operatorname{den}(\alpha_j))^{r_j}, \tag{5.31}$$

and

$$s(\beta) \leq h(p) + \sum_{j=1}^{n} (r_j s(\alpha_j) + \log(r_j + 1)). \tag{5.32}$$

Later on we will also need to estimate the denominator of the inverse of an algebraic number α ($\alpha \neq 0$). If $N(\alpha)$ denotes its norm and d a denominator of α one can use that $N(d\alpha)$ is a denominator for $(d\alpha)^{-1}$. Therefore

$$\begin{aligned}(5.33)\qquad \log \operatorname{den}(\alpha^{-1}) &\leq \log \operatorname{den}((d\alpha)^{-1}) \leq \log N(d\alpha) \\ &\leq (\deg \alpha)\, s(d\alpha) \leq 2(\deg \alpha)\, s(\alpha).\end{aligned}$$

In order to apply these inequalities to formula (5.30), we need to estimate a common denominator for all the rational numbers of the form

$$(5.34)\qquad \varrho_k := \left\langle \bar\partial \frac{1}{f}, \frac{\zeta^k}{f_{n+1}} d\zeta \right\rangle, \quad |k| \leq n(2n+1)D^n,$$

which appear as coefficients of the polynomials A_j.

Lemma 5.4. *There is a common denominator $\mathfrak{d} \in \mathbf{Z}^+$ for the rational numbers ϱ_k defined by* (5.34) *such that*

$$\log \mathfrak{d} \leq \varkappa D^{8n+3}(h + \log N + D \log D),$$

where $\varkappa = \varkappa(n)$ is an effective constant depending only on n.

Proof. We rewrite (5.34) by letting $\Phi = (\Phi_1, \ldots, \Phi_n)$, $g = (g_1, \ldots, g_n)$ and

$$\frac{1}{f^{\mathbf{1}}} = \frac{g^{\mathfrak{d}-\mathbf{1}}}{\Phi^{\mathfrak{d}}}.$$

Therefore, the rational numbers ϱ_k are linear combinations with integral coefficients of rationals of the form

$$(5.35)\qquad \left\langle \bar\partial \frac{1}{\Phi^{\mathfrak{d}}}, \frac{\zeta^k}{f_{n+1}} d\zeta \right\rangle,$$

with

$$|k| \leq n(2n+1)D^n + n(2nD^n - 1) \leq n(4n+1)D^n.$$

The coefficients of these linear combinations do not play any role because we are, for the moment, only interested in the denominators.

To compute explicitly the residues (5.35) we can use an observation from [7], Proposition 2.5 and following remark. It shows that it is enough to find n polynomials $b_1, \ldots, b_n \in \mathbf{Z}[z]$, b_j a polynomial on the single variable z_j, all of them in the ideal generated by $\Phi_1, \ldots, \Phi_n$ in $\mathbf{Z}[z]$. To find these polynomials b_j, we apply first Lemma 4.3, with $T=z_j$, $X=(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ to the family $\Phi_1, \ldots, \Phi_n$. In this way, we obtain intermediate polynomials

$$B_j(z) = B_j(z_j), \quad B_j \in \Phi_1 \mathbf{Z}[z] + \ldots, \Phi_n \mathbf{Z}[z], \tag{5.36}$$
$$\deg B_j \leqslant \varkappa D^{n+1},$$

and

$$h(B_j) \leqslant \varkappa D^{2n+1}(h + \log N + D \log D). \tag{5.37}$$

Regretfully, we have no information at this point on the degrees of the polynomials $B_{j,k}$ that appear in the representation $B_j = \Sigma_{k=1}^n B_{jk} \Phi_k$. To solve this problem we apply Rabinowitsch's trick and [32, Theorem 4]. For a fixed j, let $T \geqslant \deg B_j$, consider the polynomials in $\mathbf{Z}[z_0, z_1, \ldots, z_n]$ $(z=(z_1, \ldots, z_n)$ as always)

$$1 - z_0^T B_j(z), \ \Phi_1(z), \ldots, \Phi_n(z).$$

The first one has degree at most $2T$, the others of degree $\leqslant D+1$. We may assume $2T \geqslant D+1$ and $T \leqslant \varkappa D^{n+1}$. Their heights are bounded by $\varkappa D^{2n+1}(h + \log N + D \log D)$. By [32, Theorem 1] there exist an $a_j \in \mathbf{Z}^*$, and polynomials $S_0, \ldots, S_n \in \mathbf{Z}[z_0, z_1, \ldots, z_n]$ such that

$$\deg\left[(1 - z_0^T B_j(z)) S_0\right] \leqslant (n+4) 2T(D+1)^n \tag{5.38}$$

$$\deg(S_i \Phi_i) \leqslant (n+4) 2T(D+1)^n, \tag{5.39}$$

$$h(a_j) \leqslant \varkappa 2TD^{3n+1}(h + \log N + D \log D), \tag{5.40}$$
$$\leqslant \varkappa D^{4n+2}(h + \log N + D \log D),$$

and

$$a_j = (1 - z_0^T B_j) S_0 + S_1 \Phi_1 + \ldots + S_n \Phi_n. \tag{5.41}$$

Following [10] we decompose S_i as

$$S_i(z_0, z) = \sum_{k=0}^{T-1} S_{i,k}(z_0^T, z) z_0^k.$$

The identity (5.41) implies that

$$a_j = (1 - z_0^T B_j(z)) S_{0,0}(z_0^T, z) + S_{1,0}(z_0^T, z) \Phi_1(z) + \ldots + S_{n,0}(z_0^T, z) \Phi_n(z). \tag{5.42}$$

Replace z_0^T by X, then one has

$$\deg_X S_{i,0} \leq 2(n+4)(D+1)^n.$$

Therefore, as in Rabinowitsch's trick we let $X = 1/B_j$ and define

$$b_j = a_j B_j^{\gamma}, \quad \gamma = 2(n+4)(D+1)^n. \tag{5.43}$$

We have

$$b_j = \sum_{k=1}^{n} a_{jk} \Phi_k, \tag{5.44}$$

with $a_{jk} \in \mathbf{Z}[z]$ satisfying the estimates

$$\deg a_{jk} \leq \varkappa D^{2n+1} \tag{5.45}$$

The height of the a_{jk} is unknown but the height of b_j can be bounded using (5.37), (4.2) and (5.40).

Let us now take $M = n^2\mathfrak{p}$, to guarantee that the polynomials b_j^M are in the ideal generated by the entries of $\Phi^{\mathfrak{p}}$. We have from (5.44) that

$$b_j^M = \sum_{k=1}^{n} a_{jk}^{(M)} \Phi_k^{\mathfrak{p}}, \tag{5.46}$$

for some $a_{jk}^{(M)} \in \mathbf{Z}[z]$. Let $\Delta_M = \det(a_{jk}^{(M)})$, then

$$\begin{aligned} \deg a_{jk}^{(M)} &\leq \varkappa M D^{2n+1} \leq \varkappa D^{3n+1}, \\ \deg \Delta_M &\leq \varkappa D^{3n+1}. \end{aligned} \tag{5.47}$$

From the law of transformation of residues (2.13) we conclude that

$$\left\langle \bar{\partial} \frac{1}{\Phi^{\mathfrak{p}}}, \frac{\zeta^k}{f_{n+1}} d\zeta \right\rangle = \left\langle \bar{\partial} \frac{1}{b^M}, \frac{\Delta_M \zeta^k}{f_{n+1}} d\zeta \right\rangle_{\mathfrak{V}},$$

with $\mathfrak{V} = \{z \in \mathbf{C}^n : \Phi_1(z) = \ldots = \Phi_n(z) = 0\}$ and $b = (b_1, \ldots, b_n)$. Since Δ_M has integral coefficients and we only worry about the denominators of the algebraic numbers that appear as the residues at each point $\alpha \in \mathfrak{V}$ in the last formula, we can reduce ourselves to look for a common denominator of all the numbers

$$\left\langle \bar{\partial}\frac{1}{b^M}, \frac{\zeta^k}{f_{n+1}}d\zeta \right\rangle_\alpha, \quad \alpha \in \mathfrak{B}, \tag{5.48}$$

where $|k| \leq n(4n+1)D^n + \deg \Delta_M \leq \varkappa D^{3n+1}$.

The quantity (5.48) has the advantage that it can be computed by an iteration of the usual formulas of the Residue Calculus in one variable. We have profited already from this remark in the proof of Lemma 2.3. Let $\alpha=(\alpha_1, \ldots, \alpha_n)$ in $\mathfrak{B}$. Let ν_j be the multiplicity of α_j as a zero of B_j, then $\nu_j \geq 1$ and

$$B_j(z_j) = (z_j - \alpha_j)^{\nu_j}\theta_j(z_j),$$

with $\theta_j \in \mathbf{Z}[\alpha_j][z_j]$ defined by this identity and $\theta_j(\alpha_j) \neq 0$. Let

$$\mathfrak{a} = (a_1 \ldots a_n)^M, \tag{5.49}$$

and $\mathfrak{M} = (M\gamma\nu_1 - 1, \ldots, M\gamma\nu_n - 1)$. Then

$$|\mathfrak{M}| = M\gamma(\nu_1 + \ldots + \nu_n) - n, \quad \mathfrak{M}! = (M\gamma\nu_1 - 1)! \ldots (M\gamma\nu_n - 1)!.$$

This vector $\mathfrak{M}$ depends on the point α.

The function $\zeta^k/f_{n+1}(\zeta)$ is holomorphic in a neighborhood of α, hence

$$\left\langle \bar{\partial}\frac{1}{b^M}, \frac{\zeta^k}{f_{n+1}}d\zeta \right\rangle_\alpha = \frac{1}{\mathfrak{a}}\frac{1}{\mathfrak{M}!}\left(\frac{\partial^{|\mathfrak{M}|}\Theta_k}{\partial z^{\mathfrak{M}}}\right)(\alpha), \tag{5.50}$$

where

$$\Theta_k(z) = \frac{z^k}{f_{n+1}(z)}(\theta_1(z_1) \ldots \theta_n(z_n))^{-M\gamma}. \tag{5.51}$$

We rewrite the derivatives in (5.50) using the Leibniz product formula and obtain an expression of the form

$$\mathfrak{F}_k\left(\alpha_1, \ldots, \alpha_n, \frac{1}{f_{n+1}(\alpha)}, \frac{1}{\theta_1(\alpha_1)}, \ldots, \frac{1}{\theta_n(\alpha_n)}\right),$$

where $\mathfrak{F}_k$ is a polynomial in $\mathbf{Z}[X_1, \ldots, X_{2n+1}]$. Let $\delta_j = \deg_{X_j}\mathfrak{F}_k$, then it is easy to see that

$$\delta_j \leq \varkappa D^{3n+2} \quad (1 \leq j \leq n)$$

$$\delta_{n+1} \leq |\mathfrak{M}| + 1 \leq \varkappa D^{3n+1}$$

$$\delta_{n+1+j} \leq M\gamma + M\gamma\nu_j \leq \varkappa D^{3n+1} \quad (1 \leq j \leq n).$$

(It is hard to estimate ν_j better than by $\deg B_j$, i.e., $\nu_j \leq \varkappa D^{n+1}$.)

From the observation (5.31) we see now that a denominator of the residues

$$\left\langle \bar{\partial} \frac{1}{b^M}, \frac{\zeta^k}{f_{n+1}} d\zeta \right\rangle$$

is

$$\mathfrak{a}\mathfrak{M}! \prod_{j=1}^{n} (\operatorname{den}(\alpha_j))^{\delta_j} \times (\operatorname{den}(1/f_{n+1}(\alpha)))^{\delta_{n+1}} \times \prod_{j=1}^{n} (\operatorname{den}(1/\theta_j(\alpha_j)))^{\delta_{n+1+j}}.$$

This expression depends a priori on the multiindex k, but taking the largest possible value for the δ_j we obtain a denominator which is valid for all the k that appear in the computations. That is, we should consider the integer $\mathfrak{d}_\alpha$ given by

$$\mathfrak{d}_\alpha := \mathfrak{a}\mathfrak{M}! \left(\left(\prod_{j=1}^{n} \operatorname{den}(\alpha_j) \right)^{D} \times \operatorname{den}(1/f_{n+1}(\alpha)) \times \prod_{j=1}^{n} \operatorname{den}(1/\theta_j(\alpha_j)) \right)^{\varkappa D^{3n+1}}, \tag{5.53}$$

for some integer $\varkappa = \varkappa(n)$.

The next step is to estimate the denominators that appear in (5.53), still for a fixed $\alpha \in \mathfrak{V}$. For α_j, we use that $B_j(\alpha_j)=0$, hence $\operatorname{den}(\alpha_j)$ divides the leading term of B_j. Therefore

$$\max_j \log \operatorname{den}(\alpha_j) \leq \max_j h(B_j) \leq \varkappa D^{2n+1}(h + \log N + D \log D). \tag{5.54}$$

For the other terms we use (5.33). We need first to know the degree of the algebraic numbers $f_{n+1}(\alpha)$ and $\theta_j(\alpha_j)$. Our previous Corollary 2.2 allows us to conclude that

$$\deg(f_{n+1}(\alpha)), \deg(\theta_j(\alpha_j)) \leq (D+1)^n, \tag{5.55}$$

since $\mathfrak{V}$ is defined by equations of degree $\leq D+1$.

To find $s(\alpha_j)$ we use again the equation $B_j(\alpha_j)=0$. The conjugates of α_j are solutions of the same equation, hence their absolute can be estimated by the inequality (4.16). Then

$$\log |\tilde{\alpha}_j| \leq \log(\deg B_j) + h(B_j).$$

Hence,

$$\max_j s(\alpha_j) \leq \varkappa D^{2n+1}(h + \log N + D \log D). \tag{5.56}$$

Therefore, formula (5.32) gives the upper bounds

$$s(f_{n+1}(\alpha)) \leq h(f_{n+1}) + \sum_{j=1}^{n} (Ds(\alpha_j) + \log(D+1)) \leq \varkappa D^{2n+2}(h + \log N + D \log D).$$

The values $\theta_j(\alpha_j)$ are also explicitly given in $\mathbf{Z}[\alpha_j]$, namely

$$\theta_j(\alpha_j) = B_j^{(\nu_j)}(\alpha_j)/\nu_j!.$$

The height of the polynomial $B_j^{(\nu_j)}(t)/\nu_j!$ is at most $H(B_j) \times 2^{\deg B_j}$. Use again (5.32) to obtain

$$s(\theta_j(\alpha_j)) \leq \varkappa D^{3n+2}(h + \log N + D \log D).$$

From (5.33) we conclude

(5.57) $$\log \operatorname{den}(1/f_{n+1}(\alpha)) \leq 2 \deg(f_{n+1}(\alpha))\, s(f_{n+1}(\alpha)) \leq \varkappa D^{3n+2}(h + \log N + D \log D),$$

and, for $1 \leq j \leq n$,

(5.58) $$\log \operatorname{den}(1/\theta_j(\alpha_j)) \leq \varkappa D^{4n+2}(h + \log N + D \log D).$$

These computations lead to the following estimate for $\log \mathfrak{d}_\alpha$:

(5.59) $$\log \mathfrak{d}_\alpha \leq \varkappa D^{7n+3}(h + \log N + D \log D).$$

We know that $\#(\mathfrak{V}) \leq (2D+1)^n$ by the Bezout estimate. Moreover, the above reasoning shows that if we define

(5.60) $$\mathfrak{d} = \prod_{\alpha \in \mathfrak{V}} \mathfrak{d}_\alpha,$$

then $\mathfrak{d}$ is a denominator for any coefficient in the formula (5.30), hence it can be taken as the value in the statement of this theorem. We have

$$\log \mathfrak{d} \leq \varkappa D^{8n+3}(h + \log N + D \log D),$$

from (5.59) and (5.60). This is precisely the statement of the lemma. □

To finish the proof of Theorem 5.1 we only need to estimate $h(q_j)$. Given that we know a common denominator for all the rational numbers that appear in the formula (5.30), it is enough to find an upper bound of the absolute values of these coefficients. This can be done analytically, again by estimation of residues.

LEMMA 5.6. *The rational numbers ϱ_k defined by (5.34) satisfy the estimate*

$$\log^+|\varrho_k| \leq \varkappa D^{5n+1}(h+\log N+D\log D), \tag{5.61}$$

for $|k| \leq n(2n+1)D^n$.

Proof. We use the notation of the previous lemma, in particular, $\mathfrak{B}$ is the variety defined by $\Phi_1, \ldots, \Phi_n$. It is also exactly the set of common zeros of $f_1, \ldots, f_n$. Therefore, as we have already said in the previous lemma, $\Phi_1, \ldots, \Phi_n, f_{n+1}$ do not have any common zeros in $\mathbf{C}^n$. It follows from [11, Theorem A], applied to $\Phi_1, \ldots, \Phi_n$, that for any $\alpha \in \mathfrak{B}$

$$\log|f_{n+1}(\alpha)| \geq -(D+1)^n[11(n+1)^5(D+1)+(n+1)^2\max\{h(\Phi_j)\ (1 \leq j \leq n), h(f_{n+1})\}$$
$$+2(n+1)^2\log^+\|\alpha\|].$$

From (5.26) and (5.56) we conclude that

$$\log|f_{n+1}(\alpha)| \geq -\varkappa D^{3n+1}(h+\log N+D\log D). \tag{5.62}$$

If we take a ball $B(\alpha, \eta)$ centered at α and of radius η, $\log\eta = -\varkappa D^{3n+1}(h+\log N+D\log D)$, the same inequality (5.62) holds in $B(\alpha, \eta)$ (with a slightly different constant $\varkappa$).

There are at most $(D+1)^n$ points in $\mathfrak{B}$. Divide the ball $B(\alpha, \eta)$ in $(D+1)^n+1$ concentric shells, one of them does not contain any points in $\mathfrak{B}$. Hence, on the sphere S_α that lies half-way between the boundaries of this shell, we have

$$d(\zeta, \mathfrak{B}) \geq \frac{\eta}{2((D+1)^n+1)}, \quad \zeta \in S_\alpha.$$

We can now apply the local Nullstellen inequality in [9, Theorem A] to the family $\Phi_1, \ldots, \Phi_n$. At any point ζ_0 in S_α we obtain

$$\log \max_{1 \leq j \leq n} |\Phi_j(\zeta_0)| \geq -(2D+1)^n[11(n+1)^5(D+1)+(n+1)^2 \max_{1 \leq j \leq n} h(\Phi_j)$$
$$+2(n+1)^2\log^+\|\alpha\| - (n+1)^2 \log d(\zeta, \mathfrak{B})]$$
$$\geq -\varkappa D^{4n+1}(h+\log N+D\log D),$$

due to the choice of η. Let i be index for which $|\Phi_i(\zeta_0)| = \max_{1 \leq j \leq n}|\Phi_j(\zeta_0)|$. We have $\Phi_i(\zeta_0) = \Lambda_i(\zeta_0)\, g_i(\zeta_0)$ and

$$\log|g_i(\zeta_0)| \leqslant h(g_i) + n\log(D+1) + D\log^+\|\zeta_0\|$$
$$\leqslant \varkappa D^{2n+2}(h + \log N + D\log D).$$

Therefore

$$\log|\Lambda_i(\zeta_0)| \geqslant -\varkappa D^{4n+1}(h + \log N + D\log D).$$

Hence, recalling that $f_i = \Lambda_i^{\mathfrak{p}} g_i$ (cf. (5.21) and (5.22)), we get

$$\log|f_i(\zeta_0)| = \log|\Phi_i(\zeta_0)| + (\mathfrak{p}-1)\log|\Lambda_i(\zeta_0)|$$
$$\geqslant -\varkappa D^{5n+1}(h + \log N + D\log D).$$

We conclue that on any point $\zeta \in S_\alpha$,

$$\log\|f(\zeta)\| = \log\left(\sum_{j=1}^{n} |f_j(\zeta)|^2\right)^{1/2} \geqslant -\varkappa D^{5n+1}(h + \log N + D\log D). \tag{5.63}$$

This inequality holds for any $\alpha \in \mathfrak{B}$.

Let us consider the family of closed balls B_α such that $\partial B_\alpha = S_\alpha$. To simplify the reasoning, we order the $\alpha \in \mathfrak{B}$ so that the radii of the B_α are decreasing, $\mathfrak{B} = \{\alpha_i : 1 \leqslant i \leqslant \nu\}$. Consider the auxiliary sets $\Omega_1 = B_1$, $\Omega_2 = B_1 \setminus B_2$, $\Omega_3 = B_3 \setminus (B_1 \cup B_2)$, etc., disregarding the empty ones. These domains are disjoint, $\mathfrak{B} \subseteq \bigcup_i \mathring{\Omega}_i$, and the surface area of any $\partial\Omega_i$ can be estimated by $\omega_{2n-1}(D+1)^n \eta^{2n-1}$, ω_{2n-1}=surface area of unit sphere in $\mathbf{C}^n$.

We have that

$$\varrho_k = \sum_i \left(\sum_{\alpha \in \mathfrak{B} \cap \Omega_i} \left\langle \bar\partial \frac{1}{f}, \frac{\zeta^k d\zeta}{f_{n+1}} \right\rangle_\alpha \right). \tag{5.64}$$

Each sum between parenthesis in (5.64) can be computed using the Bochner–Martinelli formula [17]:

$$\sum_{\alpha \in \mathfrak{B} \cap \Omega_i} \left\langle \bar\partial \frac{1}{f}, \frac{\zeta^k d\zeta}{f_{n+1}} \right\rangle_\alpha = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial\Omega_i} \frac{\zeta^k}{f_{n+1}(\zeta)} \|f(\zeta)\|^{-2n} \left(\sum_{j=1}^{n} (-1)^{j-1} \bar f_j \bigwedge_{\substack{l=1 \\ l \neq j}}^{n} \overline{\partial f_l} \wedge d\zeta \right).$$

We know the behavior of every term in this integral, and the bound

$$\log^+|\varrho_k| \leqslant \varkappa D^{5n+1}(h + \log N + D\log D)$$

is now immediate. □

Let us recall that in the formula (5.30) the right hand side can be written as

$$\left\langle \bar\partial \frac{1}{f}, \frac{1}{f_{n+1}} \Delta_1(z,\zeta)\, d\zeta \right\rangle f_1(z) + \ldots + \left\langle \bar\partial \frac{1}{f}, \frac{1}{f_{n+1}} \Delta_{n+1}(z,\zeta)\, d\zeta \right\rangle f_{n+1}(z) \tag{5.65}$$
$$- \left\langle \bar\partial \frac{1}{f}, \frac{1}{f_{n+1}} \sum_{i=1}^{n} \Delta_i(z,\zeta) f_i(\zeta)\, d\zeta \right\rangle,$$

where Δ_i denote the $n \times n$ minors that appear when we develop the determinant in (5.30) along the last row. The last term is zero because the residue current is evaluated on a form which is locally in the ideal generated by $f_1, \ldots, f_n$. Therefore, let us denote by $\Sigma c_{kl} \zeta^k z^l$ the polynomial in $2n$ variables which represents the whole determinant in (5.30) or one of the minors $\Delta_1, \ldots, \Delta_{n+1}$. Since we are using the $g_{i,j}$ defined after (3.4), it follows that $c_{k,l} \in \mathbf{Z}$. The height of this polynomial can be estimated in terms of the heights $h(f_j)$, namely

$$\max_{k,l} \log |c_{k,l}| \leq \varkappa\left(\max_{1 \leq j \leq n+1} h(f_j) + \log D \right).$$

Recall $f_j = \Lambda_j^{\mathfrak{p}} g_j$, $1 \leq j \leq n$, hence

$$h(f_j) \leq \mathfrak{p}(h(\Lambda_j) + \log n) + h(g_j) \leq \varkappa D^{2n}(h + \log N + D \log D).$$

This estimate is also valid for $h(f_{n+1})$ (see (5.27)). It follows that

$$\max_{k,l} \log |c_{k,l}| \leq \varkappa D^{2n}(h + \log N + D \log D).$$

The polynomials multiplying $f_1, \ldots, f_{n+1}$ in (5.65) are in $\mathbf{Q}[z]$; they are of the form

$$\sum \gamma_l z^l = \sum_l \left(\sum_k c_{k,l} \left\langle \bar\partial \frac{1}{f}, \frac{\zeta^k}{f_{n+1}} d\zeta \right\rangle \right) z^l = \sum_l \left(\sum_k c_{k,l} \varrho_k \right) z^l.$$

In this sum, $|k| \leq n(2n+1) D^n$, hence

$$\log |\gamma_{m,l}| \leq \log \varkappa + n^2 \log D + \max_{k,l} \log |c_{k,l}| + \max_k \log |\varrho_k|$$
$$\leq \varkappa D^{5n+1}(h + \log N + D \log D).$$

Summarizing, the formula (3.40) can be written in the form

$$1 = A_1 f_1 + \ldots + A_{n+1} f_{n+1},$$

with good estimates on the degrees of the polynomials $A_j \in \mathbf{Q}[z]$. Furthermore, we have an estimate for the logarithm λ of the largest absolute value among the coefficients of all the A_j given by

$$\lambda \leq \varkappa D^{5n+1}(h+\log N+D\log D). \tag{5.66}$$

It is clear that a common denominator for all the numbers ϱ_k is also a common denominator for all the coefficients of the A_j. By Lemma 5.4 we have a common denominator $\mathfrak{d} \in \mathbf{Z}^+$ so that the polynomials defined by $\tilde{A}_j = \mathfrak{d} A_j$, will have integral coefficients and satisfy

$$h(\tilde{A}_j) \leq \log \mathfrak{d} + \lambda \leq \varkappa D^{8n+3}(h+\log N+D\log D),$$

and

$$\tilde{A}_1 f_1 + \ldots + \tilde{A}_{n+1} f_{n+1} = \mathfrak{d}. \tag{5.67}$$

Finally, we write explicitly the polynomials f_j in terms of $p_1, \ldots, p_N$, replace in (5.67) and use Lemma 5.2 to estimate the height of the resulting $q_j \in \mathbf{Z}[z]$, which therefore solve the equation

$$q_1 p_1 + \ldots + q_N p_N = \mathfrak{d}.$$

One easily sees that the above estimate for the $h(\tilde{A}_j)$ remains valid for the $h(q_i)$. This concludes the proof of Theorem 5.1. □

(1) The essential property of $\mathbf{Z}$ that we have used is that $\mathrm{Pol}(\mathbf{Z}[X_1, \ldots, X_m])$ could be equipped with a size $\mathfrak{t}$. We can replace $\mathbf{Z}$ throughout by the ring $\mathfrak{O}_K$ of integers of a number field K. The constant $\varkappa$ will depend not only on n but also on $[K:\mathbf{Q}]$.

(2) In the first version of this paper we had succeeded in proving this result with a smaller and explicit constant $\varkappa(n)$. This was done under the additional assumption that the variety of zeros at ∞ of the $p_1, \ldots, p_N$ was discrete. This indicates that the exponents in (5.1), (5.2) and (5.3) are not optimal. In fact, from [32, Theorem 1] one knows that there is a formula $\mathfrak{d} = \Sigma_{i=1}^m p_i q_i$, with $\log \mathfrak{d} \leq \varkappa D^n(h+D\log D)$.

(3) It would be particularly interesting for the case $d_1 = \ldots = d_N = 2$ to improve all the above estimates.

(4) In the related problem, given a polynomial f in the ideal generated by $p_1, \ldots, p_N$ in $\mathbf{C}[z]$, find optimal bounds for the degrees of polynomials $q_j \in \mathbf{C}[z]$ such that

$$f = p_1 q_1 = \ldots + p_N q_N,$$

it is known that in general $\max \deg q_j \geq D^{2^n}$ (essentially). One can prove by analytic methods that if $p_1, \ldots, p_N$ define a discrete variety V or, if $N<n$ and $\dim V=n-N$, then one can find q_j with $\max \deg q_j \leq \deg f + \varkappa D^\mu$ (see [8]). It would be interesting to obtain also bounds for the heights when $f, p_1, \ldots, p_N \in \mathbf{Z}[z]$.

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Received January 19, 1989