Lectures at CROCE DI MAGARA, June 98

(Alain Yger)

1. Residue symbols, Residue currents.

1. Definition in the complete intersection case.

We will start with the definition of the residue symbol in the discrete case. When $f_1, ..., f_n$ denote n holomorphic functions of n variables in some neighborhood V of the origin in $\mathbf{C}^n$, such that the origin is a simple isolated zero of the f_j, $j = 1, ..., n$, which means that

$$\mathrm{Jac}[f_1, ..., f_n](0) \neq 0,$$

one can define, for each $(n, 0)$-differential form $h d\zeta_1 \wedge \cdots \wedge d\zeta_n$, where h denotes a germ of holomorphic function at the origin, the residue symbol

$$\mathrm{Res}\begin{bmatrix} h d\zeta \\ f_1, ..., f_n \end{bmatrix} := \frac{h(0)}{\mathrm{Jac}[f_1, ..., f_n](0)}\,.$$

When the origin is still an isolated zero, but is not simple any more, we can define the residue symbol as follows: let us suppose for the moment that $\mathrm{Jac}[f_1, ..., f_n]$ is not identically equal to zero near the origin (in fact, we will see later on that this is automatic as soon as 0 is an isolated common zero of the f_j; moreover it is impossible that $\mathrm{Jac}[f_1, ..., f_n]$ lies in the ideal generated by the f_j). Then, following Sard's theorem, the set of critical values for the map $(|f_1|^2, ..., |f_n|^2)$ has Lebesgue measure 0, which implies that for almost $(\epsilon_1, ..., \epsilon_n) \in]0, \infty[^n$, close to 0, the common zeroes of

$$(f_1 - \epsilon_1 e^{i\theta_1}, ..., f_n - \epsilon_n e^{i\theta_n})$$

are μ simple isolated points in V, where μ denotes the multiplicity (or the topological degree) of the map $(f_1, ..., f_n)$. It is natural to consider, for such ϵ and any θ in $[0, 2\pi]^n$ (with $\epsilon e^{i\theta} := (\epsilon_1 e^{i\theta_1}, ..., \epsilon_n e^{i\theta_n})$)

$$I(\epsilon, \theta; h d\zeta) = \sum_{f(\alpha) = \epsilon e^{i\theta}} \frac{h(\alpha)}{\mathrm{Jac}[f_1, ..., f_n](\alpha)} = \mathrm{Tr}_{\{f = \epsilon e^{i\theta}\}}[h d\zeta]\,.$$

One can notice (with Fubini and Lebesgue's theorems) that

$$I(\epsilon; h d\zeta) := \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} I(\epsilon, \theta; h d\zeta) d\theta_1 ... d\theta_n = \frac{1}{(2i\pi)^n} \int_{\Gamma_\epsilon(f)} \frac{h(\zeta) d\zeta}{f_1 \cdots f_n}\,, \tag{1.1}$$

where $\Gamma_\epsilon(f)$ is the n-dimensional real manifold

$$\Gamma_\epsilon(f) := \{|f_1| = \epsilon_1, ..., |f_n| = \epsilon_n\},$$

with the orientation such that the differential form

$$d\arg(f_1) \wedge \ldots \wedge d\arg(f_n)$$

is a positive one when restricted to $\Gamma_\epsilon(f)$. With Stokes's theorem, one can see that $I(\epsilon; hd\zeta)$ does not depend on ϵ. This will be our definition of the residue symbol in this case

$$\operatorname{Res}\begin{bmatrix} hd\zeta \\ f_1, ..., f_n \end{bmatrix} := \frac{1}{(2i\pi)^n} \int_{\Gamma_\epsilon(f)} \frac{h(\zeta)d\zeta}{f_1 \cdots f_n}. \tag{1.2}$$

When $f_1, ..., f_p$, $p \leq n$, define a $n-p$ dimensional analytic set in V, one can define, for any $(n, n-p)$ C^1 differential form φ, with support in V, which is closed in some neighborhood of $V_f := \{f_1 = ... = f_p = 0\}$ (note that this is not incompatible with the fact that φ has compact support), the residue symbol

$$\operatorname{Res}\begin{bmatrix} \varphi \\ f_1, ..., f_p \end{bmatrix} := \frac{1}{(2i\pi)^p} \int_{\Gamma_\epsilon(f)} \frac{\varphi(\zeta)}{f_1 \cdots f_p}, \tag{1.3}$$

where this time

$$\Gamma_\epsilon(f) := \{|f_1| = \epsilon_1, ..., |f_p| = \epsilon_p\},$$

with the orientation such that the differential form

$$d\arg(f_1) \wedge \ldots \wedge d\arg(f_p)$$

is a positive one when restricted to $\Gamma_\epsilon(f)$. Here again, ϵ is taken in $]0, \infty[^p$, close to 0, and outside a set of measure zero, which corresponds to the set of critical values of the map $(|f_1|^2, ..., |f_p|^2)$.

There are two quite important difficulties when dealing with such an approach to residue symbols in analysis from the computational point of view:

• The first one is that the support of the analytic chain $\Gamma_\epsilon(f)$ is usually hard to parametrize, which makes the definitions (1.2) or (1.3) rather nonneffective from the computational point of view.

• The second one, much more involved, is that this definition does not allow us to play with smooth analytic objects and profit in a real way from analysis, for example to get rid in the definition (1.2) or (1.3) of the "rigidity constraint" which is imposed by the fact that numerators of residue symbols must be *closed* forms in a neighborhood of the origin in (1.2), or in a neighborhood of V_f in (1.3). In fact, the natural thing that one could hope would be that, when φ is any smooth test form with compact support in V, then

$$\lim_{\epsilon \mapsto 0} I(\epsilon; \varphi) \tag{1.4}$$

exists in an inconditional way. There are simple examples (due to M. Passare, A. Tsikh, J. E. Björk ([PTS],[BJ2]), showing that in general, the inconditional limit does not exist.

Nethertheless, everything is fine when the f_j define a manifold, that is if the rank of the Jacobian matrix is maximal at all points in V_f. In this situation, one obtains immediately, for example in the discrete case, that for any $(n,0)$ smooth form $\varphi = \psi d\zeta$ with compact support in V

$$\lim_{\epsilon \mapsto 0} I(\epsilon; \psi d\zeta) = \frac{\psi(0)}{\operatorname{Jac}[f_1, ..., f_n](0)} .$$

In order to superate these two difficulties, let us do the following and average (still assuming that φ is closed in a neighborhhood of V_f) the function

$$\epsilon \mapsto I(\sqrt{\epsilon}, \varphi) = I_f(\sqrt{\epsilon}; \varphi)$$

(which in fact is constant for ϵ small) on the simplex $\epsilon_1 + \ldots + \epsilon_n = \epsilon$, where $\epsilon > 0$ is given small enough and such that $\{||f||^2 = \epsilon\}$, where

$$||f||^2 := |f_1|^2 + \cdots + |f_p|^2,$$

is a smooth $2n-1$ real manifold in V. This averaging leads to

$$\operatorname{Res}\left[\begin{matrix} \varphi \\ f_1, ..., f_p \end{matrix}\right] = \frac{(p-1)!}{\epsilon^p} \int_{\eta_1 + \ldots + \eta_p = \epsilon} I(\sqrt{\eta}, \varphi) \sum_{k=1}^{p} (-1)^{k-1} \eta_k d\eta_{[k]},$$

where

$$d\eta_{[k]} := \bigwedge_{\substack{j=1 \\ j \neq k}}^{p} d\eta_j .$$

Using Fubini's and Lebesgue's theorem, toge ther with the identity

$$\Big(\sum_{k=1}^{p} (-1)^{k-1} |f_k|^2 \bigwedge_{j \neq k} d|f_j|^2\Big) \wedge \frac{\varphi}{f_1 \cdots f_p} = \sum_{k=1}^{p} (-1)^{k-1} \overline{f_k df_{[k]}} \wedge \varphi$$

and taking into account the fact that the orientation of $\mathbf{C}^n$ is the one for which the differential form $(dd^c \log(||\zeta||^2))^n$ is positive, one obtains

$$\begin{aligned} \operatorname{Res}\left[\begin{matrix} \varphi \\ f_1, ..., f_p \end{matrix}\right] &= \frac{(-1)^{\frac{p(p-1)}{2}} (p-1)!}{(2i\pi\epsilon)^p} \int_{||f||^2 = \epsilon} \sum_{k=1}^{p} (-1)^{k-1} \overline{f_k df_{[k]}} \wedge \varphi \\ &= \frac{(-1)^{\frac{p(p-1)}{2}} (p-1)!}{(2i\pi)^p} \int_{||f||^2 = \epsilon} \frac{\sum_{k=1}^{p} (-1)^{k-1} \overline{f_k df_{[k]}} \wedge \varphi}{||f||^{2p}} . \end{aligned} \tag{1.5}$$

When $p = n$, one can say more: since the differential form

$$\frac{\sum_{k=1}^{n} (-1)^{k-1} \overline{f_k df_{[k]}} \wedge h d\zeta}{||f||^{2n}}$$

is closed outside 0 (this is an easy computation), the residue symbol can be expressed by Stokes's theorem in this case as

$$\operatorname{Res}\left[\begin{matrix} hd\zeta \\ f_1, ..., f_n \end{matrix}\right] = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{\partial U} h \frac{\sum_{k=1}^{n} (-1)^{k-1} \overline{f_k df_{[k]}} \wedge d\zeta}{\|f\|^{2n}}, \tag{1.7}$$

where U is any compact subset of V with smooth boundary that contains the origin as the only common zero of the f_j. If

$$s_0(\zeta) = \frac{\overline{f}}{\|f\|^2},$$

one can also rewrite (1.7) as

$$\operatorname{Res}\left[\begin{matrix} hd\zeta \\ f_1, ..., f_n \end{matrix}\right] = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{\partial U} h \sum_{k=1}^{n} (-1)^{k-1} s_{0k} ds_{0[k]} \wedge d\zeta \tag{1.8}$$

An homotopy argument shows that one can replace s_0 in formula (1.8) by any fonction s which is defined in a neighborhood of the boundary of U, is C^1 in this neighborhhood, and satisfies

$$< s(\zeta), f(\zeta) >:= \sum_{k=1}^{n} s_k(\zeta) f_k(\zeta) \equiv 1$$

on this boundary. The general formula

$$\operatorname{Res}\left[\begin{matrix} hd\zeta \\ f_1, ..., f_n \end{matrix}\right] = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{\partial U} h \sum_{k=1}^{n} (-1)^{k-1} s_k ds_{[k]} \wedge d\zeta \tag{1.9}$$

is the Bochner-Martinelli formula.

Of course, one can use formula (1.9) a contrario and choose the section s before choosing the domain U. The only thing that one asks respect to U is to be a tubular domain around the zero set V_f. This provides interesting expressions for the residue symbol in the complete intersection case. For example, one can rewrite (1.5) as

$$\operatorname{Res}\left[\begin{matrix} \varphi \\ f_1, ..., f_p \end{matrix}\right] = \frac{(-1)^{\frac{p(p-1)}{2}}(p-1)!}{(2i\pi\epsilon)^p} \int_{<s^\epsilon, f>=1} \sum_{k=1}^{n} (-1)^{k-1} s_k^\epsilon ds_{[k]}^\epsilon \wedge \varphi$$

where ϵ is small enough and

$$s^\epsilon = \frac{\overline{f}}{\epsilon}.$$

One can also replace for example s^ϵ by

$$s^{\epsilon,q} := \frac{(\overline{f_1}|f_1|^{2q_1}, ..., \overline{f_p}|f_p|^{2q_p})}{\epsilon}.$$

This leads to the formula

$$\mathrm{Res}\left[\begin{matrix} f_1^{q_1}\cdots f_p^{q_p}\varphi \\ f_1^{q_1+1},...,f_p^{q_p+1}\end{matrix}\right] = \mathrm{Res}\left[\begin{matrix} \varphi \\ f_1,...,f_p\end{matrix}\right].$$

One can also rewrite the expression of the multidimensional residue as

$$\begin{aligned}
&\mathrm{Res}\left[\begin{matrix} \varphi \\ f_1,...,f_p\end{matrix}\right] = \\
&= \lim_{\tau\mapsto 0_+} \frac{(q_1+1)\cdots(q_p+1)}{(2i\pi)^p} \times \\
&\times \int_{\gamma_1+i\mathbf{R}}\cdots\int_{\gamma_p+i\mathbf{R}} \prod_{k=1}^{p}\Gamma(1-s_k)\Gamma(|s|+1)\Gamma((q_1+1)s_1,...,(q_p+1)s_p)\tau^{-|s|}ds_1\ldots ds_p
\end{aligned} \tag{1.10}$$

where

$$\Gamma(\underline{\lambda};\varphi) := (-1)^{\frac{p(p-1)}{2}} \int_V \prod_{k=1}^{p} |f_k|^{2(\lambda_k-1)} \bigwedge_{k=1}^{p} \overline{df_k}\wedge\varphi$$

with the $\gamma_k \in]0,1[$ for any k between 1 and p and $|s| := s_1+...+s_p$, and the q_k lie in $\mathbf{N}$. Formula (1.10) is a Mellin-Barnes representation formula for the residual symbol. In fact, such a formula holds when the test form φ is C^∞ with compact support in V and allows us to extend the action of our residue symbol on smooth test forms (non necesseraly closed near V_f). In fact, the right-hand side in (1.10) does not depend of $\vec{q}$. The local residue is then defined by (1.10). All this works fine only when the f_j define a complete intersection in V.

As for the iterated residues, we get

$$\mathrm{Res}\left[\begin{matrix} \varphi \\ f_1^{q_1+1},...,f_p^{q_p+1}\end{matrix}\right] = \frac{(-1)^{\frac{p(p-1)}{2}}}{(2i\pi)^p}\int_{<s,f>=1} s_1^{q_1}\ldots s_p^{q_p}\sum_{k=1}^{p}(-1)^{k-1}s_k ds_{[k]}\wedge\varphi. \tag{1.11}$$

When dealing with global problems (of algebraic nature rather than of analytic nature), we will also introduce global residue symbols.

• For example, let $P=(P_1,...,P_n)$ defines a quasi-regular sequence in $\mathbf{C}[X_1,...,X_n]$, that is the analytic set

$$V(P) := \{\zeta\in\mathbf{C}^n,\ P(\zeta)=0\}$$

is zero-dimensional or (which is a more algebraic point of view), whenever there exists $k\in\mathbf{N}$ and polynomials $Q_{\underline{l}}$ in $\mathbf{C}[X_1,...,X_n]$ such that

$$\sum_{\substack{\underline{l}\in\mathbf{N}^n \\ l_1+\cdots+l_n=k+1}} Q_{\underline{l}}P_1^{l_1}\cdots P_n^{l_n}\in I(P)^k,$$

where $I(P)$ is the ideal generated by the P_j, then all the $Q_{\underline{l}}$ belong to $I(P)$. Then, for any $Q \in \mathbf{C}[X_1, ..., X_n]$, one denotes

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1, ..., P_n \end{bmatrix} := \sum_{\alpha \in V(P)} \mathrm{Res}\begin{bmatrix} Q(\zeta)d\zeta \\ P_1, ..., P_n \end{bmatrix}_\alpha$$

where the sum on the right hand side is the sum of local residues at all points in $V(P)$. We will also frequently deal with total sums of residues of rational functions: if P_0 is a polynomial such that the ideal generated by $P_0, ..., P_n$ is $\mathbf{C}[X_1, ..., X_n]$, then, one can define, for any $Q \in \mathbf{C}[X_1, ..., X_n]$, the residue symbol

$$\mathrm{Res}\begin{bmatrix} \frac{Q(X)}{P_0(X)}dX \\ P_1, ..., P_n \end{bmatrix} := \sum_{\alpha \in V(P)} \mathrm{Res}\begin{bmatrix} \frac{Q(\zeta)}{P_0(\zeta)}d\zeta \\ P_1, ..., P_n \end{bmatrix}_\alpha .$$

• Another type of global situation concerns Laurent polynomials in n variables. Let $F_1, ..., F_n$ be n polynomials in $X_1^{\pm 1}, ..., X_n^{\pm 1}$ with complex coefficients, defining a zero dimensional analytic set $V^*(F)$ in $\mathbf{T}^n = (\mathbf{C}^*)^n$,

$$V^*(F) := \{\zeta \in \mathbf{T}^n,\ F_1(\zeta) = ... = F_n(\zeta) = 0\}$$

and P_0 another Laurent polynomial in $X_1^{\pm 1}, ..., X_n^{\pm 1}$; then, one can define the toric residue symbol

$$\mathrm{Res}\begin{bmatrix} \frac{Q(X)}{P_0(X)}dX \\ P_1, ..., P_n \end{bmatrix}_{\mathbf{T}} := \sum_{\alpha \in V^*(P)} \mathrm{Res}\begin{bmatrix} \frac{Q(\zeta)}{P_0(\zeta)} \frac{d\zeta}{\zeta_1 \cdots \zeta_n} \\ P_1, ..., P_n \end{bmatrix}_\alpha .$$

The reason why one uses here the differential form $\frac{d\zeta}{\zeta_1 \cdots \zeta_n}$ instead of $d\zeta$ is that one wants the monoidal change of coordinates (which are standard in the toric setting) to have a nice action on residue symbols.

This algebraic notion of residue symbol can be extended to the case when $(a_1, ..., a_n)$ is a quasi-regular sequence in a commutative $\mathbf{A}$-algebra $\mathbf{R}$, such that the quotient $\mathbf{P} := \mathbf{R}/(a_1, ..., a_n)$ is a projective $\mathbf{A}$-module finetely generated (so that $\mathrm{Hom}_{\mathbf{A}}(\mathbf{P}, \mathbf{P})$ can be equipped with a Trace). The algebraic definition of the residue symbols in this case is the pendant of the analytic definition we proposed at the beginning. If σ is a $\mathbf{A}$-linear section of the projection map

$$\pi :\ \mathbf{R} \mapsto \mathbf{P} := \mathbf{R}/(a_1, ..., a_n)\,,$$

one can associate to any $r \in \mathbf{R}$ an element $r^{\#}$ in $\mathrm{Hom}_{\mathbf{A}}(\mathbf{P}, \mathbf{P})[[a_1, ..., a_n]]$, defined as

$$r^{\#} := \sum_{\underline{l} \in \mathbf{N}^n} r_{\underline{l}} a_1^{l_1} \cdots a_n^{l_n}$$

and

$$r\sigma(u) = \sum_{\underline{l} \in \mathbf{N}^n} \sigma(r_{\underline{l}}(u)) a_1^{l_1} \cdots a_n^{l_n},\ u \in \mathbf{P},$$

is the development of $r\sigma(u)$ in the (a)-adic completion of $\mathbf{R}$. Of course, the construction of these operators $r_{\underline{l}}$ depends on the section one takes. If $r, r_1, ..., r_n$ are $n+1$ elements in $\mathbf{R}$, one can compute in $\mathrm{Hom}_{\mathbf{A}}(\mathbf{P},\mathbf{P})[[a_1,...,a_n]]$ the element

$$r^{\#} \circ \det\left[\frac{\partial r_j^{\#}}{\partial a_j}\right]_{1\le i,j\le n} = \sum_{\underline{l}\in\mathbf{N}^n} T_{\underline{l},\sigma}(r,r_1,...,r_n)a_1^{l_1}\dots a_n^{l_n}$$

(respect the rules of the determinant calculus in a non commutative setting!). Then one can define

$$\mathrm{Res}\begin{bmatrix} r dr_1 \wedge \cdots \wedge dr_n \\ a_1^{q_1+1},...,a_n^{q_n+1}\end{bmatrix} := \mathrm{Tr}(T_{\underline{q},\sigma}(r,r_1,...,r_n))\,.$$

In fact, these symbols do not depend on the choice of the section. In the local situation of $\mathcal{O}_n$ studied above, their definition fits with the definition which has been proposed at the beginning of this paragraph. This construction is due to J. Lipman [Li] and the fact that the two definitions fit together in the analytic case is proved in [Hi-Bo 1].

2. The Transformation law.

We will first study the local situation. Suppose that $f_1,....,f_p$ and $g_1,...,g_p$ are two sequences of holomorphic functions in a neighborhood V of the origin in $\mathbf{C}^n$, both defining a complete intersection in this neighborhood. Suppose also that

$$g = Af$$

where A is a matrix with holomorphic coefficients in V. Then the zero set V_f is included in V_g. Therefore, if U is a tubular neighborhood of V_g, such that the boundary of U is given by

$$\partial U = \{< s, g >\equiv 1\},$$

one can also consider U as a neighborhood of V_f, with the boundary expressed as

$$\partial U = \{< A^t s, f >\equiv 1\},$$

where A^t is the transposed of the matrix A.

Let $H(V)$ be the algebra of holomorphic functions in V. In order to settle our next proposition, we need to introduce the $H(V)$-module $\mathcal{C}^{n,n-p}$ of $(n,n-p)$ smooth differential forms in V which are closed in a neighborhood of V_g and the module $\mathcal{M} := \mathrm{Hom}_{\mathbf{C}}(\mathcal{C}^{n,n-p},\mathbf{C})$, considered as a $H(V)$-module when equipped with the external operation

$$hM(\varphi) = M(h\varphi),\ h\in H(V),\ \varphi\in\mathcal{C}^{n,n-p}\,.$$

Let σ_f and σ_g be the two homomorphisms of $H(V)$- modules from $H(V)[X_1,...,X_p]$ into $\mathcal{M}$ such that

$$\sigma_f(X_1^{q_1}\cdots X_p^{q_p}) : \varphi\mapsto q_1!\cdots q_p!\mathrm{Res}\begin{bmatrix} \varphi \\ f_1^{q_1+1},...,f_p^{q_p+1}\end{bmatrix}$$

$$\sigma_g(X_1^{q_1}\cdots X_p^{q_p}) : \varphi\mapsto q_1!\cdots q_p!\mathrm{Res}\begin{bmatrix} \varphi \\ g_1^{q_1+1},...,g_p^{q_p+1}\end{bmatrix}.$$

In this setting, one has, using formula (1.11), the following proposition:

Proposition 1.1. *For any P in $H(V)[X_1, ..., X_p]$, one has*

$$\sigma_f(P(X)) = \det A\ \sigma_g(P(A^t X))\,. \tag{2.1}$$

Remark. This is nothing that the chain-rule in differential calculus. In fact, the formula holds in the general algebraic setting where $(g_1, ..., g_n)$ and $(f_1, ..., f_n)$ are two quasi -regular sequences in the $\mathbf{A}$-algebra $\mathbf{R}$, such that

$$g = Af$$

and the modules $\mathbf{R}/(f)$ and $\mathbf{R}/(g)$ are projective and finitely generated. It is better in this case to formulate the result without denominators, namely, for any $r, r_1, ..., r_n \in \mathbf{R}$,

$$\text{Res}\begin{bmatrix} r dr_1 \wedge \ldots \wedge dr_n \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{bmatrix} = \sum_{\substack{|q_{;j}|=q_j \\ 1\le j\le n}} \prod_{i=1}^{n} \binom{\mu_i}{q_{i;}} \text{Res}\begin{bmatrix} \det A \prod_{1\le i,j\le n} a_{ij}^{q_{ij}} r dr_1 \wedge \ldots \wedge dr_n \\ g_1^{\mu_1+1}, ..., g_n^{\mu_n+1} \end{bmatrix} \tag{2.2}$$

with the notations

$$q_{;j} = (q_{1j}, ..., q_{nj}),\ q_{i;} = (q_{i1}, ..., q_{in}),\ |\mu_i| = q_{i1} + \cdots + q_{in},$$

and

$$\binom{\mu_i}{q_{i;}} = \frac{\mu_i!}{q_{i1}! \ldots q_{in}!}\,.$$

Such formulas are originally due to Kytmanov in the analytic context. For the extension to the algebraic context, we refer to [Hi-Bo 2].

There are also useful variants of the transformation law; one which happens to be quite useful is the following: suppose that $(f_0, ..., f_n)$ and $(f_0, g_1, ..., g_n)$ are two quasi-regular sequences in the $\mathbf{A}$-algebra $\mathbf{R}$, such that the modules $\mathbf{R}/(f_0, ..., f_n)$ and $\mathbf{R}/(f_0, g_1, ..., g_n)$ are projective and finitely generated. Suppose that there are relations of the form

$$f_0^{s_j} g_j = \sum_{l=1}^{n} a_{jl} f_l,\ j = 1, ..., n.$$

Then, for any $r, r_0, r_1, ..., r_n \in \mathbf{R}$, for any $q_0 \in \mathbf{N}$,

$$\text{Res}\begin{bmatrix} r\, dr_0 \wedge \ldots \wedge dr_n \\ f_0^{q_0+1}, f_1, ..., f_n \end{bmatrix} = \text{Res}\begin{bmatrix} r\det A\ dr_0 \wedge \ldots \wedge dr_n \\ f_0^{q_0+1+s_1+\ldots+s_n}, g_1, ..., g_n \end{bmatrix}. \tag{2.3}$$

There are certainly other generalisations of such an extension. One interesting to suggest is the extension of the transformation law when $g_1, ..., g_n$, instead of satisfying $g = Af$ satisfy global relations on integral dependency over $(f_1, ..., f_n)$, that is relations of the form

$$g_j^{N_j} + \sum_{k=1}^{N_j} \Big(\sum_{\substack{\underline{l} \in \mathbf{N}^n \\ l_1+\ldots+l_n=k}} a_{k\underline{l}} f_1^{l_1} \ldots f_n^{l_n} \Big) g_j^{N_j-k} = 0,\ j = 1, ..., n\,,$$

in the spirit of the work of Ostrowski. The formulas (2.2) can be extended in this context (assuming the same things as before respect to the quotients $\mathbf{R}/(f)$ and $\mathbf{R}/(g)$).

3. Duality theorems.

The first (and one of the most important respect to effectivity problems) division formula in complex analysis is the Bergman-Weil formula. Let us state it in the semilocal context. Let $f_1, ..., f_m$, $m \geq n$, be m holomorphic functions in a neighborhood V of the origin in $\mathbf{C}^n$, defining (and this is a restrictive clause which will appear to be very important) the origin as an isolated zero. Then, for $\vec{\epsilon} \in]0, \infty[^m$ with $\|\vec{\epsilon}\|$ small enough, there is one connected component Δ of the set

$$\{\zeta \in V,\ |f_1(\zeta)| \leq \epsilon_1, ..., |f_m(\zeta)| \leq \epsilon_m\}$$

which is such that $0 \in \Delta \subset \overline{\Delta} \subset V$. Furthermore, one assumes that any subfamily of $(f_1, ..., f_m)$ with cardinal n defines a quasi-regular sequence in $H(V)$. We also assume that there are holomorphic functions a_{jk}, $j = 1, ..., m$, $k = 1, ..., n$, such that

$$f_j(\zeta) - f_j(z) = \sum_{k=1}^{n} a_{jk}(z,\zeta)(\zeta_k - z_k),\ \ (\zeta, z) \in V \times V\,.$$

Then, on has in Δ, the following representation formula, valid for any function h holomorphic in Δ and continuous in $\overline{\Delta}$:

$$\begin{aligned} h(z) &= \frac{1}{(2i\pi)^n} \sum_{1\leq i_1<...<i_n\leq n} \int_{\gamma_{i_1,...,i_n}} h(\zeta) \frac{\det[a_{i_l,j}(z,\zeta)]_{1\leq l,j\leq n} d\zeta}{(f_1 - f_1(z))...(f_n - f_n(z))} = \\ &= \frac{1}{(2i\pi)^n} \sum_{1\leq i_1<...<i_n\leq n} \sum_{\underline{q}\in \mathbf{N}^n} \frac{h(\zeta)\det[a_{i_l,j}(z,\zeta)]_{1\leq l,j\leq n} d\zeta}{f_1^{q_1+1}...f_n^{q_n+1}} f_1^{q_1}(z)\dots f_n^{q_n}(z) = \\ &= \sum_{1\leq i_1<...<i_n\leq n} \sum_{\underline{q}\in \mathbf{N}^n} \operatorname{Res}\left[\begin{matrix} h\det[a_{i_l,j}(z,\zeta)]_{1\leq l,j\leq n} d\zeta \\ f_{i_1}^{q_1+1}, ..., f_{i_n}^{q_n+1}\end{matrix}\right] f_{i_1}^{q_1}(z)\dots f_{i_n}^{q_n}(z)\,, \end{aligned} \tag{3.1}$$

where $\gamma_{i_1,...,i_n}$ denotes the intersection of the n faces of Δ

$$\gamma_{i_k} := \{\zeta \in \overline{\Delta},\ |f_{i_k}(\zeta)| = \epsilon_k\},\ \ k = 1, ..., n,$$

with the orientation determined by the order of the faces. When $n = m$, the formula is just

$$\begin{aligned} h(z) &= \frac{1}{(2i\pi)^n} \int_{\Gamma_\epsilon(f)} \frac{h\det[g_{jk}(z,\zeta)]_{1\leq l,j\leq n} d\zeta}{(f_1 - f_1(z))\dots(f_n - f_n(z))} = \\ &= \sum_{\underline{q}\in \mathbf{N}^n} \operatorname{Res}\left[\begin{matrix} h\det[g_{jk}(z,\zeta)]_{1\leq l,j\leq n} d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1}\end{matrix}\right] f_1^{q_1}(z)\dots f_n^{q_n}(z)\,, \end{aligned} \tag{3.2}$$

where the g_{jk} are defined by the formulas

$$f_j(z) - f_j(\zeta) = \sum_{k=1}^{n} g_{jk}(z,\zeta)(z_k - \zeta_k),\ j = 1, ..., n\,.$$

When $f_1, ..., f_p$ define a complete intersection in a neighborhood V of the origin ($0 \in V(f)$), one can construct generic linear forms $L_{p+1}, ..., L_n$, such that $f_1, ..., f_p, L_{p+1}, ..., L_n$, define the origin as an isolated zero, with

$$L_j(\zeta) = \sum_{k=1}^{n} \lambda_{jk}\zeta_k,\ j = p+1, ..., n\,.$$

Then, for $\vec{\epsilon} \in]0, \infty[^n$ such that $\|\vec{\epsilon}\|$ is small enough and one can apply Sard's theorem, for any h holomorphic in Δ and continuous in $\overline{\Delta}$, one has

$$h(z) =$$

$$= \frac{1}{(2i\pi)^n} \int_{\substack{|f_j|=\epsilon_j, j=1,...,p \\ |L_k|=\epsilon_k, k=p+1,...,n}} \frac{h(\zeta) \bigwedge_{j=1}^{p} (\sum_{k=1}^{n} g_{jk}(z,\zeta) d\zeta_k) \wedge (\bigwedge_{j=p+1}^{n} \lambda_{jk} d\zeta_k)}{\prod_{j=1}^{p} (f_j - f_j(z)) \prod_{j=p+1}^{n} (L_j - L_j(z))} =$$

$$\equiv \frac{1}{(2i\pi)^n} \int_{\substack{|f_j|=\epsilon_j, j=1,...,p \\ |L_k|=\epsilon_k, k=p+1,...,n}} \frac{h(\zeta) \bigwedge_{j=1}^{p} (\sum_{k=1}^{n} g_{jk}(z,\zeta) d\zeta_k) \wedge (\bigwedge_{j=p+1}^{n} \lambda_{jk} d\zeta_k)}{f_1 \dots f_p \prod_{j=p+1}^{n} (L_j - L_j(z))} \quad (\mathrm{mod}(f_1, ..., f_p))\,. \tag{3.3}$$

The main consequence of the Bergman-Weil formula is the duality theorem:

Theorem 1.1. *Let $f_1, ..., f_p$, be p functions which define a complete intersection in a neighborhood V of the origin in $\mathbf{C}^n$. Then, an holomorphic function h belongs to the ideal generated by $f_1, ..., f_p$ in $H(V)$ if and only if for any $(n, n-p)$ smooth differential form with compact support in V, d-closed in a neighborhood of $V(f)$, one has*

$$\mathrm{Res}\left[\begin{matrix} h\varphi \\ f_1, ..., f_p \end{matrix}\right] = 0\,. \tag{3.4}$$

Proof. When $p = n$, this is just a consequence of the Bergman-Weil formula (3.2). When $p < n$, the idea is to use formula (3.3) and to introduce a sequence of smooth fonctions $(\chi_s)_s$, $s \in \mathbf{N}$, on $[0, \infty]$ that converges to the characteristic function of $[1, \infty[$. One has, if

$$\vec{\epsilon} = (\vec{\epsilon}', \epsilon_{p+1}, ..., \epsilon_n),$$

$$\int_{\substack{|f_j|=\epsilon_j, j=1,\dots,p\\ |L_k|=\epsilon_k, k=p+1,\dots,n}} \frac{h(\zeta)\bigwedge_{j=1}^{p}(\sum_{k=1}^{n} g_{jk}(z,\zeta)d\zeta_k)\wedge(\bigwedge_{j=p+1}^{n}\lambda_{jk}d\zeta_k)}{f_1\dots f_p\prod_{j=p+1}^{n}(L_j-L_j(z))} =$$

$$= \pm\lim_{s\mapsto\infty}\int_{\Gamma_{\vec{\epsilon}'}(f)} \frac{h(\zeta)\bigwedge_{j=1}^{p}(\sum_{k=1}^{n} g_{jk}(z,\zeta)d\zeta_k)\wedge(\bigwedge_{j=p+1}^{n}\lambda_{jk}d\zeta_k)\wedge\bigwedge_{j=p+1}^{n}\overline{\partial}\chi_s(\frac{|L_j|^2}{\epsilon_j^2})}{f_1\dots f_p\prod_{j=p+1}^{n}(L_j-L_j(z))}. \tag{3.5}$$

If condition (3.4) is satisfied, then it follows from (3.5) that

$$\int_{\substack{|f_j|=\epsilon_j, j=1,\dots,p\\ |L_k|=\epsilon_k, k=p+1,\dots,n}} \frac{h(\zeta)\bigwedge_{j=1}^{p}(\sum_{k=1}^{n} g_{jk}(z,\zeta)d\zeta_k)\wedge(\bigwedge_{j=p+1}^{n}\lambda_{jk}d\zeta_k)}{f_1\dots f_p\prod_{j=p+1}^{n}(L_j-L_j(z))} = 0,$$

which implies (following (3.3)) that h lies in the ideal generate by $f_1, \dots, f_p$. The converse is just a consequence of Stokes's theorem. $\diamondsuit$

The situation is much more involved in the non complete intersection case. We will just briefly mention the ideas in this case. The key point is that the action of the residue symbols that have been introduced in section 1 can be extended to smooth differential forms with compact support (that is we can get rid of the fact that the differential form is closed in a neighborhood of the zero set of the f_j). The objects that one can define in this way are currents, that is linear functionals acting on spaces of smooth differential forms with compact support (or also, which is an equivalent point of view, differential forms with coefficients distributions). Let us be more precise: given $f_1, \dots, f_p$, p holomorphic functions in an open subset V of $\mathbf{C}^n$, one can associate to them a large family of currents.

First, we pick up a *weight* $(q_1, \dots, q_p)$ in $\mathbf{N}^p$ and define, for any $\epsilon > 0$, the map $s^{\epsilon,q}$ as

$$s^{\epsilon,q} := \frac{(\overline{f_1}|f_1|^{2q_1}, \dots, \overline{f_p}|f_p|^{2q_p})}{\epsilon}.$$

Then, we pick up a subset $\mathcal{I} := \{i_1, \dots, i_k\}$ in $\{1, \dots, p\}$, with cardinal k, $\mathrm{codim} V(f) \le k \le \inf(n,p)$. For any smooth test φ form with type $(n, n-p)$, one can define the residue symbol

$$\mathrm{Res}\begin{bmatrix} \varphi \\ f_{i_1}, \dots, f_{i_k} \\ f_1, \dots, f_p \end{bmatrix}^{(q)} := \lim_{\epsilon\mapsto 0}\frac{(-1)^{\frac{k(k-1)}{2}}}{(2i\pi)^k}\int_{<s^{\epsilon,q},f>=1}\Big(\sum_{l=1}^{k}(-1)^{l-1}s_{i_l}ds_{[i_l]}\Big)\wedge\varphi. \tag{3.6}$$

Of course, the main difficulty here is to show that such a limit exists, which is not evident at all (the proof involves deeply Hironaka's main theorem about resolution of singularities over a field of characteristic 0 as well as the $\mathcal{D}$-module theory developped by M. Kashiwara and J. E. Bjork). We refer to [PTY] for a detailed proof. Note that if the f_j define a complete intersection, there is just one value of k which is allowed (namely $k=p$) and one can show then that in this case the residue symbol does not depend on the weight q. It corresponds with the definition of residue symbols introduced in section 1. Otherwise, it depends deeply on the choice of this weight. For example, when $p=k=n$, one can show that

$$\text{Res}\begin{bmatrix} \varphi \\ f_1,...,f_p \\ f_1,...,f_p \end{bmatrix}^{(q)} := \lim_{\tau\mapsto 0_+} \frac{(q_1+1)\cdots(q_p+1)}{(2i\pi)^p} \times$$

$$\times \int_{\gamma_1+i\mathbf{R}} \cdots \int_{\gamma_p+i\mathbf{R}} \prod_{k=1}^{p} \Gamma(1-s_k)\Gamma(|s|+1)\Gamma((q_1+1)s_1,...,(q_p+1)s_p)\tau^{-|s|}ds_1\dots ds_p$$

where

$$\Gamma(\underline{\lambda};\varphi) := (-1)^{\frac{p(p-1)}{2}} \int_V \prod_{l=1}^{p} |f_l|^{2(\lambda_l-1)} \bigwedge_{l=1}^{p} \overline{df_l} \wedge \varphi$$

with the $\gamma_l \in]0,1[$ for any l between 1 and p and $|s| := s_1+...+s_p$. The key point here is that this Gamma function has a polar set that contains the origin in $\mathbf{C}^p$, which explains why the limit may depend on the choice of the weight q.

In order to describe an attempt to extend the duality theorem to the case of non complete intersections, we will assume from now on that the number of variables is strictly superior to the number of functions (p). We will consider p functions $f_1,...,f_p$, holomorphic in a neighborhood V of the origin in $\mathbf{C}^n$ and defining an analytic set V_f with codimension d in this neighborhood. We will assume as before that there exist holomorphic functions g_{jk}, $j=1,...,p$, $k=1,...,n$, in $V\times V$, such that

$$f_j(z)-f_j(\zeta)=\sum_{k=1}^{n} g_{jk}(z,\zeta)(z_k-\zeta_k),\ j=1,...,p$$

and for technical reasons, we will introduce the $(1,0)$ differential forms in $V\times V$

$$G_j(z,\zeta) := \sum_{k=1}^{n} g_{jk}(z,\zeta)d\zeta_k,\ j=1,...,p.$$

We will fix a function φ from V to $\mathbf{C}$, which is smooth, with compact support in V, identically equal to 1 near 0, and a $\mathbf{C}^n$-valued smooth application s defined in a neighborhood U of the support of $\overline{\partial}\varphi$ and such that

$$< s(\zeta),\zeta - z > \neq 0,\ \zeta\in U, z\in W$$

where W is a neighborhood of 0 which is contained in the set where $\varphi\equiv 1$. We note

$$\tilde{s}(\zeta,z) = \frac{s(\zeta)}{< s(\zeta),\zeta-z >},\ \zeta\in U,\ z\in W\,.$$

Note that this $\mathbf{C}^n$-valued function $\tilde{s}$ (defined in $U\times W$) depends in an holomorphic way of the variables z. One has the following proposition

Proposition 1.2. *For any $z \in W$, for any $q \in \mathbf{N}^n$, for any h holomorphic in V,*

$$h(z) \equiv -\sum_{k=d}^{p} \sum_{1\le i_1<\ldots<i_k\le p} \sum_{1\le j_1<\ldots<j_{n-k}\le n} \frac{(-1)^{\frac{(n-k)(n-k-1)}{2}}(n-k)!}{(2i\pi)^{n-k}}$$

$$\mathrm{Res}\begin{bmatrix} h\overline{\partial}\varphi \wedge \big(\sum_{l=1}^{n-k}(-1)^{l-1}\tilde{s}_{j_l} d\tilde{s}_{[j_l]}\big)(z,\zeta)\wedge d\zeta_{\mathcal{J}} \wedge G_{\mathcal{I}}(z,\zeta) \\ f_{i_1},\ldots,f_{i_k} \\ f_1,\ldots,f_p \end{bmatrix}^{(q)} \quad \mathrm{mod}\ (f_1,\ldots,f_p). \tag{3.7}$$

where

$$G_{\mathcal{I}} := \bigwedge_{l=1}^{k} G_{i_l}(z,\zeta),\ \mathcal{I} = \{i_1,\ldots,i_k\}$$

$$d\zeta_{\mathcal{J}} := \bigwedge_{l=1}^{n-k} d\zeta_{j_l},\ \mathcal{J} = \{j_1,\ldots,j_{n-k}\}\,.$$

Remark. The right-hand side of (3.7) defines a function which is holomorphic in W. Nethertheless, this proposition does not gives us a duality theorem as in the complete intersection case. The only thing that can be checked about the residue symbols introduced in (3.7) is that

$$\mathrm{Res}\begin{bmatrix} h\psi \\ f_{i_1},\ldots,f_{i_k} \\ f_1,\ldots,f_p \end{bmatrix}^{(q)} = 0 \tag{3.8}$$

when ψ is a smooth $(n, n-k)$ form with support in V and h is locally (at any point z_0 in V) in the ideal $\overline{(f_1,\ldots,f_p)^k_{z_0}}$ where the bar denotes the integral closure (that will be discussed in the next lesson). The set of holomorphic functions in V which satisfy this property is an ideal in $H(V)$ which is in general strictly included in $(f_1,\ldots,f_p)$. There is still the hope that, for a convenient choice of $q_1,\ldots,q_p$, one has (3.8) for any h in $(f_1,\ldots,f_p)$. Is this was the case (3.7) would be the formulation of a duality result.

Proof. We will not give here the proof of this theorem; it is based on more involved arguments dealing with integral kernels in complex analysis. One can find the proof (at least in the case $q = 0$) in section 5 of [DGSY], proposition 5.6.

4. Intersection and division.

An important ingredient in intersection theory is the notion of integration current associated with an analytic cycle on a n-dimensional complex manifold. When this cycle is purely dimensional (with codimension d) and decomposed as

$$\mathcal{Z} = \sum_j k_j \mathcal{Z}_j$$

where the $\mathcal{Z}_j$ are irreducible cycles and the k_j positive integers (we are interested here in effective cycles), the integration current $[\mathcal{Z}]$ is a (d,d) current whose action on $(n-d, n-d)$

test forms with compact support is given by

$$< [\mathcal{Z}], \varphi >:= \sum_j k_j \int_{X_j \setminus \mathrm{Sing}(X_j)} \varphi$$

if X_j denotes the support of the irreducible cycle $\mathcal{Z}_j$. This definition can be extended (just by linearity) to cycles which are not purely dimensional.

When $\mathcal{Z}$ is a cycle which is defined as a complete intersection, let say $\mathcal{Z} = \{f_1 = \ldots = f_p = 0\}$ (taking into account multiplicities) in an open set V of $\mathbf{C}^n$, then one has the Monge-Ampere equation

$$(dd^c)^p \log(|f_1|^2 + \ldots + |f_p|^2) = [\mathcal{Z}]$$

that can be written also

$$< [\mathcal{Z}], \varphi >= \mathrm{Res} \begin{bmatrix} df_1 \wedge \ldots \wedge df_p \wedge \varphi \\ f_1, ..., f_p \end{bmatrix}, \ \varphi \in \mathcal{D}^{(n-p,n-p)}(V). \tag{4.1}$$

When $f_1, ..., f_p$ do not define a complete intersection anymore, but define a codimension d analytic set, it is interesting to notice that residue symbols can be paired with the df_j in order to construct, for any choice of weight $q \in \mathbf{N}^p$, closed positive currents $[\mathcal{Z}]_d^q$,..., $[\mathcal{Z}]_\mu^q$, where $\mu := \inf(n, p)$ with respective types (d, d),...,(μ, μ) defined as

$$< [\mathcal{Z}]_k^q, \varphi >:= \sum_{1 \leq i_1 < \ldots < i_k \leq p} \mathrm{Res} \begin{bmatrix} df_{i_1} \wedge \ldots \wedge df_{i_k} \wedge \varphi \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_p \end{bmatrix}^{(q)}, \ \varphi \in \mathcal{D}^{n-k,n-k}(V), \ d \leq k \leq \mu\,.$$

It is a good exercice in differential calculus to check these currents are closed and positive. We will not do that here. Computing such currents seems to be in general a hard job; the simplest situations, where it seems possible to deal with such computations are the situations where the f_j are monomials (the normal crossing situation). The question that arises naturally is whether there exists particular choices of q such that, for any $k \in \{d, \ldots \mu\}$, the current $[\mathcal{Z}]_k^q$ coincides with the integration current (with multiplicities) associated with the cycle attached to the codimension k embedded components in the decomposition of $\mathcal{Z}$. These ideas are part of some work in progress with M. Passare and A. Tsikh. There seem to be quite a lot of questions to explore in this direction.

I would like to conclude this first lesson with a question in relation with the well known Lojasiewicz inequality: if $f_1, ..., f_p$ are p analytic function in an open subset V of $\mathbf{C}^n$, then, for any W relatively compact in V, there exists a positive optimum exponent $\delta_W(f) > 0$ such that

$$\forall \zeta \in W, \ \mathrm{Max}_{1 \leq j \leq p} |f_j(\zeta)| \geq \gamma(\min(1, \mathrm{distance}(\zeta, V(f)))^{\delta_W(f)}$$

for some $\gamma > 0$. The question that arises naturally, and that would be quite interesting respect to effective problems treated with analytic methods, is to clarify the relation between this Lojasiewicz exponent and the order of the different residue currents

$$\varphi \mapsto \mathrm{Res} \begin{bmatrix} \varphi \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_p \end{bmatrix}^{(q)}, \ d \leq k \leq \mu, \ q \in \mathbf{N}^n\,.$$

When $f_1, ..., f_n$ define a discrete complete intersection, the order of the residue current

$$\varphi \mapsto \text{Res}\left[\begin{matrix} \varphi \\ f_1, ..., f_n \end{matrix}\right]$$

is less than the maximum of all $\mu(\alpha) - 1$, where $\mu(\alpha)$ denotes the multiplicity at the common zero α of $f_1, ..., f_n$.

[BJ2] J. E. Bjork, Residue calculus and $\mathcal{D}$-modules on complex manifolds (manuscript, April 1996).

[DGYS] A. Dickenstein, R. Gay, C. Sessa, A. Yger, Analytic functionals annihilated by ideals, manuscripta math. 90, 175-223 (1996).

[Hi-Bo 2] J. Y. Boyer, M. Hickel, Une generalisation de la loi de transformation pour les residus, Bulletin Soc. Math. France, 1997.

[Pts] M. Passare, A. Tsikh, Defining the residue current of a compete intersection, in *Complex Analysis, Harmonic Analysis and Applications*, Pitman Research Notes 347, Longman, 1996.

[PTY] M. Passare, A. Tsikh, A. Yger, Residue currents of the Bochner-Martinelli type, Preprint, Bordeaux 1998.

2. Briançon-Skoda theorem, residue symbols and properness.

1. Briançon-Skoda theorem.

Let R be a commutative ring and I an ideal in A; an element $h \in R$ is algebraically dependent over I ($h \in \overline{I}$) if there exists a relation of integral dependency of the form

$$h^N + a_1 h^{N-1} + \ldots + a_N = 0, \ a_j \in I^j, \ j = 1, ..., N. \tag{1.1}$$

In fact, the set of such h is an ideal $\overline{I}$ which lies (from the inclusion point of view) between I and its radical.

For example, in the n-dimensional local ring $\mathcal{O}_n$, when $I = (f_1, ..., f_p)$, any h which satisfies a relation of the form (1.1) is such that there exists a constant C such that, in some neighborhood of the origin,

$$|h(\zeta)| \leq C\|f(\zeta)\|, \ \|f(\zeta)\| := (|f_1(\zeta)|^2 + \ldots + |f_p(\zeta)|^2)^{\frac{1}{2}}. \tag{1.2}$$

It is a deep result of M. Lejeune and B. Teissier [LT] that in this case ($R = \mathcal{O}_n$), the two conditions (1.1) and (1.2) are equivalent. In fact, they are equivalent to a third one, which is known as the valuative criterion:

Valuative criterion. *Whenever γ is a germ of curve at the origin such that the valuation at* 0 *of*

$$t \mapsto f(\gamma(t))$$

is greater than ν for any f in I, then the valuation at 0 *of*

$$t \mapsto h(\gamma(t))$$

is also greater than ν.

As a classical example of this criterion, we can see that, whenever f is a germ of holomorphic function at the origin which is in the maximal ideal in $\mathcal{O}_n$, then f is in the integral closure of its Jacobian ideal $(\partial f/\partial x_1, ..., \partial f/\partial x_n)$.

When I is an ideal generated by monomials (still in the local ring $\mathcal{O}_n$), let say $I = (\zeta^{\gamma_1}, ..., \zeta^{\gamma_p})$, h belongs to the integral closure of I if h lies in the ideal which is generated by the monomials ζ^{γ}, where γ is in the convex hull $\mathrm{conv}(E(I))$ of the staircase $E(I)$ of I, defined as

$$E(I) := \bigcup_{j=1}^{p} (\gamma_j + \mathbf{N}^n).$$

Since $E(I)$ is a semigroup in an affine space with dimension n, it follows from the theorem of Caratheodory in convex analysis that

$$\overbrace{E(I) + \ldots + E(I)}^{n \text{ times}} \subset \mathrm{conv}(E(I)).$$

In terms of ideal inclusions, this means that $\overline{I}^n \subset I$. As a matter of fact, this is a particular case of a very important result, that we will state for the moment in the case of the local ring $\mathcal{O}_n$.

Theorem 1.1 (Briançon-Skoda). *Let I be an ideal in the local ring $\mathcal{O}_n$, generated by p elements. Then, for any $\lambda \geq 1$, one has*

$$\overline{I^{\lambda-1+\inf(p,n)}} \subset I^{\lambda}\,. \tag{1.3}$$

Proof. Let us suppose first that $p \leq n$ and that the f_j define a regular sequence in $\mathcal{O}_n$. We will suppose that the f_j are holomorphic in a neighborhood V of the origin in $\mathbf{C}^n$ and that V_f contains the origin. The first thing to use is that, if I is generated by a regular sequence $f_1, ..., f_p$, one has, for any strictly positive integer λ

$$I^{\lambda} = \bigcap_{\substack{\underline{\lambda}\in\mathbf{N}^p \\ \lambda_1+\ldots+\lambda_p=\lambda+p-1}} (f_1^{\lambda_1}, ..., f_p^{\lambda_p})$$

(for the proof, which just uses the standard definition of quasi regularity, see [LiT], p. 106). Let φ be a $(n, n-p)$ test form with support in V, which is assumed to be closed in a neighborhood of V_f. It follows from the coaerea formula ([Fe], theorem 3.2.11, p.248) that the function

$$\epsilon \in]0,\infty[^p \mapsto \mathrm{mes}_{2n-p}(\mathrm{Supp}\,\varphi \cap \mathrm{Supp}\,\Gamma_f(\epsilon)) = \theta_{f,\varphi}(\sqrt{\epsilon}).$$

is integrable in $]0,\infty[^p$. Moreover, one has

$$\lim_{\eta\mapsto 0}\int_{\epsilon_1+\ldots+\epsilon_p\leq\eta} \theta_{f,\varphi}(\sqrt{\epsilon})d\epsilon_1\ldots d\epsilon_p = 0\,.$$

Then, it is possible to find a sequence $(\epsilon^{(k)})_k$ in $]0,\infty[^p$ which tends to 0, is such that

$$0 < \gamma \leq \frac{\epsilon_i^{(k)}}{\epsilon_j^{(k)}} \leq \Gamma < 0\,,\; 1 \leq i,j \leq p, \tag{1.4}$$

and is such that $\Gamma_f(\epsilon^{(k)})$ corresponds to a smooth real analytic chain and

$$\lim_{k\mapsto\infty} \mathrm{mes}_{2n-p}(\mathrm{Supp}\,\varphi \cap \mathrm{Supp}\,\Gamma_f(\epsilon^{(k)})) = 0\,.$$

Let us fix $\lambda \in \mathbf{N}^*$ and take h in the integral closure of the ideal $I^{\lambda-1+p}$. Then, on the set $\Gamma_f(\epsilon^{(k)}) \cap \mathrm{Supp}\,\varphi$, one has

$$|h(\zeta)| \leq C|\max(\epsilon_j^{(k)})|^{\lambda-1+p}$$

for some constant $C > 0$. This follows from the fact that h satisfies a relation of integral dependency over the ideal $I^{\lambda-1+p}$. Take now $\lambda_1, ..., \lambda_p$ in $\mathbf{N}$ such that $\lambda_1 + \ldots + \lambda_p = \lambda + p - 1$. Then, one has on $\Gamma_f(\epsilon^{(k)}) \cap \mathrm{Supp}\,\varphi$

$$|f_1^{\lambda_1}\ldots f_p^{\lambda_p}| \geq (\min(\epsilon_j^{(k)}))^{\lambda-1+p}\,.$$

Therefore, one has, taking into account (1.4),

$$\lim_{k\mapsto\infty}\int_{\Gamma_f(\epsilon^{(k)})}\frac{h\varphi}{f_1^{\lambda_1+1}\dots f_p^{\lambda_p+1}}=0,$$

which means exactly

$$\mathrm{Res}\left[\frac{h\varphi}{f_1^{\lambda_1+1},\dots,f_p^{\lambda_p+1}}\right]=0\,.$$

Using the duality theorem, we get that $h\in(f_1^{\lambda_1},\dots,f_p^{\lambda_p})$ and the conclusion of our theorem follows.

Since any ideal J in $\mathcal{O}_n$ is such that

$$J=\bigcap_{k>0}(J+\mathcal{M}^k),$$

where $\mathcal{M}$ is the maximal ideal, in order to prove that

$$\overline{I^{\lambda+(n-1)}}\subset I^\lambda$$

for any ideal, it is enough to do it for any ideal I such that the radical of I is the maximal ideal. In this case, a classical result of Northcott-Rees [NR] asserts that if $\sqrt{I}=\mathcal{M}$ and $I=(f_1,\dots,f_p)$, $p>n$, then any system $(g_1,\dots,g_n)$ of generic linear combinations of the f_j (with complex coefficients) is a *reduction* of I, that is an ideal contained in I which has the same integral closure than I. Therefore, for any ideal in $\mathcal{O}_n$, for any $\lambda\in\mathbf{N}^*$, one has

$$\overline{I^{\lambda+(n-1)}}\subset I^\lambda\,.$$

The proof is not complete, since we still have to deal with the general case when $p<n$ and the f_j do not define a complete intersection. A direct proof in this case can be obtained using the weighted version of the Bochner-Martinelli formula. In fact, the proof when $\lambda=1$ is a consequence of our duality theorem (Proposition 1.2 in chapter 1). A more careful use of these ideas inspired from the work of Berndtsson-Andersson (see for example [Elk]) leads to the result for arbitrary λ. We will not do it here. $\diamondsuit$

In fact, the result of Briançon-Skoda holds in any regular local ring and can be stated as follows: if $\mathbf{R}$ is a regular local ring with dimension n and I any ideal in $\mathbf{R}$, then, for any $\lambda\in\mathbf{N}^*$, one has

$$\overline{I^{\lambda+(n-1)}}\subset I^\lambda\,.$$

Moreover, if I is generated by p elements, with $p\le n$, we have

$$\overline{I^{\lambda+(p-1)}}\subset I^\lambda\,,\ \lambda\in\mathbf{N}^*\,.$$

This is a result of Lipman-Sathaye [LS]. In this algebraic setting, assuming $\mathbf{R}$ noetherian, the fact that an element in $\mathbf{R}$ lies in the integral closure of the ideal I can be tested again with the valuative criterion whose formulation in this case is:

Valuative criterion (algebraic version). *Whenever θ is an homomorphism from* $\mathbf{R}$ *in a discrete valuation ring, then*

$$\nu(h) \geq \nu(I)$$

where ν is the order function on $\mathbf{R}$ *obtained from this valuation.*

In order to conclude this section, we would like to mention an interesting formulation of Briançon-Skoda theorem in the analytic case, in terms of residue calculus. Suppose that $f_1, ..., f_n$ are n elements in $\mathcal{O}_n$ defining a regular sequence and that $g_1, ..., g_n$ are n elements lying in the integral closure of the ideal (f). Then, for any $\underline{q} \in \mathbf{N}^n$, one has, for any $h \in \mathcal{O}_n$,

$$\mathrm{Res}\begin{bmatrix} hg_1^{q_1+1} \cdots g_n^{q_n+1} d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{bmatrix} = 0\,.$$

This can be expressed just saying that the formal power series

$$\sum_{\underline{q}\in\mathbf{N}^n} \mathrm{Res}\begin{bmatrix} hg_1^{q_1+1} \cdots g_n^{q_n+1} d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{bmatrix} u_1^{q_1} \ldots u_n^{q_n} \tag{1.4}$$

is identically zero. This power series is

$$\mathrm{Res}\begin{bmatrix} hg_1 \ldots g_n d\zeta \\ f_1 - u_1 g_1, ..., f_n - u_n g_n \end{bmatrix}$$

where the residue calculus is understood over the $\mathbf{C}[[u]]$- algebra $\mathbf{C}[[u_1, ..., u_n]][[\zeta_1, ..., \zeta_n]]$.

2. Residues and properness.

In this section, we will first been interested into global problems. We start with a well known theorem of Jacobi.

Theorem 2.1 (Jacobi). *Let $P_1, ..., P_n$, be n polynomials in n variables with respective degrees $D_1, ..., D_n$, such the homogeneous parts of higher degree define the origin (the corresponding hypersurfaces in* $\mathbf{P}^n$ *intersect only in* $\mathbf{C}^n$*). Then, for any polynomial Q such that*

$$\deg Q \leq D_1 + \ldots + D_n - n - 1,$$

one has

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1, ..., P_n \end{bmatrix} = 0\,. \tag{2.1}$$

Proof. Let us give two proofs, an algebraic one and a geometric one. The algebraic one goes as follows: consider the homogeneizations $\mathcal{P}_1$,...,$\mathcal{P}_n$ of the polynomials P_j. The corresponding hypersurfaces define a complete intersection in $\mathbf{P}^n$ (this is the geometric vision) or the homogeneous ideal $(\mathcal{P}_1, ..., \mathcal{P}_n)$ in $\mathbf{C}[X_0, ..., X_n]$ has no embedded component at the origin (in fact these polynomials define a regular sequence in $\mathbf{C}[X_0, ..., X_n]$, which is the algebraic vision of the situation). From the Hilbert Nullstellensatz, one knows that there are polynomials $R_1(X_1)$,..., $R_n(X_n)$ (with respective degrees $r_1, ..., r_n$), such that

$$R_j(X_j) = \sum_{k=1}^{n} R_{jk}(X) P_k(X),\ k = 1, ..., n\,.$$

At any prime in $\mathbf{C}[X_0, ..., X_n]$ different from $(X_0, ..., X_n)$, the homogeneizations $\mathcal{R}_1, ..., \mathcal{R}_n$, of the R_j are locally in the ideal generated by the $\mathcal{P}_j$, $j = 1, ..., n$. In fact, since the maximal ideal in $\mathbf{C}[X_0, ..., X_n]$ is not an embedded prime in $(\mathcal{P}_1, ..., \mathcal{P}_n)$, one has

$$\mathcal{R}_j = \sum_{k=1}^{n} \mathcal{R}_{jk} \mathcal{P}_k\,,\ j = 1, ..., n, \tag{2.2}$$

for some homogeneous polynomials $\mathcal{R}_{jk}$. Dezhomogeizing (2.2), we get

$$R_j(X) = \sum_{k=1}^{n} \widetilde{R}_{jk} P_k,\ j = 1, ..., n,$$

with

$$\mathrm{deg} R_{jk} + D_k = r_j,\ 1 \leq j, k \leq n\,.$$

Is we use the transformation law (namely its global version), we obtain, for any $Q \in \mathbf{C}[X_1, ..., X_n]$,

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1, ..., P_n \end{bmatrix} = \mathrm{Res}\begin{bmatrix} Q(X)\det[\widetilde{R}_{jk}]dX \\ R_1(X_1), ..., R_n(X_n) \end{bmatrix}\,. \tag{2.3}$$

We are led to computations of total sums of residues in one variable and conclude that the residue symbol is zero provided

$$\mathrm{deg} Q + r_1 + \ldots + r_n - D_1 - \ldots - D_n \leq r_1 + \ldots + r_n - n - 1,$$

that is

$$\mathrm{deg} Q \leq D_1 + \ldots + D_n - n - 1\,,$$

which is our result.

The geometric proof (which inspired the original proof of Jacobi) can be described as follows. Let $\mathcal{P}_0, ..., \mathcal{P}_n$ be $n+1$ homogeneous polynomials in $n+1$ variables defining cycles $\mathcal{Z}_j$ which do not intersect in $\mathbf{P}^n$. Then

$$\mathbf{P}^n = \mathcal{U}_0 \cap \ldots \cap \mathcal{U}_n,$$

where

$$\mathcal{U}_j := \mathbf{P}^n \setminus \mathcal{Z}_j\,.$$

For any homogeneous polynomial $\mathcal{Q}$ with degree $D_0+\ldots+D_n-n-1$, the $(n,0)$ globally defined meromorphic differential form

$$\omega_{\mathcal{P},\mathcal{Q}} = \frac{\mathcal{Q}}{\mathcal{P}_0\ldots\mathcal{P}_n}\Big(\sum_{j=0}^{n}(-1)^{j-1}X_j dX_{[j]}\Big)$$

defines a Cech cochain in $\mathcal{C}^n(\mathcal{U},\Omega^n_{\mathbf{P}^n})$, that is an element in $H^n(\mathcal{U},\mathbf{P}^n)$. Such a class $[\omega_{\mathcal{P},\mathcal{Q}}]$ acts (by duality) on the finitely dimensional space $H_n(\mathbf{P}^n,\mathbf{C})$; if we define

$$\begin{bmatrix} \mathcal{Q}\big(\sum\limits_{j=0}^{n}(-1)^{j-1}X_j dX_{[j]}\big) \\ \mathcal{P}_0,\ldots,\mathcal{P}_n \end{bmatrix} := \mathrm{Tr}[\omega_{\mathcal{P},\mathcal{Q}}],$$

we can state the general residue formula in $\mathbf{P}^n$: for any $k\in\{0,\ldots,n\}$,

$$\begin{bmatrix} \mathcal{Q}\big(\sum\limits_{j=0}^{n}(-1)^{j-1}X_j dX_{[j]}\big) \\ \mathcal{P}_0,\ldots,\mathcal{P}_n \end{bmatrix} = (-1)^k \sum_{\alpha\in\bigcap\limits_{j\neq k}\mathcal{Z}_j} \mathrm{Res}_{\alpha}^{[\mathcal{Z}_0,\ldots,\widehat{\mathcal{Z}_k},\ldots,\mathcal{Z}_n]}(\omega_{\mathcal{P},\mathcal{Q}}), \qquad (2.5)$$

where

$$\mathrm{Res}_{\alpha}^{[\mathcal{Z}_0,\ldots,\widehat{\mathcal{Z}_k},\ldots,\mathcal{Z}_n]}(\omega_{\mathcal{P},\mathcal{Q}}) = \mathrm{Res}_{\alpha}^{[\mathcal{Z}_0,\ldots,\widehat{\mathcal{Z}_k},\ldots,\mathcal{Z}_n]}\Big(\frac{1}{\mathcal{P}_k}\,\frac{\Theta_{\mathcal{P},\mathcal{Q}}}{\prod\limits_{j\neq k}\mathcal{P}_j}\Big)$$

denotes the computation of the local residue respect to the divisors $\mathcal{Z}_j$, $j\neq k$ (one expresses everything in local coordinates and is led to computations of local residues as in chapter 1 in local charts). Coming back now to our problem, if we express in homogeneous coordinates the differential form in $\mathbf{C}^n$

$$\omega = \frac{Q}{P_1\ldots P_n}dX,$$

we get

$$\frac{X_0^{D_1+\ldots+D_n-n-1-\deg Q}\mathcal{Q}}{\mathcal{P}_1\ldots\mathcal{P}_n}\Big(\sum_{j=0}^{n}(-1)^{j-1}X_j dX_{[j]}\Big).$$

If we set $\mathcal{P}_0=X_0$, we have immediately

$$\begin{bmatrix} \mathcal{Q}X_0^{D_1+\ldots+D_n-\deg Q-n}\big(\sum\limits_{j=0}^{n}(-1)^{j-1}X_j dX_{[j]}\big) \\ \mathcal{P}_0,\ldots,\mathcal{P}_n \end{bmatrix} = 0$$

if $\deg(Q)\leq D_1+\ldots+D_n-n-1$. Jacobi's theorem follows from formula (2.5) applied with $k=0$.

Of course, the hypothesis in Jacobi's theorem deeply involve the fact that the hypersurfaces defined by the P_j do not intersect at infinity in some particular compactification of $\mathbf{C}^n$, namely here $\mathbf{P}^n$. We can think about other compactifications and therefore other statements in the Jacobi spirit. The most interesting one concerns the toric point of view. In order to state this theorem (due to Khovanski), let us define, if F is a Laurent polynomial in n variables

$$F(X) = \sum_{\alpha \in \mathcal{A}(F) \subset \mathbf{Z}^n} c_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n},\ c_\alpha \neq 0,$$

the support of F as the set $\mathcal{A}(F)$ and, if ξ is a direction in $(\mathbf{R}^n)^*$,

$$F^{(\xi)}(X) := \sum_{\substack{\alpha \in \mathcal{A}(F) \\ <\alpha,\xi> \text{ minimal}}} c_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}\,.$$

Theorem 2.2 (Khovanskii [Kh]). *Let $F_1, ..., F_n$ be n Laurent polynomials in n variables such that for any direction $\xi \in (\mathbf{R}^n)^*$, one has*

$$\mathbf{T}^n \cap \{F_1^{(\xi)} = ... = F_n^{(\xi)} = 0\} = \emptyset\,. \tag{2.6}$$

Then, for any Laurent polynomial G which support lies in the relative interior of

$$\overline{\text{conv}(\text{Supp}\,(F_1))} + \cdots + \overline{\text{conv}(\text{Supp}\,(F_n))}$$

(that is the interior in the affine subspace generated by this convex polyedra), one has

$$\text{Res}\left[\begin{matrix} G dX \\ F_1, ..., F_n \end{matrix}\right]_{\mathbf{T}} = 0\,. \tag{2.7}$$

Proof. The idea of the proof is inspired by the geometric proof of Jacobi formula. Instead of $\mathbf{P}^n$ as a compactification of $\mathbf{C}^n$, we use a smooth toric variety which is compatible with all the polyedra $\Delta_j := \overline{\text{conv}(\text{Supp}\,F_j)}$, $j = 1, ..., n$. Such a variety can be obtained from a refinement of the fan associated with the convex polyedron

$$\Delta := \overline{\text{conv}(\text{Supp}\,(F_1))} + \cdots + \overline{\text{conv}(\text{Supp}\,(F_n))}\,.$$

The Laurent polynomials $F_1, ..., F_n$ induce Cartier divisors on this smooth n-dimensional complex manifold; the conditions (2.6) mean that these divisors intersect only in the torus $\mathbf{T}^n$. One can construct (see [Cox2]) a ring of homogeneous coordinates $\mathbf{C}[x_1, ..., x_s]$ associated with this manifold; each coordinate x_j is in correspondance with a one dimensional edge of the fan, directed by a primitive vector η_j. We can (as in the previous proof) express the differential form

$$\frac{G(X)}{F_1(X) \dots F_n(X)} \frac{dX}{X_1 \dots X_n}$$

in homogeneous coordinates, using the parametrisation of the torus, namely

$$X_j = \prod_{i=1}^{s} x_j^{\eta_{ij}} := x^{<e_j,\eta>}\,,$$

which leads to the differential form

$$\frac{G(x^{<e_1,\eta>},...,x^{<e_n,\eta>})}{\prod_{j=1}^{n} F_j(x^{<e_1,\eta>},...,x^{<e_n,\eta>})} \frac{\Omega(x)}{x_1\ldots x_s} \tag{2.8}$$

where Ω is the Euler form on the toric variety (exactly like the Euler form in $\mathbf{P}^n$). Now, we use the fact that it is possible to construct [Cox1] a toric residue on this toric variety $\mathcal{X}$, exactly as we did in the case of $\mathbf{P}^n$. Let us just recall that the grading of $\mathbf{C}[x_1,...,x_s]$ is a A_{n-1}-grading (here A_{n-1} is the $n-1$ Chow group on the manifold), given by

$$\deg(x_1^{q_1}\ldots x_s^{q_s}) := [q_1\mathcal{D}_1 + \ldots + q_s\mathcal{D}_s],$$

where the Weil divisors $\mathcal{D}_j$ are the closed orbits in correspondence with the 1-dimensional faces of the fan that was used to construct the toric variety (we refer to [Fu] for the details about divisors and Chow groups on a toric variety). If $\mathcal{F}_0,...,\mathcal{F}_n$ are homogeneous polynomials in $x_1,...,x_s$, with degrees $\delta_0,...,\delta_n$ respect to this grading, such that the corresponding sections of the line bundles $\mathcal{O}_{\mathcal{X}}(\delta_j)$, $j=0,...,n$, do not have common zeroes on $\mathcal{X}$, and $\tilde{\mathcal{G}}$ is a polynomial with degree $\beta = \sum_{j=0}^{n}\delta_j - [\sum_{j=1}^{s}\mathcal{D}_j]$, one can define the total toric residue

$$\begin{bmatrix} \tilde{\mathcal{G}}\Omega \\ \mathcal{F}_0,...,\mathcal{F}_n \end{bmatrix}$$

as the trace of $[\mathcal{G}\Omega/\mathcal{F}_0\ldots\mathcal{F}_n] \in H^n(\mathcal{X},\Omega^n_{\mathcal{X}})$, considered as acting on $H_n(\mathcal{X},\Omega^n_{\mathcal{X}})$. One has a similar residue formula than in the case of $\mathbf{P}^n$ (see 2.5). Here the homogeneizations of the F_j, $j=1,...,n$, are

$$\mathcal{F}_j(x) = \sum_{\alpha\in\mathcal{A}(F_j)} c_{j\alpha} \prod_{i=1}^{s} x_i^{<\alpha,\eta_i> - \min_{\xi\in\Delta_j\cap\mathbf{Z}^n} <\xi,\eta_i>}$$

(see for example [CD]). The argument in order to prove Khovanskii's theorem follows the geometric argument used in order to prove Jacobi's theorem. The only thing to check is that the differential form (2.8) can be written in that case as

$$\frac{\tilde{\mathcal{G}}(x)}{\mathcal{F}_1(x)\ldots\mathcal{F}_n(x)}\,\Omega(x)$$

(in fact one can take $\mathcal{F}_0 = 1$). We use then the residue theorem on the toric variety. $\diamond$

Despite of such theorems, there are situations when it is difficult to compactify either $\mathbf{C}^n$ or the torus $\mathbf{T}^n$ in order to avoid points at infinity, which prevents us from such a result of the Jacobi type. Nethertheless, a very interesting situation is the situation of proper maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. In this case, one can find certainly, by means of blowing-ups, a compactification of $\mathbf{C}^n$ such that the divisors induced by the P_j on this compactification do not intersect at infinity. This was suggested to us by A. Dimca. Nethertheless,such a construction is not effective and we need some more precise vanishing theorem for total sums of residues. The case when the variety at infinity is discrete has been studied in [Pl]. For the case of proper maps, we can state the following

Theorem 2.3. *Let $P = (P_1, ..., P_n)$ be a polynomial map from $\mathbf{C}^n$ to $\mathbf{C}^n$, such that there exists strictly positive constants $\delta_1, ..., \delta_n, \gamma, K$ and such that $0 < \delta_j \leq D_j$, $D_j = \deg(P_j)$, with*

$$\max_{1 \leq k \leq n} \frac{|P_j(X)|}{\|X\|^{\delta_j}} > \gamma > 0, \ \|X\| \geq K\,. \tag{2.9}$$

Then, for any $Q \in \mathbf{C}[X_1, ..., X_n]$, one has

$$\operatorname{Res}\left[\frac{Q(X)dX}{P_1^{q_1+1}, ..., P_n^{q_n+1}}\right] = 0 \tag{2.10}$$

whenever

$$\deg Q \leq (q_1 + 1)\delta_1 + \ldots + (q_n + 1)\delta_n - n - 1\,.$$

Proof. A theorem of this kind, less precise, was proved in [BY1], [Y], [FPY], using Bochner-Martinelli formulas. It was also proved in a more algebraic setting in [BY2], [BY3], under the restrictive conditions $\delta_i = \delta_j$ for any $1 \leq i, j \leq n$ and the $1 - \delta_i/D_i$ smaller than $\frac{1}{n(n+1)}$. The key argument used in [BY2], [BY3] was the Briançon-Skoda theorem. The final step (eliminate all restrictive conditions) is a result due to M. Hickel.

We will give here an analytic proof, in the spirit of [Y] (where the result was proved when all δ_j are equal), based on the representation of sum of residues with Bochner-Martinelli formulas. It seems reasonable to think that such a method could be extended in toric varieties. The defect of this analytic proof is that it holds only in the analytic setting. The alternate proof of Hickel, based on Briançon-Skoda theorem, can be extended to the algebraic situation (thanks to Lipman-Teissier theorem), which is a capital advantage.

Clearly, it is enough to prove the result when $q = 0$. We can also assume without restriction that the δ_j, $j = 1, ..., n$, are integers, since one can always make the basis change

$$X_j = Y_j^N, \ j = 1, ..., n$$

where N is a positive integer (in fact a common denominator for the δ_j if those numbers are assumed to be rational, which of course is always possible).

We will just briefly sketch the proof here. Let M be an integer such that, for any $k \in \{1, ..., n\}$,

$$M + \delta_k - D_k \geq 0\,.$$

We know from section 1 in chapter 1 that the residue symbol

$$\operatorname{Res}\left[\frac{Q(X)dX}{P_1, ..., P_n}\right]$$

can be expressed as

$$\operatorname{Res}\left[\frac{Q(X)dX}{P_1, ..., P_n}\right] = \gamma_n \int_{\sum_{j=1}^{n} \frac{|P_j|^2}{(1+\|X\|^2)^{\delta_j+M}} = R} \left(\sum_{k=1}^{n} (-1)^{k-1} s_k^{(\delta,R)} ds_{[k]}^{(\delta,R)}\right) \wedge Q d\zeta\,,$$

where

$$\gamma_n := \frac{(-1)^{\frac{n(n-1)}{2}}}{(2i\pi)^n}$$

and

$$s^{(\delta,R)} := \frac{1}{R}\Big(\frac{\overline{P_1}}{(1+\|X\|^2)^{\delta_1+M}},\dots\frac{\overline{P_n}}{(1+\|X\|^2)^{\delta_n+M}}\Big)\,.$$

Using the same tricks than in section 1 in chapter 1, we have also

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1,\dots,P_n\end{bmatrix} = \gamma_n \int_{\|\zeta\|=R} \Big(\sum_{k=1}^n (-1)^{k-1}\tilde{s}_k^{(\delta)} d\tilde{s}_{[k]}^{(\delta)}\Big)\wedge Q d\zeta\,,$$

where

$$\tilde{s}^{(\delta)} := \frac{1}{\sum\limits_{j=1}^n \frac{|P_j|^2}{(1+\|X\|^2)^{\delta_j+M}}}\ \Big(\frac{\overline{P_1}}{(1+\|X\|^2)^{\delta_1+M}},\dots\frac{\overline{P_n}}{(1+\|X\|^2)^{\delta_n+M}}\Big)\,.$$

We can rewrite this as

$$\begin{aligned}\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1,\dots,P_n\end{bmatrix} &= \gamma_n \left[\int_{\|\zeta\|=R} \|P\|_\delta^{2\lambda}\Big(\sum_{k=1}^n (-1)^{k-1}\tilde{s}_k^{(\delta)} d\tilde{s}_{[k]}^{(\delta)}\Big)\wedge Q d\zeta\right]_{\lambda=0} \\ &= \gamma_n \left[\int_{\|\zeta\|=R} \|P\|_\delta^{2(\lambda-n)}\Big(\sum_{k=1}^n (-1)^{k-1}s_k^{(\delta,1)} ds_{[k]}^{(\delta,1)}\Big)\wedge Q d\zeta\right]_{\lambda=0}\,.\end{aligned}$$

where

$$\|P\|_\delta^2 := \sum_{j=1}^n \frac{|P_j|^2}{(1+\|X\|^2)^{\delta_j+M}}\,.$$

We now express in homogeneous coordinates $\widetilde{X} := (X_0,\dots,X_n)$ the differential form

$$\|P\|_\delta^{2(\lambda-n)}\Big(\sum_{k=1}^n (-1)^{k-1}s_k^{(\rho,1)} ds_{[k]}^{(\rho,1)}\Big)\wedge Q d\zeta$$

(when λ is a fixed complex number such that $\mathrm{Re}\lambda >> 1$). This leads to a differential form

$$\Omega_{P,Q;\lambda}^{(\rho)}\,,$$

which is a $(n,n-1)$ form in $\mathbf{P}^n$, which can be expressed as

$$\Omega_{P,Q;\lambda}^{(\rho)}(\widetilde{X}) = X_0^{nM+\delta_1+\dots+\delta_n-\deg Q-n-1}\ \frac{1}{\overline{X_0}}\left(\frac{\sum\limits_{j=1}^n |\mathcal{P}_j|^2|X_0|^{2(\delta_j-D_j+M)}\|\widetilde{X}\|^{2\delta_{[j]}}}{\|\widetilde{X}\|^{2(M+\Delta)}}\right)^\lambda \times$$

$$\times\ \frac{\Theta(\widetilde{X})}{\Big(\sum\limits_{j=1}^n |\mathcal{P}_j|^2|X_0|^{2(\delta_j-D_j+M)}\|\widetilde{X}\|^{2\delta_{[j]}}\Big)^n}$$

where Θ is a smooth form,

$$\Delta = \delta_1 + \ldots + \delta_n ,\ \delta_{[j]} := \Delta - \delta_j \,.$$

We now introduce a $2n$ chain Σ in $\mathbf{P}^n$ (for example the complement of a union of balls which are all included in $\mathbf{C}^n$), such that the support of Σ does not contain any of the common zeros of the P_j, but contains the hyperplane at infinity. We have, using Stokes's theorem, that for $\mathrm{Re}\lambda >> 1$,

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1, ..., P_n \end{bmatrix} = -\int_{\partial\Sigma} \Omega^{(\rho)}_{P,Q;\lambda}(\widetilde{X}) = -\int_{\Sigma} \overline{\partial}\Omega^{(\rho)}_{P,Q;\lambda}(\widetilde{X}) \,. \tag{2.11}$$

We now follow the analytic continuation of the two members in (2.11) as functions of the parameter λ. In fact, one can show that the value at $\lambda = 0$ of the right-hand side of (2.11) is well defined. Here we use the properness hypothesis, which tells us that in a neighborhood of infinity,

$$|X_0|^M \leq C \sum_{k=1}^{n} |X_0|^{M+\delta_k - D_k} \mathcal{P}_k(\widetilde{X}) \,, \tag{2.12}$$

which means that locally X_0^M lies in the integral closure of the ideal generated by the $\mathcal{P}_k X_0^{M+\delta_k - D_k}$, $k = 1, ..., n$. The theorem of Briançon-Skoda, applied locally near any point at infinity, asserts that X_0^{nM} lies in this ideal near all such points. We do not use this result here, but just notice (using resolutions of singularities) that if the condition

$$nM \leq nM + \delta_1 + \ldots + \delta_n - n - 1 - \deg Q$$

is fullfilled, then the right-hand side of (2.11) (computed following the analytic continuation at $\lambda = 0$) is zero. This is exactly our result, and the proof is completed. For the details (at least when all δ_j are equal), we refer to [Y]. $\diamondsuit$

As a consequence of this result, we can check immediately that if $(P_1, ..., P_n)$ is a proper map with separated Lojasiewicz exponents $\delta_1, ..., \delta_n$, then, for any polynomial Q which is in $\mathbf{C}[X_1, ..., X_n]$, the rational function

$$u \mapsto \mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1 - u_1, ..., P_n - u_n \end{bmatrix}$$

(this object is computed using residue theory in $\mathbf{C}(u)[X_1, ..., X_n]$) is a polynomial in u; since all residue symbols

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1^{q_1+1}, ..., P_n^{q_n+1} \end{bmatrix}$$

are zero as soon as $< \delta, q + \underline{1} > -n$ is strictly bigger than $\deg(Q)$, one can write

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P_1 - u_1, ..., P_n - u_n \end{bmatrix} = \sum_{\substack{\underline{q} \in \mathbf{N}^n \\ \delta_1(q_1+1)+\ldots+\delta_n(q_n+1) \leq \deg Q + n}} \gamma_{\underline{q}} u_1^{q_1} \ldots u_n^{q_n} \,.$$

In particular, if

$$P_j(Y) - P_j(X) = \sum_{k=1}^{n} G_{jk}(X,Y)(Y_k - X_k),\ j = 1, ..., n$$

and Bez(X,Y) denotes the determinant of the matrix $[G_{jk}(X,Y)]$, one has

$$\text{Bez}(X,Y) = \sum_{|\underline{\alpha}|+|\underline{\beta}| \leq D_1+...+D_n-n} \gamma_{\underline{\alpha},\underline{\beta}} X^{\underline{\alpha}} Y^{\underline{\beta}}$$

and the residue symbol

$$\text{Res}\left[\begin{matrix} Q(X)\text{Bez}(X,Y)dX \\ P_1 - u_1, ..., P_n - u_n \end{matrix}\right] = \sum_{|\underline{\alpha}|+|\underline{\beta}| \leq D_1+...+D_n-n} \gamma_{\underline{\alpha},\underline{\beta}} \text{Res}\left[\begin{matrix} X^{\underline{\alpha}}\text{Bez}(X,Y)dX \\ P_1 - u_1, ..., P_n - u_n \end{matrix}\right] Y^{\underline{\beta}}$$

is a polynomial in Y, u, which contains monomials $u^{\underline{\mu}} Y^{\underline{\nu}}$ such that

$$< \mu + \underline{1}, \delta > + \nu \leq D_1 + \ldots + D_n + \deg Q\,.$$

This applies (as a consequence) to the Kronecker's formula for global polynomials maps. If $(P_1, ..., P_n)$ is a proper polynomial map with separated Lojasiewicz exponents $\delta_1, ..., \delta_n$, one can write any polynomial Q as

$$Q(Y) = \text{Res}\left[\begin{matrix} Q(X)dX \\ P_1, ..., P_n \end{matrix}\right] + \sum_{\substack{\mu \in (\mathbf{N}^n)^*,\ \nu \in \mathbf{N}^n \\ <\mu+\underline{1},\delta>+\nu \leq D_1+\ldots+D_n+\deg Q}} \tilde{\gamma}_{\mu,\nu} Y^{\nu} (P_1(Y))^{\mu_1} \ldots (P_n(Y))^{\mu_n}\,. \tag{2.13}$$

When

$$\text{Res}\left[\begin{matrix} Q(X)dX \\ P_1, ..., P_n \end{matrix}\right] = 0,$$

formula (2.13) is an explicit formula in order to express Q in the ideal generated by the P_j; this is the most precise one in this context. The first formula in this direction appeared in [FPY].

We will conclude here this lesson with the caracterizations of the notion of properness in terms of residue symbols, both in the global and local case.

• Let $\mathbf{K}$ be a commutative field. As we have seen above, a dominant polynomial map from $\mathbf{K}^n$ to $\mathbf{K}^n$, $P = (P_1, ..., P_n)$, is proper if and only if all residue symbols of the form

$$\text{Res}\left[\begin{matrix} QdX \\ P_1 - u_1, ..., P_n - u_n \end{matrix}\right],$$

computed taking as reference algebra the $\mathbf{K}(u)$-algebra $\mathbf{K}(u)[X_1, ..., X_n]$, are in $\mathbf{K}[u]$. This result was proved for the first time by G. Biernat [Bi]. All these ideas are the basic tools in [FPY] and [Hi-Bo].

• Let now $\mathbf{R}$ be a regular local ring with dimension n (assume that $\zeta_1, ..., \zeta_n$ are such that their classes in $\mathcal{M}/\mathcal{M}^2$, where $\mathcal{M}$ is the maximal ideal, generate $\bigoplus_s \mathcal{M}^s/\mathcal{M}^{s+1}$). Let $(f_1, ..., f_n)$ be a quasiregular sequence in $\mathbf{R}$. In general, when $r, h \in \mathbf{R}$, one can remark that

$$\mathrm{Res}\left[\begin{matrix} r d\zeta_1 \wedge d\zeta_n \\ f_1 - hu_1, ..., f_n - hu_n \end{matrix}\right],$$

computed taking as the reference algebra the $\mathbf{R}/\mathcal{M}[[u]]$-algebra $\mathbf{R}[[u]]$, is an element in $(\mathbf{R}/\mathcal{M})[[u]]$, namely the formal power series

$$F(u) := \mathrm{Res}\left[\begin{matrix} r d\zeta_1 \wedge d\zeta_n \\ f_1 - u_1 h, , ..., f_n - u_n h \end{matrix}\right] = \sum_{\underline{q} \in \mathbf{N}^n} \mathrm{Res}\left[\begin{matrix} r h^{|\underline{q}|} d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{matrix}\right] u_1^{q_1} \dots u_n^{q_n}.$$

If h is in the integral closure of $(f_1, ..., f_n)$, there is a relation of the form

$$h^N + \sum_{k=1}^{n} \Big(\sum_{|\underline{q}|=k} a_{k,\underline{q}} f_1^{q_1} \dots f_n^{q_n} \Big) h^{N-k} = 0,$$

where the $a_{k,\underline{q}} \in \mathbf{R}$. If the $\alpha_{k,\underline{q}}$ denote the classes of the $a_{k,\underline{q}}$ modulo the maximal ideal, one can rewrite this relation as

$$h^N + \sum_{k=1}^{n} \Big(\sum_{|\underline{q}|=k} \alpha_{k,\underline{q}} f_1^{q_1} \dots f_n^{q_n} \Big) h^{N-k} \in (f_1, ..., f_n)^{N+1}.$$

Let us fix $r \in \mathbf{R}$ and note

$$\theta(r; \underline{q}) := \mathrm{Res}\left[\begin{matrix} r h^{|\underline{q}|} d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{matrix}\right].$$

Then, one has, for any $\underline{q}$ such $|\underline{q}| \geq N$,

$$\begin{aligned} &\theta_{r,\underline{q}} + \sum_{k=1}^{N} \sum_{|\underline{l}|=k} \alpha_{k,\underline{l}} \theta_{r,\underline{q}-\underline{l}} = \\ &= \mathrm{Res}\left[\begin{matrix} r\Big(h^{|\underline{q}|} + \sum_{k=1}^{n} \big(\sum_{|\underline{l}|=k} \alpha_{k,\underline{l}} f_1^{l_1} \dots f_n^{l_n}\big) h^{|\underline{q}|-k}\Big) d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{matrix}\right] = 0, \end{aligned} \tag{2.14}$$

since the numerator of this residue symbol lies in $(f_1, ..., f_n)^{|\underline{q}|+1} \subset (f_1^{q_1+1}, ..., f_n^{q_n+1})$. Since the coefficients of the formal series F satisfy such a difference equation, F is the formal power series that corresponds to a rational function $F \in (\mathbf{R}/\mathcal{M})[[u]]$ with no pole at 0. Note that this rational function is such that the maximum of the degrees of the numerator and denominator is less than $2N$ and therefore does not depend of r. Moreover the denominator of this rational function is independent of r.

What is interesting here is that this assertion has a converse, as noticed by M. Hickel: suppose that there exists an integer M such that for any $r \in \mathbf{R}$,

$$\mathrm{Res}\left[\begin{matrix} r d\zeta \\ f_1 - u_1 h, ..., f_n - u_n h \end{matrix}\right]$$

is a rational function with degrees of the numerator and denominator bounded by M, with no poles at the origin, with denominator independent of r, let us say

$$\mathrm{Res}\left[\begin{matrix} r d\zeta \\ f_1 - u_1 h, ..., f_n - u_n h \end{matrix}\right] = \frac{N_r(u)}{D(u)}\,.$$

Then, one has

$$D(u)\mathrm{Res}\left[\begin{matrix} r d\zeta \\ f_1 - u_1 h, ..., f_n - u_n h \end{matrix}\right] = N_r(u)\,.$$

Then, if

$$D(u) = 1 + \sum_{\substack{\underline{l} \in (\mathbf{N}^n)^* \\ |\underline{l}| \le M}} \xi_{\underline{l}} u_1^{l_1} \ldots u_n^{l_n}\,,$$

one has, for any $\underline{q}$ such that $|\underline{q}| = 2M$,

$$\mathrm{Res}\left[\begin{matrix} r\Big(h^{2M} + \sum\limits_{k=1}^{M} \big(\sum\limits_{|\underline{l}|=k} \xi_{\underline{l}} f_1^{l_1} \ldots f_n^{l_n}\big) h^{2M-k}\Big) d\zeta \\ f_1^{q_1+1}, ..., f_n^{q_n+1} \end{matrix}\right] = 0\,.$$

But, since $(f_1, ..., f_n)$ is a regular sequence, we have

$$(f_1, ..., f_n)^{M+1} = \bigcap_{|\underline{\lambda}| = M+n} (f_1^{\lambda_1}, ..., f_n^{\lambda_n})\,.$$

Therefore, it follows from the duality theorem that

$$h^{2M} + \sum_{k=1}^{M} \big(\sum_{|\underline{l}|=k} \xi_{\underline{l}} f_1^{l_1} \ldots f_n^{l_n}\big) h^{2M-k} \in \bigcap_{|\underline{\lambda}| = M+n} (f_1^{\lambda_1}, ..., f_n^{\lambda_n}) = (f_1, ..., f_n)^{M+1}\,.$$

This proves that h is in the integral closure of $(f_1, ..., f_n)$.
We have proved the following in this local context:

Proposition 2.1. *h is in the integral closure of $(f_1, ..., f_n)$ if and only if there exists $M \in \mathbf{N}$ such that, for any $r \in R$, the formal power series*

$$\mathrm{Res}\left[\begin{matrix} r d\zeta \\ f_1 - h u_1, ..., f_n - h u_n \end{matrix}\right]$$

is a rational function with degree at most M, denominator independent of r, and without poles at* 0.

Properness and residue symbols, from the global point of view as well as from the local point of view, are deeply connected. The key points to elucidate are related with the geometric interpretation of this concept of multi-valued Lojasiewicz exponent.

[Bi] G. Biernat, On the sum of residues for a polynomial mapping, Bull. Soc. Sci. Lett. Lotz, Ser. Rech. Deform. 9, 1990, 73-83.

[BY1] C. A. Berenstein, A. Yger, A formula of Jacobi and its consequences, Annales de l'Ecole Normale Superieure, 24, 1991, 363-377.

[BY2] C. A. Berenstein, A. Yger, Residue calculus and effective Nullstellensatz, preprint.

[BY3] C. A. Berenstein, A. Yger, Residues and effective Nullstellensatz, Electronic Research Annoucements of the AMS, 2, 2, 1996.

[CD] E. Cattani, A. Dickenstein, A global view of residues in the torus, Journal of Pure and Applied Algebra 117-118, 1997.

[Cox1] D. Cox, Toric residues, Arkiv for Mathematik 34, 1996, 73-96.

[Cox2] D. Cox, The homogeneous coordinate ring of a toric variety, Journal of Algebraic Geometry, 4, 1995, 17-50.

[Elk] M. Elkadi, Une version effective du theorème de Briançon-Skoda dans le cas algébrique discret, Acta Arithmetica, 66, 1994, 201-220.

[FPY] A. Fabiano, G. Pucci, A. Yger, Effective Nullstellensatz and geometric degree for zero-dimensional ideals, Acta Arithmetica, 78(2), 1996.

[Fe] Federer, *Geometric measure theory*, Springer-Verlag, New-York, 1969.

[Fu] W. Fulton, *Introduction to Toric varieties*, Princeton Univ. Press, Princeton, 1993.

[Hi-Bo] J. Y. Boyer, M. Hickel, Extension dans un cadre algébrique d'une formule de Weil, preprint.

[Kh] A. Khovanskii, Newton polyedra and the Euler-Jacobi formula, Russian Math. Surveys 33, 1978, 237-238.

[LS] B. Teissier, A. Sathaye, Jacobian ideals and a theorem of Briançon-Skoda, Michigan Math Journal 28, 1981, 199-222.

[LT] M. Lejeune, B. Teissier, Clôture intégrale des idéaux et équisingularité, Publications de l'institut Fourier, F38402, St Martin d'Hères, 1975.

[LiT] J. Lipman, B. Teissier, Pseudo-Rational local rings and a theorem of Briançon-Skoda about integral closures of ideals, Michigan Math. J. 28, 1981, 97-116.

[NR] D. G. Northcott, D. Rees, Reductions of ideals in local rings, Proc. Cambridge. Philos. Soc. 50 (1954), 145-158.

[Pl] A. Ploski, The Noether exponent and Jacobi formula, Bull. Soc. Sci. Lett. Lotz, Ser. Rech. Deform. 15, 1993, 53-57.

[Y] A. Yger, How to compute multidimensional residues in the toric context, Conf. Cetraro, 1994 (to appear).

* This last condition is in fact not necessary.

1. Kronecker's formula for proper polynomial maps.

Let $P_1, ..., P_n$ be a polynomial map from $\mathbf{C}^n$ to $\mathbf{C}^n$ such that

$$\max_{1 \le j \le n} \frac{|P_j(X)|}{\|X\|^{\delta_j}} \ge \gamma, \ \|X\| \ge K .$$

Then, as we have seen in the preceeding chapter, if

$$\mathrm{Bez}(X,Y) = \sum_{\substack{\underline{\alpha},\underline{\beta} \in \mathbf{N}^n \\ |\underline{\alpha}|+|\underline{\beta}| \le D_1+\ldots+D_n-n}} \gamma_{\underline{\alpha},\underline{\beta}} X^{\underline{\alpha}} Y^{\underline{\beta}},$$

one can write

$$\begin{aligned} 1 &= \sum_{\substack{\underline{q} \in \mathbf{N}^n, \underline{\alpha}\underline{\beta} \in \mathbf{N}^n \\ <\underline{q},\delta+\underline{1}>+|\beta| \le D_1+\ldots+D_n \\ |\underline{\alpha}|+|\underline{\beta}| \le D_1+\ldots+D_n-n}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} X^{\underline{\alpha}} dX \\ P_1^{q_1+1}, ..., P_n^{q_n+1} \end{matrix}\right] Y^{\underline{\beta}} P_1(Y)^{q_1} \ldots P_n(Y)^{q_n} = \\ &= \mathrm{Res}\left[\begin{matrix} \mathrm{Bez}(X,Y) dX \\ P_1, ..., P_n \end{matrix}\right] + \\ &+ \sum_{\substack{\underline{q} \in (\mathbf{N}^n)^*, \underline{\alpha}\underline{\beta} \in \mathbf{N}^n \\ <\underline{q},\delta+\underline{1}>+|\beta| \le D_1+\ldots+D_n \\ |\underline{\alpha}|+|\underline{\beta}| \le D_1+\ldots+D_n-n}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} X^{\underline{\alpha}} dX \\ P_1^{q_1+1}, ..., P_n^{q_n+1} \end{matrix}\right] Y^{\underline{\beta}} P_1(Y)^{q_1} \ldots P_n(Y)^{q_n} . \end{aligned} \tag{1.1}$$

Let us now consider a polynomial P_0 which does not vanish on the set of common zeroes of the P_j, $j = 1, ..., n$. Let $a_{01}, ..., a_{0n}$ be n polynomials in $2n$ variables (X, Y) such that

$$P_0(X) - P_0(Y) = \sum_{k=1}^{n} a_{0k}(X,Y)(X_k - Y_k)$$

and Λ_k, $k = 1, ..., n$, be the determinant obtained replacing the column with index k in the Bezoutian determinant $\mathrm{Bez}(X,Y)$ by the column

$$\begin{pmatrix} a_{01}(X,Y) \\ \vdots \\ a_{0n}(X,Y) \end{pmatrix} .$$

One can rewrite the identity (1.1) as

$$\begin{aligned} 1 &= \mathrm{Res}\left[\begin{matrix} \mathrm{Bez}(X,Y) dX \\ P_1, ..., P_n \end{matrix}\right] P_0(X) + \sum_{k=1}^{n} \mathrm{Res}\left[\begin{matrix} \Lambda_k(X,Y) dX \\ P_1, ..., P_n \end{matrix}\right] P_k(X) + \\ &+ \sum_{\substack{\underline{q} \in (\mathbf{N}^n)^*, \underline{\alpha}\underline{\beta} \in \mathbf{N}^n \\ <\underline{q},\delta+\underline{1}>+|\beta| \le D_1+\ldots+D_n \\ |\underline{\alpha}|+|\underline{\beta}| \le D_1+\ldots+D_n-n}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} X^{\underline{\alpha}} dX \\ P_1^{q_1+1}, ..., P_n^{q_n+1} \end{matrix}\right] Y^{\underline{\beta}} P_1(Y)^{q_1} \ldots P_n(Y)^{q_n} . \end{aligned} \tag{1.2}$$

This leads to a Bézout identity of the form

$$1 = \sum_{k=0}^{n} Q_k(Y) P_k(Y).$$

(see [BY2], [BY3]). If we now suppose that Q is a polynomial which vanishes on the set of common zeroes of $P_1, ..., P_n$ and ν denotes the maximum of local Noether exponents of the polynomial map $(P_1, ..., P_n)$ at these points (note that the local Noether exponent is less than the multiplicity), one can write, using the same ideas

$$Q^{\nu}(Y) = \sum_{\substack{\underline{q}\in(\mathbf{N}^n)^*, \underline{\alpha}\underline{\beta}\in\mathbf{N}^n \\ <\underline{q},\delta+\underline{1}>+|\beta|\leq D_1+\ldots+D_n+\nu\deg Q \\ |\underline{\alpha}|+|\underline{\beta}|\leq D_1+\ldots+D_n-n}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} Q^{\nu}(X) X^{\underline{\alpha}} dX \\ P_1^{q_1+1}, ..., P_n^{q_n+1}\end{matrix}\right] Y^{\underline{\beta}} P_1(Y)^{q_1} \ldots P_n(Y)^{q_n}. \tag{1.3}$$

This is an explicit version of the Hilbert's Nullstellensatz in this case (see [FPY]). One can also write such a version of the Nullstellensatz when $(P_1, ..., P_k)$ is *strictly quasi-regular* in the sense of Ploski [CaPl]. This means that one can find $L_1, ..., L_{n-k}$ such that $(P_1, ..., P_k, L_1, ..., L_{n-k})$ defines a proper map. This means that there exists Lojasiewicz exponents $\delta_1, ..., \delta_n$ such that

$$\sum_{j=1}^{k} \frac{|P_j(X)|}{\|X\|^{\delta_j}} + \sum_{j=1}^{n-k} \frac{|L_j(X)|}{\|X\|^{\delta_j}} \geq \gamma, \ \|X\| \geq K.$$

Then, one can use the Cauchy-Weil's formula in a bounded connected component Δ of the set

$$\{|P_j(X)| \leq R_j, \ j = 1, ..., k; \ |L_j(X)| \leq \widetilde{R}_j, \ j = 1, ..., n-k\}$$

that contains all common zeroes of $(P_1, ..., P_k, L_1, ..., L_{n-k})$. If

$$\Gamma_R := \{\zeta \in \Delta, \ |P_j(X)| = R_j, \ j = 1, ..., k; \ |L_j(X)| = \widetilde{R}_j, \ j = 1, ..., n-k\},$$

one has, for z in this component

$$1 = \int_{\Gamma_R} \frac{\mathrm{Bez}^{(L)}(\zeta, z) d\zeta}{\prod_{j=1}^{k} (P_j(\zeta) - P_j(z)) \prod_{j=1}^{n-k} (L_j(\zeta) - L_j(z))},$$

where

$$\mathrm{Bez}^{(L)}(X, Y) = \sum_{|\underline{\alpha}|+|\underline{\beta}|\leq D_1+\ldots+D_k-k} \gamma^{(L)}_{\underline{\alpha},\underline{\beta}} X^{\underline{\alpha}} Y^{\underline{\beta}}$$

denotes a Bézoutian of $(P_1, ..., P_n, L_1, ..., L_{n-k})$. This formula can be transformed as follows (exactly as in the case $k = n$ studied before). For any P_0 that does not vanish on the zero set defined by the P_j, with

$$P_0(X) - P_0(Y) = a_{01}(X, Y)(X_1 - Y_1) + \ldots + a_{0n}(X, Y)(X_n - Y_n),$$

one has the Bézout identity:

$$
\begin{aligned}
1 = & \operatorname{Res}\left[\begin{matrix} \frac{\mathrm{Bez}^{(L)}(X,Y)}{P_0(X)}dX \\ P_1,...,P_k,L_1,...,L_{n-k}\end{matrix}\right] P_0(X) + \sum_{k=1}^{n} \operatorname{Res}\left[\begin{matrix} \Lambda_k^{(L)}(X,Y)dX \\ P_1,...,P_k,L_1,...,L_{n-k}\end{matrix}\right] P_k(X) + \\
& + \sum_{\substack{\underline{q}\in\mathbf{N}^n,(q_1,\dots,q_k)\neq\underline{0},\ \underline{\alpha}\underline{\beta}\in\mathbf{N}^n \\ <\underline{q},\delta+\underline{1}>+|\beta|\leq D_1+\dots+D_k+n-k \\ |\underline{\alpha}|+|\underline{\beta}|\leq D_1+\dots+D_k-k}} \gamma_{\underline{\alpha},\underline{\beta}}^{(L)} \operatorname{Res}\left[\begin{matrix} X^{\underline{\alpha}}dX \\ P_1^{q_1+1},...,P_k^{q_k+1},L_1^{q_{k+1}+1},...,L_{n-k}^{q_n+1}\end{matrix}\right] \times \\
& \times Y^{\underline{\beta}} P_1(Y)^{q_1}\dots P_k(Y)^{q_k} L_1^{q_{k+1}}(Y)\dots L_{n-k}^{q_n}(Y)\,.
\end{aligned}
\tag{1.4}
$$

In the above formula, $\mathrm{Bez}^{(L)}$ is a Bézoutian of $P_1,...,P_k,L_1,...,L_{n-k}$ and the $\Lambda_k^{(L)}$ are obtained as before substituting to the column with intex k in the determinant $\mathrm{Bez}^{(L)}$ the column

$$
\begin{pmatrix} a_{01}(X,Y) \\ \vdots \\ a_{0n}(X,Y) \end{pmatrix}.
$$

Of course, in this kind of construction, one could think about an optimum choice of the L_j in order that the Lojasiewicz exponents $(\delta_1,...,\delta_n)$ are (if possible) all maximal. We do not know which corresponds to the optimum choice here.

Such a Kronecker's formula holds in the sparse situation, when one deals with Laurent polynomials instead of polynomials.

Let $F_1,...,F_n$, be n Laurent polynomials, with respective supports $\mathcal{A}_1$,...,$\mathcal{A}_n$ (with convex closed enveloppes $\Delta_1,...,\Delta_n$). Suppose that all relative interiors of the Δ_j, $j=1,...,n$, contain the origin. Then, if the condition that ensures that the Cartier divisors induced by the F_j on the toric variety $\mathcal{X}(\Delta_1+\ldots+\Delta_n)$ intersect only in the torus are satisfied, and G is a Laurent polynomial with support in

$$
\bigcap_{j=1}^{n}\bigcup_{l\in\mathbf{N}} \text{relative interior } [\Delta_1+\ldots+(l+1)\Delta_j+\ldots+\Delta_n],
$$

then, the toric residue symbols

$$
\operatorname{Res}\left[\begin{matrix} G(X)dX \\ F_1^{q_1+1},...,F_n^{q_n+1}\end{matrix}\right]_{\mathbf{T}},\ \underline{q}\in\mathbf{N}^n
$$

are zero as soon as

$$
\operatorname{Supp} G \subset \text{rel. int. } [(q_1+1)\Delta_1+\ldots+(q_n+1)\Delta_n]\,.
$$

It follows then that

$$
\operatorname{Res}\left[\begin{matrix} G(X)dX \\ F_1-u_1,...,F_n-u_n\end{matrix}\right]_{\mathbf{T}},
$$

where the u_j are complex parameters (this residue symbol makes sense since the $F_j - u_j$, $j = 1, ..., n$, define a discrete variety in the torus) can be expanded as

$$\begin{aligned} &\mathrm{Res}\left[\begin{matrix} G(X)dX \\ F_1 - u_1, ..., F_n - u_n \end{matrix}\right]_{\mathbf{T}} = \\ &= \sum_{\substack{\underline{q}\in\mathbf{N}^n \\ \mathrm{Supp}\, G \not\subset \mathrm{rel.\ int.}\ [(q_1+1)\Delta_1+\ldots+(q_n+1)\Delta_n]}} \mathrm{Res}\left[\begin{matrix} G(X)dX \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{matrix}\right]_{\mathbf{T}} u_1^{q_1}\ldots u_n^{q_n}, \end{aligned} \tag{1.5}$$

that is, as a polynomial in $u = (u_1, ..., u_n)$.

Let us suppose also that the Δ_j are polyedra with dimension n and let us introduce the "saturated" polyedra $\widetilde{\Delta}_j$ in $\mathbf{R}^{2n}$ which are the closed convex enveloppes of the sets

$$\bigcup_{\xi\in\Delta_j, \eta\in\Delta_j} [(\xi, 0), (0, \eta)]$$

Then, as in the affine case, we can introduce a Bézoutian such that

$$\mathrm{Bez}(X, Y)X_1\ldots X_n = \sum_{\substack{\underline{\alpha},\underline{\beta}\in\mathbf{N}^n \\ (\underline{\alpha},\underline{\beta})\in\widetilde{\Delta}_1+\ldots+\widetilde{\Delta}_n}} \gamma_{\underline{\alpha},\underline{\beta}} X^{\underline{\alpha}} Y^{\underline{\beta}}$$

and get the following Kronecker's identity

$$\begin{aligned} 1 = &\mathrm{Res}\left[\begin{matrix} X_1\ldots X_n \mathrm{Bez}(X, Y)dX \\ F_1, ..., F_n \end{matrix}\right]_{\mathbf{T}} + \\ + &\sum_{\substack{\underline{q}\in(\mathbf{N}^n)^*, (\underline{\alpha}\underline{\beta})\in\widetilde{\Delta}_1+\ldots+\widetilde{\Delta}_n \\ \underline{\alpha}\not\in \mathrm{rel.\ int.}\ [(q_1+1)\Delta_1+\ldots+(q_n+1)\Delta_n]}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} X^{\underline{\alpha}}dX \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{matrix}\right]_{\mathbf{T}} Y^{\underline{\beta}} F_1(Y)^{q_1}\ldots F_n(Y)^{q_n}. \end{aligned} \tag{1.6}$$

If F_0 is a Laurent polynomial which does not vanish on the set of common zeroes of the F_j, $1 \leq j \leq n$, in the torus and the Λ_k, $k = 1, ..., n$, are Laurent polynomials constructed as before when substituting to the column with index k in the Bézoutian determinant $\mathrm{Bez}(X, Y)$ the column

$$\begin{pmatrix} a_{01}(X, Y) \\ \vdots \\ a_{0n}(X, Y) \end{pmatrix},$$

where the a_{0k} satisfy

$$F_0(X) - F_0(Y) = \sum_{k=1}^{n} a_{0k}(X, Y)(X_k - Y_k),$$

one deduces from (1.5) the Bézout identity

$$1 = \mathrm{Res}\left[\begin{matrix} X_1 \ldots X_n \mathrm{Bez}(X,Y)dX \\ F_1, ..., F_n \end{matrix}\right]_{\mathbf{T}} F_0(X) + \sum_{k=1}^{n} \mathrm{Res}\left[\begin{matrix} X_1 \ldots X_n \Lambda_k(X,Y)dX \\ F_1, ..., F_n \end{matrix}\right]_{\mathbf{T}} F_k(X) +$$

$$+ \sum_{\substack{\underline{q} \in (\mathbf{N}^n)^*, (\underline{\alpha}\underline{\beta}) \in \widetilde{\Delta_1} + \ldots + \widetilde{\Delta_n} \\ \underline{\alpha} \notin \mathrm{rel.\ int.}\ [(q_1+1)\Delta_1 + \ldots + (q_n+1)\Delta_n]}} \gamma_{\underline{\alpha},\underline{\beta}} \mathrm{Res}\left[\begin{matrix} X^{\underline{\alpha}} dX \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{matrix}\right]_{\mathbf{T}} Y^{\underline{\beta}} F_1(Y)^{q_1} \ldots F_n(Y)^{q_n} . \tag{1.7}$$

One can also state in this case a version of the algebraic Nullstellensatz, exactly with the same idea one uses in the affine case.

Nethertheless, such statements dealing with systems of sparse polynomials are deeply connected with the fact that n of them, namely $F_1, ..., F_n$, do not have common zeroes at infinity. If one wants to lower this condition, it is natural to introduce a notion of properness in the torus. We will use an analytic approach in order to precise this notion in a particular case.

Suppose that F_1,...,F_n are n Laurent polynomials with respective supports $\mathcal{A}_1$,...,$\mathcal{A}_n$; suppose that the closed convex enveloppes of the $\mathcal{A}_j$, $j = 1, ..., k$, are all equal to a polyedron Δ, which contains the origin as an interior point. Another way to express that the divisors induced by the F_j on the toric variety $\mathcal{X}(\Delta)$ do not intersect at infinity is the following: there are two constants $\gamma > 0$, $K \geq 0$, such that, for any $\zeta \in \mathbf{C}^n$,

$$\|\mathrm{Re}(\zeta)\| \geq K \Longrightarrow \max_{1 \leq k \leq n} |F_k(e^{\zeta_1}, ..., e^{\zeta_n})| \geq \gamma e^{\max_{\xi \in \Delta} \mathrm{Re} <\xi,\zeta>} .$$

When $F_1, ..., F_n$ induce divisors that intersect at infinity on the toric variety $\mathcal{X}(\Delta)$, it is natural to introduce the weaker following concept: the map $F = (F_1, ..., F_n)$ will be proper (respect to the torus embedded in the toric variety $\mathcal{X}(\Delta)$), if one can find a convex polyedron δ, also containing the origin as an interior point, with $\delta \subset \Delta$, such that there exists two constants $\gamma > 0$, $K \geq 0$, such that, for any $\zeta \in \mathbf{C}^n$,

$$\|\mathrm{Re}(\zeta)\| \geq K \Longrightarrow \max_{1 \leq k \leq n} |F_k(e^{\zeta_1}, ..., e^{\zeta_n})| \geq \gamma e^{\max_{\xi \in \delta} \mathrm{Re} <\xi,\zeta>} .$$

If such is the case, one can show that, for any Laurent polynomial G, for $\underline{q} \in \mathbf{N}^n$ such that $|\underline{q}|$ is large enough, one has

$$\mathrm{Res}\left[\begin{matrix} G(X)dX \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{matrix}\right]_{\mathbf{T}} = 0 .$$

As in the affine case, for any Laurent polynomial G, Kronecker's formula,

$$G(Y) = \mathrm{Res}\left[\begin{matrix} G(X) X_1 \ldots X_n \mathrm{Bez}(X,Y)dX \\ F_1 - F_1(Y), ..., F_n - F_n(Y) \end{matrix}\right]$$

provides an algebraic identity

$$G(Y) = \sum_{\underline{q}\in \mathbf{N}^n, |\underline{q}|\leq C(G)} \text{Res} \begin{bmatrix} G(X)\text{Bez}(X,Y) \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{bmatrix}_{\mathbf{T}} P_1(Y)^{q_1} \dots P_n(Y)^{q_n} \,.$$

Such an identity can be used in order to solve Bézout identities or explicit formulations of the Hilbert's Nullstellensatz. It seems an interesting problem to understand this properness condition in algebraic terms (that is in terms of integral dependence at infinity), using the ring of homogeneous coordinates associated with the toric variety $\mathcal{X}(\Delta)$. The natural result one could predict would be the following:

Question. *Suppose that there exists n convex polyedra with rational vertices, containing the origin as an interior point, and such that there exists two constants $\gamma > 0$, $K \geq 0$, such that, for any $\zeta \in \mathbf{C}^n$*

$$\|\text{Re}(\zeta)\| \geq K \Longrightarrow \max_{1\leq k\leq n} \frac{|F_k(e^{\zeta 1}, ..., e^{\zeta_n})|}{e^{\max\limits_{\xi\in\delta_k} \text{Re}<\xi,\zeta>}} \geq \gamma.$$

Is it true that, for $\underline{q} \in \mathbf{N}^n$,

$$\text{Supp}\, G \subset \text{int.}\ ((q_1+1)\delta_1 + \ldots + (q_n+1)\delta_n) \Longrightarrow \text{Res} \begin{bmatrix} G(X)dX \\ F_1^{q_1+1}, ..., F_n^{q_n+1} \end{bmatrix}_{\mathbf{T}} = 0 \ ?$$

2. Nullstellensatz and degree estimates.

Suppose that $P_1, ..., P_n$ are n polynomials in $\mathbf{C}[X_1, ..., X_n]$ defining a discrete variety and let ν be the sum of the local Noether exponents at all common zeroes of $P_1, ..., P_n$. Then, one can find, for each $j \in \{1, ..., n\}$, a polynomial with degree at most ν in the variables $(X_1, ..., X_n)$ which lies in the ideal generated by $P_1, ..., P_n$. Let us suppose that δ is the Lojesiewicz exponent at infinity, that is

$$\delta := \max\{r \in \mathbf{R},\ \liminf_{\|\zeta\|\mapsto\infty} \frac{\|P(\zeta)\|}{\|\zeta\|^r} > 0\}\,.$$

We will suppose here that $\delta \leq 0$ (the case $\delta > 0$ has been studied in our second lesson).

Taking the homogeneizations $\mathcal{R}_1(X_0, X_1)$,...,$\mathcal{R}_n(X_0, X_n)$, of the R_j, $j = 1, ..., n$, one has, for some constant $C > 0$ and in a compact neighborhood U of the origin in $\mathbf{C}^{n+1}$,

$$|X_0|^{|\delta|+D} |\mathcal{R}_j(X_0, X_j)| \leq C \sum_{k=1}^{n} |X_0^{D-D_k} \mathcal{P}_k(X_0, ..., X_n)|,$$

where $\mathcal{P}_k$ is the homogeneized version of P_k, $D_k = \deg P_k$, $k = 1, ..., n$, $D = \max D_k$. We use Briançon-Skoda's theorem, which allows us to write

$$(X_0^{|\delta|+D} \mathcal{R}_j(X_0, X_j))^n = \sum_{k=1}^{n} \mathcal{U}_{jk}(X_0, ..., X_n) X_0^{\max(|\delta|,D)-D_k} \mathcal{P}_k(X_0, ..., X_n)\,,\ j = 1, ..., n\,. \tag{2.1}$$

If we deshomogeneize (2.1), we get

$$R(X_j)^n = \sum_{k=1}^{n} R_{jk}(X)P_k(X), \ j = 1, ..., n, \tag{2.2}$$

where we have

$$\deg R_{jk} \leq n(\nu + |\delta|) + (n-1)D\,, \ 1 \leq j, k \leq n\,. \tag{2.3}$$

The same argument can be used for inconsistent systems: suppose that $(P_1, ..., P_m)$ define a sequence of polynomials without common zeroes, and such that

$$\delta := \max\{r \in \mathbf{R}, \ \liminf_{\|\zeta\| \mapsto \infty} \frac{\|P(\zeta)\|}{\|\zeta\|^r} > 0\}\,.$$

Then, one can find polynomials $Q_1, ..., Q_m$ such that

$$1 = P_1Q_1 + \ldots + P_mQ_m\,, \ \deg(P_jQ_j) \leq n(\max(-\delta, 0) + D)\,.$$

3. Perron's theorem and size estimates.

Our goal in this section is to compute residue symbols of the form

$$\mathrm{Res}\left[\frac{Q(X)dX}{p_1^{q_1+1}, ..., p_n^{q_n+1}}\right]\,. \tag{3.1}$$

when the polynomials $Q, p_1, ..., p_n$ have coefficients in a regular factorial domain $\mathbf{A}$. Since we are interested here into arithmetic problems, we will deal with situations where the algebra $\mathbf{A}[X_1, ..., X_n]$ is equipped with a logarithmic size: the two important examples we will treat here are the example of $\mathbf{A} = \mathbf{Z}[u_1, ..., u_q]$, the sizes induced on $\mathrm{Pol}\,\mathbf{A} := \mathbf{A}[(Y_i)_{i\in\mathbf{N}}]$ being the the sizes

$$t_C(P) := C\deg_u P + \int_{[0,2\pi]^m} \log|P(e^{i\theta_1}, ..., e^{i\theta_n})| d\theta_1 \ldots d\theta_n, \ P \in \mathrm{Pol}\,\mathbf{A}$$

and the example $\mathbf{A} := \mathbf{F}_p[u_1, ..., u_q]$, the size induced on $\mathrm{Pol}\,\mathbf{A}$ in this case being

$$t(P) := \deg_u P\,, \ P \in \mathrm{Pol}\,\mathbf{A}\,.$$

Let us deal with the case $\mathbf{A} = \mathbf{Z}$ and consider a quasi-regular sequence $(p_1, ..., p_n)$, $p_j \in \mathbf{Z}[X_1, ..., X_n]$, the quasiregularity being unterstood over $\mathbf{C}$. The first idea in order to compute express residue symbols of the form (3.1) is to use the algebraic Nullstellensatz. For example, one can use the transformation law and the set of formulas (2.2). Of course, using plain linear algebra it is possible to assume that such relations are with rational coefficients, or, raising denominators, that they can be written

$$S_j(X_j) = \sum_{k=1}^{n} S_{jk}p_k\,, \ S_j, S_{jk} \in \mathbf{Z}[X_1, ..., X_n]\,.$$

Since a rough estimate for the degrees of the S_j and the S_{jk} is $\kappa(n)D^n$ (using Brownawell's estimate for $|\delta|$, see [Br1]), the estimate one can predict for the size of the polynomials S_j, S_{jk} is $\tilde{k}(n)D^{n^2} \max_{1\leq j\leq n} t(p_j)$, if t is the size on $\mathrm{Pol}(\mathbf{Z})$.

Question. *It is a natural question to ask whether the well known Perron's theorem [Pe, Satz 57] can be precised if for example it is settled over a field with parameters such that* $\mathbf{C}(u_1, ..., u_q)$, *the* P_j, $j = 0, ..., n$, *being in* $\mathbf{C}[u_1, ..., u_q]$) *whether, given* $n+1$ *polynomials in* $\mathbf{C}[u_1, ..., u_q][X_1, ..., X_n]$, $p_0, ..., p_n$, *it is possible to find an integral dependency relation*

$$\mathcal{Q}(u, p_0, ..., p_n) = 0$$

with coefficients in $\mathbf{C}[u_1, ..., u_q]$, *such that the weighted degree (as in Perron's theorem) of the polynomial* $\mathcal{Q}(u, Y)$ *in* Y *(with the weight in* Y_j *being* $D_j = \deg_X(p_j)$*) is at most* $D_0 \dots D_n$, *and the weighted degree of* $\mathcal{Q}(u, Y)$ *in* u *(with the weight in* Y_j *being* $\mu_j = \deg_u(p_j)$, $j = 0, ..., n$*) is at most* $\mu_0 \dots \mu_n$.

At the moment, we have no idea about such a result. It may be possible that the construction of [Elk-Mo], where the relations of integral dependence are given by non zeroes minors with maximal rank in the Bézoutian matrix, could provide relations of integral dependency with balanced control respect to degree and logarithmic size (respectively degree in X and degree in u in our example). Something else that seems to be noticiable in this direction is the fact that if one considers the $q + n + 1$ polynomials $(u_1, ..., u_q, p_0, p_1, ..., p_n)$ in $n + q$ variables, one can find a non trivial relation of integral dependence

$$\mathcal{Q}(u_1, ..., u_q, p_0, ..., p_n) = 0$$

with total degree at most $\prod_{j=0}^{n} (\mu_j + D_j)$. If the p_j, $j = 1, ..., n$, define a regular sequence on $\mathbf{C}(u)[X_1, ..., X_n]$, such a relation is a relation of integral dependency of p_0 over $\mathbf{C}(u, p_1, ..., p_n)$.

Nethertheless, if $p_1, ..., p_n$ are polynomials with coefficients in a factorial regular ring $\mathbf{A}$ (with fraction field $\mathbf{K}$), equipped with a size, defining a dominant sequence in $\mathbf{K}[X_1, ..., X_n]$, we can use the theorem of P. Philippon [Ph] which ensures one has estimates in accord with arithmetic Bézout theory [BGS] for the size of an element $\delta_j \in \mathbf{A}[v_0, ..., v_n]^*$ such that

$$\delta_j(v_0, ..., v_n) = \sum_{k=1}^{n} (p_j - v_j) q_{jk}(v, X) + (X_j - v_0) q_{j0}(u, X), \; q_{jk} \in \mathbf{A}[v_0, ..., v_n, X] \,.$$

If the degrees of the polynomials are bounded by D and the sizes bounded by h (to fix ideas, we take here $\mathbf{A} = \mathbf{Z}$), we have the following control, in $\kappa_1(n)D^n$, where $D = \max_{1\leq j\leq n} \deg(p_j)$ for the degrees (in v) and $\kappa_2(n)D^n(h + D)$ for the sizes. From the relations

$$\delta_j(X_j, P_1, ..., P_n) \equiv 0, \; j = 1, ..., n \,, \tag{3.2}$$

it is possible to compute (and therefore estimate the size) the residue symbols

$$\mathrm{Res}\left[\begin{matrix} Q(X)dX \\ p_1^{q_1+1},...,p_n^{q_n+1}\end{matrix}\right].$$

This is done using the identities

$$\mathrm{Res}\left[\begin{matrix} Q(X)dt\wedge dX \\ t^{|\underline{q}|+1},p_1-\alpha_1 t,...,p_n-\alpha_n t\end{matrix}\right] = \sum_{|\underline{l}|=|\underline{q}|}\mathrm{Res}\left[\begin{matrix} Q(X)dX \\ p_1^{l_1+1},...,p_n^{l_n+1}\end{matrix}\right]\alpha_1^{l_1}\dots\alpha_n^{l_n}\,,\ \alpha\in\mathbf{C}^n,$$

the identities (3.2) rewritten as

$$t^{s_j}(R_j(\alpha,X_j)-tS_j(t,\alpha,X_j)) = \sum_{k=1}^{n} B_{jk}(t,\alpha,X_j,p_1,...,p_n)(p_j-\alpha_j t)\,,$$

and the generalisation of the transformation law proposed in the first chapter (see formula (2.3) in chapter 1). The estimates we get for numerators and denominators of such residue symbols are of the form $\kappa(n)|\underline{q}|D^n(h+D)+h(Q)$ (see [BY2], [BY3]).

4. An explicit version of the effective Nullstellensatz.

Consider $P_1,...,P_m$ m polynomials without common zeroes (let us say for the moment with complex coefficients). Let $(\tilde{p}_1,...,\tilde{p}_n)$ be n generic linear combinations of the P_j defining a *normal* system, that is any subfamily of the family $(\tilde{p}_1,...,\tilde{p}_n)$ is quasiregular). We know (from Kollár's theorem [JKS]) that there is a constant γ such that for any such family $(\tilde{p}_i)_{i\in\mathcal{I}}$, one has

$$\max_{i\in\mathcal{I}}|f_i(\zeta)|\geq\gamma\Big(\frac{\min(1,d(\zeta,V_{\mathcal{I}}))}{1+\|\zeta\|}\Big)^{\prod_{i\in\mathcal{I}}\deg\tilde{p}_i}$$

(with the restriction that the degrees are at least 2). With the Noether normalization lemma, one can show that it is possible to find n linear forms (independent), $L_1,...,L_n$, such that the map

$$(L_1^{D^n+1}\tilde{p}_1,...,L_n^{D^n+1}\tilde{p}_n)$$

is proper (with Lojasiewicz exponent at least 1). In fact, one can show later on that the choice of these linear forms is in fact generic.

Once these forms have been chosen, one can find a linear combination (also generic) of the P_j, $j=1,...,n$, which does not vanish on the zero set of the $p_j=L_j\tilde{p}_j$, $j=1,...,n$. We can use the formulas in section 1 of this chapter with the system of polynomials $P_0,\widetilde{P}_j, j=1,...,n$, where $\widetilde{P}_j := L_j^{D^n+1}p_j$, $j=1,...,n$ and make explicit a Bézout identity

$$1=P_0\widetilde{Q}_0+\widetilde{P}_1\widetilde{Q}_1+\ldots+\widetilde{P}_n\widetilde{Q}_n\,.$$

We obtain with this process an effective Bézout identity

$$1=P_1Q_1+\ldots+P_mQ_m$$

which happens to be economic respect to degree and size estimates. This identity respects the field over which the P_j are defined. Moreover, if the entries P_j have degree estimates in D and logarithmic size estimates in h, the degree estimate for the Q_j is in $\kappa_1(n)D^n$, while the size estimate for them is in $\kappa_2(n)D^{\mathbf{0}(n)}(h+D+\log m)$. These estimates are far to be optimum, since an arithmetic version of Bézout's theorem (as in [BGS]) predicts there could be height estimates in $\kappa(n)D^n(h+D+m)$. It seems that the Cauchy-Weil's formula, though used as here in some rather artisanal way, could provide an interesting joint solution to the arithmetic and geometric division problem. Note that the pairing of arithmetic cycles and Green currents (that is analytic tools) provides already a solution to the arithmetic and geometric intersection problem (see [GS]).

[BGS] J. B. Bost, H. Gillet, C. Soulé, Heights of projective varieties and positive Green currents, J. Amer. Math. Soc. 7 (1994), 903-1027.
[Br1] W. D. Brownawell, Bounds for the degrees in the Nullstellensatz, Annals of Math. 126 (1987), 577-591.
[BY2] C. A. Berenstein, A. Yger, Residue calculus and effective Nullstellensatz, preprint.
[BY3] C. A. Berenstein, A. Yger, Residues and effective Nullstellensatz, Electronic Research Annoucements of the AMS, 2, 2, 1996.
[CaPl] P. Cassou-Noguès, A. Ploski, Un théorème des zéros effectif, Bull. Pol. Ac. Math, 44 (1996), 61-70.
[Elk-Mo] M. Elkadi, B. Mourrain, Some applications of Bézoutians in effective algebraic geometry, *Preprint.*
[FPY] A. Fabiano, G. Pucci, A. Yger, Effective Nullstellensatz and geometric degree for zero-dimensional ideals, Acta Arithmetica, 78(2), 1996.
[JKS] S. Ji, J. Kollár, B. Shiffman, A global Lojasiewicz inequality for algebraic varieties, Trans. Amer. Math. Soc. 329 (1992), 813-818.
[GS] H. Gillet , C. Soulé, Arithmetic intersection theory, Publ. Inst. Hautes Études Sci. 72(1990), 93-74.
[Ph] P. Philippon, Dénominateurs dans le théorème des zéros de Hilbert, Acta Arithm. LVIII. 1 (1991), 1-25.
[Pe] O. Perron, *Algebra* I (Die Grundlagen), Göschens Lehbücherei, Berlin et Leipzig, 1932.