Green currents and analytic continuation[1]

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1. Introduction. Inspired by the work of Arakelov and Faltings, H. Gillet and C. Soulé developed a method to express arithmetic heights of cycles in $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, considered as an arithmetic variety over $\mathbf{Z}$ ([GS1], [GS2], [BGS]). This was done in terms of a multiplication operation between pairs $(\mathcal{Z}, G_Z)$, where $\mathcal{Z}$ is an arithmetic cycle of codimension p in $\mathbf{P}^n$, $Z = \mathcal{Z}(\mathbf{C})$ the corresponding algebraic cycle in $\mathbf{P}^n(\mathbf{C})$, and G_Z a $(p-1, p-1)$ current in $\mathbf{P}^n(\mathbf{C})$. This current must satisfy the Green equation

$$dd^c G_Z + \delta_Z = f\,, \tag{1}$$

where f is a smooth (p,p) form and δ_Z is the integration current on the cycle Z. (We recall $d^c = (\partial - \overline{\partial})/4\pi i$.) Such a current G_Z is usually called a Green current for Z. The multiplication between such pairs is formally defined by the relation

$$(\mathcal{Z}_1, G_{Z_1}) \bullet (\mathcal{Z}_2, G_{Z_2}) = (\mathcal{Z}_1 \cdot \mathcal{Z}_2, \delta_{Z_2} \wedge G_{Z_1} + f_1 \wedge G_{Z_2})\,, \tag{2}$$

where $\mathcal{Z}_1 \cdot \mathcal{Z}_2$ is the arithmetic intersection of the two cycles [GS1]. In order for such a definition to make sense one needs additional constraints on the Green currents. Gillet-Soulé assume that the Green current is chosen to be C^∞ outside the support $|Z|$ of the cycle and having logarithmic singularities (after resolving the singularities of Z) on Z. This allows them to prove that the wedge product $\delta_{Z_2} \wedge G_{Z_1}$ makes sense. The product thus defined has to be understood modulo some equivalence relations, namely, it is defined in the p-Chow group of $\mathbf{P}^n$, i.e., in the quotient group of the additive group of pairs $(\mathcal{Z}, G_Z)$ modulo the subgroup generated by elements of the form $(0, du + d^c v)$, with u, v currents in $\mathbf{P}^n(\mathbf{C})$, and elements of the form $(\mathrm{div}\, h, -i_*(\log |h|^2))$, where h is a rational function on a subscheme Y of codimension $p-1$, the divisor $\mathrm{div}\, h$ is a divisor in Y, and $i : Y(\mathbf{C}) \to \mathbf{P}^n(\mathbf{C})$ is the canonical embedding. The corresponding product of classes turns out to be commutative.

The concept of Green currents also makes sense on any smooth arithmetic variety X, not only $\mathbf{P}^n$. We denote by $X(\mathbf{C})$ the corresponding complex manifold. On the other hand, $\mathbf{P}^n(\mathbf{C})$ is equipped with a Kähler form, namely the Fubini-Study metric and corresponding form

$$\omega = dd^c \log(|x_0|^2 + \cdots |x_n|^2) = \frac{i}{2\pi} \partial\overline{\partial} \log(||x||^2)\,. \tag{3}$$

An Arakelov variety is a pair (X, ω), where X is a projective arithmetic variety and ω is a Kähler form on $X(\mathbf{C})$. For a codimension p arithmetic cycle Z on an Arakelov variety, we

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have the notion of normalized Green current, namely, the $(p-1,p-1)$ current G_Z (unique up to currents of the form $du + d^c v$) which is a solution both of the Lelong-Poincaré equation

$$dd^c G_Z + \delta_Z = H(\delta_Z)\,, \tag{4}$$

and of

$$H(G_Z) = 0,$$

where H is the harmonic projection relative to the Hodge decomposition on $\mathbf{P}^n(\mathbf{C})$. For example, in the case of $\mathbf{P}^n$, if $\mathcal{Z}$ is defined by p homogeneous equations $Q_1 = \cdots = Q_p = 0$, of respective degrees D_j, the zeros counted with multiplicities, and such that the sequence $Q_1, \ldots, Q_p$ is a regular sequence in $\mathbf{P}^n(\mathbf{C})$, then a normalized Green current solves the equation

$$dd^c G_Z + \delta_Z = D_1 \cdots D_p \omega^p\,. \tag{5}$$

It is shown in [GS1] that one can find such normalized current G_Z with the additional properties of being smooth outside $|Z|$ and of logarithm growth at $|Z|$, as required above for the product (2) to make sense. Normalized Green currents are used to get representative for the Chow classes (as described below.) Note that positive Green currents are not normalized. The currents we will deal with in this paper will in general be positive, the normalization will appear as an auxiliary step for the expression of the arithmetic height of cycles.

Let us recall basic facts about arithmetic intersection theory. When $\mathcal{Z}$ is a codimension p arithmetic cycle in $\mathbf{P}^n$, its Chow class $\widehat{\mathcal{Z}}$ is the element of the p-Chow group of $\mathbf{P}^n$ defined by the class of a pair $(\mathcal{Z}, G_Z)$, where G_Z is precisely a normalized Green current. We need also to define the 1-Chow class $\hat{c}_1(\mathbf{P}^n)$. This is done as follows: given $\mathcal{Z}_0$ and a generic hyperplane $< u, x >= u_0x_0 + \cdots + u_nx_n = 0$, $u \in \mathbf{Z}^{n+1}$, one can take

$$\Gamma_{Z_0} = -\log \frac{|< u, x >|^2}{||x||^2}$$

as a Green current for Z_0. The 1-Chow class defined as the class of the pair $(\mathcal{Z}_0, \Gamma_{Z_0})$ doesn't depend on the choice of u. This class will be $\hat{c}_1(\mathbf{P}^n)$. In this case, it is easy to compute any power $(\hat{c}_1(\mathbf{P}^n))^k$ (with respect to the previously defined product (2)), $1 \leq k \leq n$, using as representative the cycle $\mathit{\Pi}_u = \{< u^{(0)}, x >= \cdots =< u^{(k-1)}, x >= 0\}$ ($u^{(j)}$ linearly independent in $\mathbf{Z}^{n+1}$) and the locally integrable Green current L, introduced by H. Levine,

(6)
$$L(x) = -\log\left(\frac{\sum_{j=0}^{k-1} |< u^{(j)}, x >|^2}{||x||^2}\right)\left(\sum_{j=0}^{k-1}\left(dd^c \log \sum_{j=0}^{k-1} |< u^{(j)}, x >|^2\right)^j \wedge \omega^{k-1-j}\right)$$

One can associate to a codimension p arithmetic cycle $\mathcal{Z}$ in $\mathbf{P}^n$ a height, which is defined as follows: compute the product

$$\widehat{\mathcal{Z}} \bullet \hat{c}_1(\mathbf{P}^n)^{n+1-p} \tag{7}$$

choosing vectors $u^{(j)}, 0 \leq j \leq n-p$, such that $|\mathit{\Pi}_u|_{\mathbf{Q}} \cap |\mathcal{Z}|_{\mathbf{Q}} = \emptyset$, and choosing a normalized Green current G_Z which is smooth outside $|Z|$. Formula (2) provides a representative for (7). The first component is a codimension $n+1$ cycle in the scheme $\mathbf{P}^n$, i.e., a cycle of the form

$$\sum_{\tau \text{ prime}} n_\tau [\tau] .$$

The second component is the (n, n) current

$$\delta_{\Pi_u} \wedge G_Z + H(\delta_Z) \wedge L ,$$

where $\Pi_u = \mathit{\Pi}_u(\mathbf{C})$ is the corresponding linear variety in $\mathbf{P}^n(\mathbf{C})$. Note that there is no problem in defining the first summand, since the singular supports of the two factors are disjoint. Moreover, from Wirtinger's theorem [Sto]

$$H(\delta_Z) = deg(Z)\, \omega^p ,$$

so that the second component of (7) is

$$\delta_{\Pi_u} \wedge G_Z + deg(Z)\, \omega^p \wedge L .$$

The logarithmic height of $\mathcal{Z}$ is defined by

$$h(\mathcal{Z}) = \sum_{\tau \text{ prime}} n_\tau \log \tau + \frac{1}{2} \int_{\mathbf{P}^n(\mathbf{C})} (\delta_{\Pi_u} \wedge G_Z + deg(Z)\, \omega^p \wedge L) \tag{8}$$

and it is independent of the choices made so far. As proved in [St] (see also [BGS, (1.4.4)]),

$$\int_{\mathbf{P}^n(\mathbf{C})} \omega^p \wedge L = \sum_{k=p}^{n} \sum_{j=1}^{k} \frac{1}{j} ,$$

so that

$$h(\mathcal{Z}) = \sum_{\tau \text{ prime}} n_\tau \log \tau + \frac{deg(Z)}{2} \sum_{k=p}^{n} \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{2} \int_{\Pi_u} G_Z . \tag{9}$$

There is a great difficulty in computing explicitly logarithmic heights, even for the case of hypersurfaces. Nevertheless, in this case the expression (9) can be given a simpler representation [BGS, (3.3.1)]. When $\mathcal{Z}$ is an hypersurface in $\mathbf{P}^n$, which is defined by some homogeneous polynomial Q with degree D, the normalized Green current one can take for Z is

$$-\log \frac{|Q(x)|^2}{\|x\|^{2D}} + \int_{\mathbf{P^n}(\mathbf{C})} \log \frac{|Q(x)|^2}{\|x\|^{2D}} \omega^n .$$

Using the commutativity of the product $\bullet$, one gets for such a hypersurface,

$$h(\mathcal{Z}) = \frac{D}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} + \int_{\mathbf{P^n(C)}} \log\frac{|Q(x)|}{\|x\|^D}\omega^n, \tag{10}$$

that is,

$$h(\mathcal{Z}) = \frac{D}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} + \int_{\mathbf{S}^{2n+1}} \log|Q(t)|d\nu(t), \tag{11}$$

where ν is the uniform probability measure (that is invariant with respect to the unitary group $\mathbf{U}(n+1)$) on the unit sphere $\mathbf{S}^{2n+1}$. The integral that appears in (10) (or (11)) can be interpreted as the derivative at $s = 0$ of a zeta function, namely,

$$\zeta_Q(s) = \int_{\mathbf{P^n(C)}} \left(\frac{|Q(x)|}{\|x\|^D}\right)^s \omega^n = \int_{\mathbf{S}^{2n+1}} |Q(t)|^s d\nu(t)\,. \tag{12}$$

Using the homogeneity of Q one can rewrite the last integral to obtain, for any $\rho > 0$, for any s with $Re\, s > 0$,

$$\zeta_Q(s) = \frac{n!}{\pi^{n+1}\Gamma(n+1+Ds/2)}\int_{\mathbf{C}^{n+1}} \exp(-||z||^2)|Q(z)|^s dm(z)\,.$$

Note that the function ζ_Q can be analytically continued as a meromorphic function in the whole complex plane, with poles in $\mathbf{Q}^-$ (see [At]).

In this paper, we will show how one can express positive Green currents in terms of such zeta functions. We will then normalize them and obtain an explicit expression for the logarithmic heigth of the arithmetic p-cycle $\{Q_1 = \cdots = Q_p = 0\}$ in $\mathbf{P}^n$, where the Q_j are homogeneous polynomials in $\mathbf{Z}[x_0, \cdots, x_n]$ (with degree D) such that the corresponding divisors intersect properly in $\mathbf{P}^n(\mathbf{C})$. If we assume $\{x_0 = \cdots = x_{n-p} = Q_1(x) = \cdots = Q_p(x) = 0\}$ is the empty set in $\mathbf{P}^n(\mathbf{C})$, then the logarithmic size of $\mathcal{Z}$ is the sum of the "arithmetic" contribution

$$\sum_{\tau \text{prime}} n_\tau \log \tau$$

(where $\sum_{\tau\text{prime}} n_\tau$ is the $n+1$ arithmetic cycle $\Pi \cdot \mathcal{Z}$, where $\Pi := \{x_0 = \cdots = x_{n-p} = 0\}$), and of an "analytic" contribution, which can be reached as the "value" at $\lambda = 0$ of the following zeta function

$$\lambda \mapsto \frac{D^p}{2}\sum_{k=p}^{n}\sum_{j=1}^{k}\frac{1}{j} - \frac{1}{2}\int_{(x,y)\in\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \omega(x)^{n-p+1}U_\lambda(x,y) + \frac{1}{2}\int_{\Pi\times\mathbf{P}^n(\mathbf{C})} U_\lambda\big|_{\Pi\times\mathbf{P}^n(\mathbf{C})}$$

with

$$U_\lambda(x,y) = I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x,y),$$

with

$$I_\lambda(y) := \frac{i}{2\pi}\lambda\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)^\lambda \partial\log\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)\wedge\overline{\partial}\log\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)\wedge$$

$$\wedge\left(dd^c\log\left(\sum_{j=1}^p |Q_j(y)|^2\right)\right)^{p-1}$$

and

$$\Upsilon_\mu(x,y) := \int_{\beta\in\mathbf{P}} \pi^*(L_\mu)(x,y,\beta)\,,$$

where L_μ is the (n,n) current in $\mathbf{P}^{2n+1}(\mathbf{C})$ defined in homogeneous ccordinates as

$$L_\mu := \frac{-1}{\mu}\left(\frac{||x-y||^2}{||x||^2+||y||^2}\right)^\mu\left(\sum_{k=0}^n\left(dd^c\log||x-y||^2\right)^k\wedge(dd^c\log(||x||^2+||y||^2))^{n-k}\right)$$

and π the map obtained by taking quotients from the map

$$((\mathbf{C}^{n+1})^*)^2\times(\mathbf{C}^2)^*\mapsto(\mathbf{C}^{n+2})^*:\ (x,y,(\beta_0,\beta_1))\mapsto(\beta_0x,\beta_1y)\,.$$

Note that such a method could be used whenever exists an expression of the integration current on $\mathcal{Z}(\mathbf{C})$ as the value at the origin of a zeta function involving the functions $\lambda\mapsto|Q_j|^\lambda$ or $\lambda\mapsto\|Q\|^{2\lambda}$, where the Q_j are the homogeneous polynomials (supposed with the same degree) which define $\mathcal{Z}$. Since this question remains open when $\mathcal{Z}(\mathbf{C})$ is not defined as a complete intersection, we will deal in this paper mostly with the complete intersection case; in this case, our inspiration goes back to a classical construction of Levine [Le], which has been extented in [GK].

Of course, such an approach does not solve entirely the problem of computing logarithmic heights but it has two advantages, the first is that one can use the functional equation of Bernstein-Sato in order to compute a functional equation satisfied by ζ, the second is that the formulas are expressed directly in terms of the polynomials defining the cycle, without any information on its decomposition into irreducible cycles. The method we develop here is based on our approach to the theory of multidimensional residue currents through the principle of analytic continuation [BGVY]. Some of our results were announced in several conferences, like the Analytic Geometry conference held in Paris in June 1992. We would like to thank Patrice Philippon and Christophe Soulé for many useful discussions about their work on heights.

2. Green currents and analytic continuation in $\mathbf{C}^n$. In this section we would like to profit from the factorization property of the integration current relative to a complete intersection, in order to construct Green currents. It is well known that, if $f_1,\dots,f_p$ are holomorphic functions defining a complete intersection variety Z in an open set $\Omega\subseteq\mathbf{C}^n$ and

δ_Z denotes the integration current with multiplicity, i.e., the integration current associated to the corresponding cycle, then [CH]

$$\delta_Z = \overline{\partial}\frac{1}{f} \wedge df_1 \wedge \cdots \wedge df_p \, , \tag{13}$$

where $\overline{\partial}(1/f)$ is the $(0,p)$ residue current associated to $f_1,\dots,f_p$. In the monograph [BGVY] we consider different methods to represent such a residue current in terms of zeta-functions of one or several variables. Let us recall the two main one variable ways to do this. The first one, [BGVY, Theorem 3.18], is the following: for any $(n,n-p)$ test form φ,

$$< \overline{\partial}\frac{1}{f}, \varphi >= \frac{(-1)^{p(p-1)/2}}{(2i\pi)^p}\left(\lambda^p \int_{\mathbf{C}^n} |f_1 ... f_p|^{2(\lambda-1)}\, \overline{\partial f} \wedge \varphi\right)_{\lambda=0} \tag{14}$$

where

$$\overline{\partial f} = \bigwedge_{j=1}^{p} \overline{\partial f_j} = \overline{\partial f_1} \wedge \cdots \wedge \overline{\partial f_p} \tag{15}$$

and the evaluation at $\lambda = 0$ means that one takes the meromorphic continuation of the right hand side of (14) (considered as a holomorphic function of λ for $\mathrm{Re}\,(\lambda)$ large enough) and follows this analytic continuation up to the origin. Note that we proved in [BGVY, Theorem 3.18] that the poles of the zeta function defined that way are all in $\mathbf{Q}^-$. It follows from (13) that the action of the integration current δ_Z on a $(n-p,n-p)$ test form can be expressed as

$$< \delta_Z, \varphi >= \frac{(-1)^{p(p-1)/2}}{(2i\pi)^p}\left(\lambda^p \int_{\mathbf{C}^n} |f_1 ... f_p|^{2(\lambda-1)}\, \overline{\partial f} \wedge \partial f \wedge \varphi\right)_{\lambda=0} \tag{16}$$

with ∂f having the obvious meaning similar to (15). The following lemma provides a construction for a Green current based on the equation (16).

Lemma 1. *The current-valued holomorphic map $\lambda \mapsto \Psi_\lambda$ defined for* $\mathrm{Re}\,\lambda >> 0$ *by*

$$\Psi_\lambda = \frac{(-1)^{p(p+1)/2}}{(2i\pi)^{p-1}} \frac{|f_1|^{2\lambda}}{\lambda} \bigwedge_{j=2}^{p} \overline{\partial}\left(\frac{|f_j|^{2\lambda}}{f_j}\right) \wedge \bigwedge_{j=2}^{p} \partial f_j$$

can be analytically continued as a meromorphic function in $\mathbf{C}$. *The Laurent development of this function at the origin is*

$$-\frac{\delta_{Z_1}}{\lambda} + G + \lambda H_\lambda \, , \tag{17}$$

where Z_1 is the cycle corresponding to the ideal $(f_2,\dots,f_p)$, $\lambda \mapsto H_\lambda$ is holomorphic near the origin, and G is a $(p-1,p-1)$ current which satisfies the Green equation

$$dd^c G + \delta_Z = 0 \, . \tag{18}$$

Proof. One can easily compute $dd^c\Psi_\lambda$ for $\mathrm{Re}\,\lambda >> 0$ and obtain exactly the right hand side of (16). Since the action of dd^c (or any differential operator with constant coefficients) commutes with the process of analytic continuation, it is clear that the coefficient G of λ^0 in the Laurent development of Ψ_λ about 0 satisfies the equation (18). That the pole $\lambda = 0$ is simple and contributes $-\delta_{Z_1}$ follows from the hypothesis that Z is a complete intersection, as it was shown in [BGVY, p.73]. This depends on the fact that the meromorphic function of two complex variables

$$(\lambda_1, \lambda_2) \mapsto |f_1|^{2\lambda_1} \bigwedge_{j=2}^{p} \overline{\partial} \left(\frac{|f_j|^{2\lambda_2}}{f_j} \right) \wedge \bigwedge_{j=2}^{p} \partial f_j$$

is holomorphic near the origin in $\mathbf{C}^2$.

□

There is a second way to define the residue current that has been introduced in [BGVY, Proposition 5.21]. Let us recall that for any $(n, n-p)$ test form φ,

$$(19) \qquad < \overline{\partial} \frac{1}{f}, \varphi >= \frac{(-1)^{p(p-1)/2}(p-1)!}{(2i\pi)^p} \left(\lambda \int_{\mathbf{C}^n} ||f||^{2(\lambda-p)} \overline{\partial f} \wedge \varphi \right)_{\lambda=0}$$

where $||f||^2 = |f_1|^2 + \cdots + |f_p|^2$. The integration current can be recovered as follows

$$(20) \qquad < \delta_Z, \varphi >= \frac{(-1)^{p(p-1)/2}(p-1)!}{(2i\pi)^p} \left(\lambda \int_{\mathbf{C}^n} ||f||^{2(\lambda-p)} \overline{\partial f} \wedge \partial f \wedge \varphi \right)_{\lambda=0}$$

Lemma 2. *Let A be the differential form*

$$A = \sum_{k=1}^{p} (-1)^{k-1} f_k \partial f_1 \wedge \ldots \widehat{\partial f_k} \wedge \ldots \wedge \partial f_p \,.$$

The current-valued holomorphic map $\lambda \mapsto \Xi_\lambda$ defined for $\mathrm{Re}\,\lambda >> 0$ by

$$(21) \qquad \Xi_\lambda = \frac{(-1)^{p(p+1)/2}(p-1)!}{(2i\pi)^{p-1}} \left(\frac{||f||^{2(\lambda-p)} \overline{A} \wedge A}{\lambda} \right)$$

can be analytically continued as a meromorphic function in $\mathbf{C}$ with a simple pole at $\lambda = 0$. The coefficient of λ^0 in the Laurent development of this function at the origin is a $(p-1, p-1)$ current S, which satisfies the Green equation

$$(22) \qquad dd^c S + \delta_Z = 0 \,.$$

Proof. The possibility of analytic continuation of Ξ_λ to the whole complex plane as a meromorphic function with a simple pole at the origin appears in the proof of [BGVY, Theorem 3.25]. Let us proceed to show that S is a solution of the equation (22). An immediate computation shows that for $\mathrm{Re}\,\lambda >> 0$ one has

$$\overline{\partial}\left(||f||^{2(\lambda-p)}\overline{A}\wedge A\right) = \lambda||f||^{2(\lambda-p)}\overline{\partial f}\wedge A$$

and thus,

$$\partial\overline{\partial}\left(||f||^{2(\lambda-p)}\overline{A}\wedge A\right) = (-1)^p\lambda^2||f||^{2(\lambda-p)}\overline{\partial f}\wedge\partial f$$

Dividing the last expression by λ, one recognizes in the right hand side (up to a multiplicative constant) the current-valued function of λ that gives the integration current in (20). Here we use again the fact that analytic continuation commutes with dd^c.

□

Remark. It is easy to verify that S is C^∞ outside the support $|Z|$ of the cycle and has a logarithmic singularity in the sense of [GS1], [BGS] on $|Z|$. This is not the case for the current G. We only know that its singular support is contained in the union U of the supports of divisors of the f_j, and that it has a logarithmic singularity on U.

The main advantage of the construction of G is that it preserves the multiplicative properties of residue calculus. We will use this feature in the next section. One could also use multivariable zeta functions to factorize the integration current and thus to construct explicitly solutions of the Green equation. This idea appears in [BY2]. Namely, the action of the integration current on a test form is given by

$$(23)\qquad <\delta_Z,\varphi> = \frac{(-1)^{p(p-1)/2}(p-1)!}{(2i\pi)^p}\left(\left(\sum_{j=1}^p\lambda_j\right)\int_{\mathbf{C}^n}\prod_{j=1}^p|f_j|^{2\lambda_j}\frac{\overline{\partial f}\wedge\partial f}{||f||^{2p}}\wedge\varphi\right)_{\lambda=0}$$

The function of p complex variables λ_j is a meromorphic function in $\mathbf{C}^p$, whose polar set is contained in a finite union of hyperplanes not passing through the origin [BY1, Theorem 2]. We can transform the meromorphic function in (23) to a multiplicative expression by means of the Mellin transform [BY2, Lemma 2.2]. Namely, choose $p-1$ strictly positive numbers γ_j such that $|\gamma| := \sum\gamma_j < p-1$, then one can rewrite the right hand side of (23) as

$$(24)\qquad \left(C_p(\lambda)\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma_p^*(s)\left(\int|f_1|^{2(\lambda_1-p+|s|)}\prod_{j=2}^p|f_j|^{2(\lambda_j-s_j)}\overline{\partial f}\wedge\partial f\wedge\varphi\right)ds\right)_{\lambda=0}$$

where we have used the notation

$$C_p(\lambda) = \frac{(-1)^{p(p-1)/2}}{(2i\pi)^{2p-1}}\sum_{j=1}^p\lambda_j\,,$$

$$\Gamma_p^*(s) = \Gamma(s_1)\cdots\Gamma(s_{p-1})\Gamma(p-|s|)\,,$$

and, finally,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \cdots\, ds = \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \cdots \int_{\gamma_{p-1}-i\infty}^{\gamma_{p-1}+i\infty} \cdots\, ds_1 \cdots ds_{p-1}\,.$$

It is easy to obtain Green currents from (24). For example, we let f' represent the system $f_2, \ldots, f_p$, we set $C'_p(\lambda) = 2\pi i(-1)^p C_p(\lambda)$, and introduce the current-valued holomorphic function (for $\mathrm{Re}\lambda_j >> 0$),

$$C'_p(\lambda) \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma_p^*(s) \left(\int \frac{|f_1|^{2(\lambda_1-p+|s|+1)}}{(\lambda_1-p+|s|)(\lambda_1-p+|s|+1)} \prod_{j=2}^{p} |f_j|^{2(\lambda_j-s_j)} \overline{\partial f'} \wedge \partial f' \wedge \varphi \right) ds$$

This function can be analytically continued as a meromorphic function $\Upsilon(\lambda)$ to the whole space $\mathbf{C}^p$. In order to get a Green current, one fixes a generic $t \in (\mathbf{R}^+)^p$ and keeps the coefficient of μ^0 in the Laurent expansion of $\Upsilon(\mu t)$ about the origin as a meromorphic function of the single complex variable μ. If we choose another index j, we can proceed with $f' = (f_1, \ldots, \hat{f}_j, \ldots, f_n)$, and there is a sign change in C'_p.

In [BGVY, Theorem 3.18] there is a different representation of the integration current that will be used in the proof of Proposition 6. For $t \in (\mathbf{R}^+)^p$ we have

$$(25) \qquad < \delta_Z, \varphi >= t_1 \cdots t_p \frac{(-1)^{p(p-1)/2}}{(2i\pi)^p} \left(\lambda^p \int_{\mathbf{C}^n} \prod_{k=1}^{p} |f_k|^{2(t_k\lambda-1)}\, \overline{\partial f} \wedge \partial f \wedge \varphi \right)_{\lambda=0}$$

3. Construction of normalized Green currents. In this section we work on a n-dimensional complex manifold X. Consider a collection of effective divisors $\mathcal{D}_1, \ldots, \mathcal{D}_p, 1 \leq p \leq n$. The intersection product of these divisors defines an analytic cycle Z, equipped with its integration current δ_Z. Assume that the corresponding line bundles $[\mathcal{D}_j]$ have global holomorphic sections s_j, and let ρ_j be C^∞ metrics on these line bundles. Furthermore, let us assume that the divisors intersect properly, in fact, a bit more: given any local chart U_α, we assume that the s_j expressed in this chart as $s_{j,\alpha}$ define a regular sequence, independently of the order (i.e., they define a normal system in U_α.) Let $c(\rho_1), \ldots, c(\rho_p)$ be corresponding (first) Chern forms, ($c(\rho_j) = dd^c \log \rho_j$). In this section, we give a procedure to construct via analytic continuation methods a normalized Green current associated to the collection of divisors. That is, a solution G of the Green equation

$$(26) \qquad dd^c G + \delta_Z = c(\rho_1) \wedge \cdots \wedge c(\rho_p)\,.$$

In order to do this, we try to follow the earlier construction in Lemma 1. The problem is to take into account the correction terms corresponding to globalization of local formulas. It is here that the Chern forms appear. For this purpose we introduce the current-valued holomorphic function which is defined locally by

$$(27) \qquad \Gamma_\lambda = c_p \lambda^{p-2} \big(\|s_1\|_{\rho_1}^2 \cdots \|s_p\|_{\rho_p}^2\big)^\lambda \bigwedge_{j=2}^{p} \overline{\partial} \log \|s_j\|_{\rho_j}^2 \wedge \bigwedge_{j=2}^{p} \partial \log \|s_j\|_{\rho_j}^2\,,$$

where we have suppressed the index α corresponding to the local chart U_α and

$$c_p = \frac{(-1)^{p(p+1)/2}}{(2\pi i)^{p-1}} \,.$$

In fact, this makes sense since it is clear that the form Γ_λ is globally defined on X.

Our process of inductive construction of G relies on the following lemma. Let us denote by $Z_k, Z_{k,l}$ the cycles defined as

$$Z_k = \prod_{j\neq k} \mathcal{D}_j$$

$$Z_{k,l} = \prod_{j\neq k,l} \mathcal{D}_j \,.$$

Lemma 3. *The current-valued map $\lambda \mapsto \Gamma_\lambda$, defined by (27), can be analytically continued to* **C** *as a meromorphic current-valued map, with a simple pole at the origin. The Laurent development of this map about $\lambda = 0$ is*

$$-\frac{\delta_{Z_1}}{\lambda} + \Gamma_0 + \lambda H_\lambda$$

where $\lambda \mapsto H_\lambda$ is holomorphic about the origin, and Γ_0 is a $(p-1, p-1)$ current on X such that

$$dd^c\Gamma_0 + \delta_Z = \left(c(\rho_1) - \sum_{k=2}^{p} c(\rho_k) \right) \wedge \delta_{Z_1} + \sum_{k=2}^{p} c(\rho_k)^2 \wedge \delta_{Z_{1,k}} \,. \tag{28}$$

In the case when $p = 2$, the last formula has to be interpreted as

$$dd^c\Gamma_0 + \delta_Z = (c(\rho_1) - c(\rho_2)) \wedge \delta_{Z_1} + c(\rho_2)^2 \,. \tag{28$'$}$$

Proof. We start the proof by developping, for $\mathrm{Re}\lambda >> 0$, the big wedge products in the definition of Γ_λ into three types of terms. Namely,

$$\Gamma_\lambda = c_p(\rho_1 \cdots \rho_p)^{-\lambda}(R_\lambda + S_\lambda + T_\lambda) \,, \tag{29}$$

where

$$R_\lambda = \frac{1}{\lambda} \left(\lambda^{p-1} |s_1|^{2\lambda} |s_2 \cdots s_p|^{2(\lambda-1)} \bigwedge_{j=2}^{p} \overline{\partial s_j} \wedge \bigwedge_{j=2}^{p} \partial s_j \right) . \tag{30}$$

Similarly,

$$
\begin{aligned}
S_\lambda = \lambda^{p-2}|s_1...s_p|^{2\lambda}\Bigg(& (-1)^{(p-1)(p-2)/2}\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\frac{\partial\rho_k}{\rho_k}\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_p}{s_p} \\
& -\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\cdots\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\partial s_p}{s_p} \\
& -\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\partial\rho_k}{\rho_k}\wedge\cdots\wedge\frac{\partial s_p}{s_p}\Bigg),
\end{aligned} \tag{31}
$$

where it is understood that in each sum the ρ_k term replaces the corresponding s_k term.

The remaining term, i.e., T_λ, appears only when $p > 2$. In this case, it is a sum of terms of the form

$$\gamma_{k_1,k_2}(\lambda)\wedge\omega_{k_1,k_2}\,,$$

where $2\leq k_1<k_2\leq p$ and ω_{k_1,k_2} is a smooth form defined locally, and

$$\gamma_{k_1,k_2}(\lambda) := \lambda^{p-2}\,|s_1...s_p|^{2\lambda}\frac{\partial s_{k_1}}{s_{k_1}}\wedge\frac{\overline{\partial s_{k_2}}}{\overline{s_{k_2}}}\wedge\left(\bigwedge_{\substack{2\leq k\leq p\\ k\neq k_1,k_2}}\frac{\partial s_k}{s_k}\wedge\frac{\overline{\partial s_k}}{\overline{s_k}}\right). \tag{32}$$

The fact that Γ_λ has an analytic continuation as a meromorphic function is a consequence, as always, of Atiyah's theorem. The first thing we have to show is that the terms appearing in T_λ are holomorphic at the origin and vanish there. In order to do that, we need to study the function

$$\lambda\mapsto\int_{\mathbf{C}^n}\gamma_{k_1,k_2}(\lambda)\wedge\varphi\,, \tag{33}$$

where φ is a $(n-p+2,n-p+2)$ test form, since the ω_{k_1,k_2} can be incorporated into it. We start with a procedure that we introduced in [BGY, Theorem 1.3] and that it was further developped in the proofs of Proposition 3.6 and Theorem 3.18 of [BGVY]. Let us write

$$\varphi=\sum_\tau\xi_\tau\wedge\overline{\omega}_\tau\,,$$

where ξ_τ are $(n-p+2,0)$ smooth forms and ω_τ are $(0,n-p+2)$ forms with constant coefficients. We use a local resolution of singularities

$$\mathcal{X}\xrightarrow{\ \pi\ }U\subseteq X$$

for the hypersurface $s_1\cdots s_p=0$. In the local coordinates w in $\mathcal{X}$, one can write

$$\pi^*s_j(w)=u_j(w)w_1^{\alpha_{j,1}}\cdots w_n^{\alpha_{j,n}}=u_j(w)w^{*\alpha_j},\ j=1,\ldots,p.$$

The functions u_j do not vanish. Note the symbol $w^{*\alpha_j}$, which is defined in the last statement. The exponents $\alpha_{j,k}$ are all non negative integers. This is the notation from [BGVY]. In case the components of the base vector w are strictly positive, we can allow the exponents to be complex numbers (as we will do in the next paragraph).

The expression (33) is a linear combination of two kinds of terms. The first kind, and hardest to deal with, is the following. Denote $\alpha = \sum_{j=1}^{p} \alpha_j$ and $|w| = (|w_1|, \ldots, |w_n|)$, these terms are of the form

$$(34) \qquad \lambda^{p-2} \int |w|^{*2\lambda\alpha} \frac{\partial w_{i_0}}{w_{i_0}} \wedge \frac{\overline{\partial w_{j_0}}}{\overline{w_{j_0}}} \wedge \left(\bigwedge_{i \in I} \frac{\partial w_i}{w_i} \right) \wedge \left(\bigwedge_{j \in J} \frac{\overline{\partial w_j}}{\overline{w_j}} \right) \wedge \theta(w, \lambda) \overline{\pi^*(\omega_\tau)} \wedge \xi_\tau$$

where I, J are subsets of $\{1, \ldots, n\}$, of cardinality $p-3$, $i_0 \notin I$, and $j_0 \notin J$. Remark that the fact that such a term appears implies that $\alpha_{k_1,i_0} > 0$, $\alpha_{k_2,j_0} > 0$, and that for any $k \neq k_1, k_2$ there exists at least one $j \in J$, $i \in I$ with $\alpha_{k,i}\alpha_{k,j} > 0$. The function θ is C^∞ in all the variables, with compact support in w and entire as a function of λ. Moreover, if we write

$$\overline{\pi^*(\omega_\tau)} = \sum_{\substack{J' \subset \{1,\ldots,n\} \\ \# J' = n-p+2}} \overline{\omega}_{\tau,J'} d\overline{w}_{J'} \ , \ d\overline{w}_{J'} = \bigwedge_{j \in J'} d\overline{w}_j \ ,$$

the functions $\omega_{\tau,J'}$ are holomorphic in the local chart because the coefficients of ω_τ were holomorphic (in fact, constant). Moreover, we can replace in (34) $\pi^*(\omega_\tau)$ by $\omega_{\tau,K} dw_K$, where K is the complementary index set of $J \cup \{j_0\}$, since all the other coordinates already appeared elsewhere in (34). Let

$$\mathcal{P} = \{w \ : w_j = 0 \ \text{ for all}\, j \in J \cup \{j_0\}\} \, .$$

There are two possibilities. Either $\pi(\mathcal{P})$ is contained in $|Z_1| = \{s_2 = \ldots = s_p = 0\}$ or it is not. In the first case, since ω_τ is an $(n-p+2, 0)$ form, its restriction to the codimension $p-1$ analytic variety $|Z_1|$ is zero (here is the point where we use the complete intersection conditions), and this implies that $\omega_{\tau,K}$ vanishes on $\mathcal{P}$, i.e., there are holomorphic functions $y_j, j \notin K$, such that

$$\omega_{\tau,K} = \sum_{j \notin K} y_j w_j \ .$$

Therefore, in this case, the number of $\overline{w}_j$ that one has to eliminate from the denominator in (34), using integration by parts, does not exceed $p-3$. Each time we do an integration by parts, we use up a factor λ in (34). Thus, at the end of the process, there are no $\overline{w}_j$ in the denominator of (34), while at least one factor λ remains. Such a term has an analytic continuation of the form $\lambda h(\lambda)$, h holomorphic about the origin. In the other case, we already know from the remark following (34) that all $\pi^* s_k$, $2 \leq k \leq n$, $k \neq k_1$ vanish on $\mathcal{P}$, because they have at least one w_j, $j \in J \cup \{j_0\}$ as a factor. Since we are in the second case, it is impossible that $\pi^* s_{k_1}$ also vanishes on $\mathcal{P}$. This implies that $i_0 \notin J \cup \{j_0\}$. Hence, with exactly $p-3$ integrations by parts (each one using up one factor λ) we can get rid of the w_i, $i \in I$, in the denominators. Since there is no $\overline{w}_{i_0}$ in the denominator, the expression we are left with is holomorphic in λ and vanishes at $\lambda = 0$. The second kind of terms are those that contain in the denominator either at most $p-3$ factors w_j or at most $p-3$ factors $\overline{w}_j$. Since, in this case the number of integrations by parts, to get a holomorphic function of λ about the origin, does not exceed $p-3$, we still have a factor λ remaining, which is what we wanted to prove. Summarizing, we have completely proved that the current-valued map

$$\lambda \mapsto c_p(\rho_1 \cdots \rho_p)^{-\lambda} T_\lambda$$

can be analytically continued to a neighborhood of the origin as a holomorphic function vanishing at $\lambda = 0$.

Exactly the same argument shows that the function S_λ is holomorphic in a neighborhood of the origin. In fact, the same proof shows that one does not change its value S_0 at the origin, if one replaces in the definition (31) of S_λ the factor $|s_1 \cdots s_p|^{2\lambda}$ by $|s_2 \cdots s_p|^{2\lambda}$. (See, for instance, the proof of Proposition 5.21 in [BGVY].)

Now we consider the behaviour of R_λ. In order to apply Lemma 1, we remark that a simple computation shows that $c_p R_\lambda$ is exactly the same as Ψ_λ in that lemma, when we replace f_j by s_j. Thus, near $\lambda = 0$ and locally in X,

$$c_p R_\lambda = -\frac{\delta_{Z_1}}{\lambda} + G + \lambda \Phi_\lambda$$

where G is a locally defined $(p-1, p-1)$ current satisfying

$$dd^c G + \delta_Z = 0$$

and $\lambda \mapsto \Phi_\lambda$ is holomorphic near the origin.

Therefore, we can write the globally defined Γ_λ in a local chart as

$$\Gamma_\lambda = (1 - \lambda \log(\rho_1 \cdots \rho_p) + \lambda^2 u_\lambda) \left(-\frac{\delta_{Z_1}}{\lambda} + G + c_p S_0 + \lambda \Theta_\lambda \right) , \tag{35}$$

after we develop $(\rho_1 \cdots \rho_p)^{-\lambda}$ about $\lambda = 0$ and incorporate the previous considerations. The current-valued functions u_λ and Θ_λ are holomorphic in a neighborhood of the origin. G and S_0 are global currents. We can rewrite (35) as

$$\Gamma_\lambda = -\frac{\delta_{Z_1}}{\lambda} + G + c_p S_0 + \log(\rho_1 \cdots \rho_p)\delta_{Z_1} + \lambda H_\lambda ,$$

which is the statement of Lemma 3 with

$$\Gamma_0 = G + c_p S_0 + \log(\rho_1 \cdots \rho_p)\delta_{Z_1} .$$

We have

$$dd^c \Gamma_0 = -\delta_Z + c_p dd^c S_0 + \sum_{k=1}^{p} c(\rho_k) \wedge \delta_{Z_1} . \tag{36}$$

To conclude the proof, we need to compute dd^cS_0. Using once more the fact that dd^c commutes with the process of analytic continuation, and the earlier remark that to compute S_0 we could suppress the factor $|s_1|^{2\lambda}$ in (31), we need to compute the coefficient of λ^0 in the Laurent development about $\lambda = 0$ of $\lambda \mapsto dd^c\Upsilon_\lambda$, where

$$\Upsilon_\lambda = c_p\lambda^{p-2}|s_2...s_p|^{2\lambda}\Bigg((-1)^{\frac{(p-1)(p-2)}{2}}\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\frac{\partial s_2}{s_2}\wedge\cdot\cdot\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\frac{\partial\rho_k}{\rho_k}\wedge\cdot\cdot\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_p}{s_p}$$
$$-\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\cdots\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\partial s_p}{s_p}$$
$$-\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\partial\rho_k}{\rho_k}\wedge\cdots\wedge\frac{\partial s_p}{s_p}\Bigg),$$

which can be rewritten as

$$\Upsilon_\lambda = \Upsilon_\lambda^0 - \Upsilon_\lambda^1 - \Upsilon_\lambda^2\,.$$

The function Υ_λ^0 is given by

$$\Upsilon_\lambda^0 = -\frac{\lambda^{p-2}}{(2\pi i)^{p-1}}|s_2...s_p|^{2\lambda}\left(\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\frac{\partial s_2}{s_2}\wedge\cdot\cdot\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\frac{\partial\rho_k}{\rho_k}\wedge\cdot\cdot\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_p}{s_p}\right)$$

and its value at the origin can be computed using formula (16), namely,

$$U_0 := \Upsilon_{\lambda=0}^0 = -\frac{1}{2\pi i}\sum_{k=2}^{p}\delta_{Z_{1,k}}\wedge\frac{\overline{\partial}\rho_k}{\rho_k}\wedge\frac{\partial\rho_k}{\rho_k}\,.$$

Therefore,

$$dd^cU_0 = \sum_{k=2}^{p}c(\rho_k)^2\wedge\delta_{Z_{1,k}}\,. \tag{37}$$

We consider now the function Υ_λ^1. Its value at $\lambda = 0$ will be denoted later on as U_1 (similarly for the current U_2.)

$$\Upsilon_\lambda^1 = c_p\lambda^{p-2}|s_2...s_p|^{2\lambda}\left(\sum_{k=2}^{p}\frac{\overline{\partial s_2}}{\overline{s_2}}\wedge\cdots\wedge\overline{\partial}\log\rho_k\wedge\cdots\wedge\frac{\overline{\partial s_p}}{\overline{s_p}}\wedge\frac{\partial s_2}{s_2}\wedge\cdots\wedge\frac{\partial s_p}{s_p}\right) \tag{38}$$

We compute succesively $\partial\Upsilon_\lambda^1$ and $\overline{\partial}\partial\Upsilon_\lambda^1$, using the identities

$$\begin{aligned}\overline{\partial}|s_l|^{2\lambda} &= \lambda|s_l|^{2\lambda}\frac{\overline{\partial s_l}}{\overline{s_l}}\\ \partial|s_l|^{\lambda} &= \lambda|s_l|^{\lambda}\frac{\partial s_l}{s_l}\end{aligned} \tag{39}$$

We get first

$$\partial \Upsilon^1_\lambda = c_p \lambda^{p-2} |s_2 ... s_p|^{2\lambda} \left(\sum_{k=2}^{p} (-1)^k \partial\overline{\partial} \log \rho_k \wedge \bigwedge_{\substack{l=2 \\ l\neq k}}^{p} \frac{\overline{\partial s_l}}{\overline{s_l}} \wedge \bigwedge_{l=2}^{p} \frac{\partial s_l}{s_l} \right)$$

Then,

$$\overline{\partial}\partial \Upsilon^1_\lambda = c_p \lambda^{p-1} |s_2 \cdots s_p|^{2\lambda} \left(\sum_{k=2}^{p} \partial\overline{\partial} \log \rho_k \wedge \bigwedge_{l=2}^{p} \frac{\overline{\partial s_l}}{\overline{s_l}} \wedge \bigwedge_{l=2}^{p} \frac{\partial s_l}{s_l} \right),$$

that is,

$$dd^c \Upsilon^1_\lambda = -c_p \lambda^{p-1} |s_2 \cdots s_p|^{2\lambda} \left(\sum_{k=2}^{p} dd^c \log \rho_k \wedge \bigwedge_{l=2}^{p} \frac{\overline{\partial s_l}}{\overline{s_l}} \wedge \bigwedge_{l=2}^{p} \frac{\partial s_l}{s_l} \right) \tag{40}$$

One can now compute the value at $\lambda = 0$ of this last expression using (16), we get

$$\begin{aligned} dd^c U_1 = dd^c \Upsilon^1_{\lambda=0} &= -(-1)^{\frac{(p-1)(p-2)}{2}} c_p (2\pi i)^{p-1} \left(\sum_{k=2}^{p} dd^c \log \rho_k \right) \wedge \delta_{Z_1} \\ &= \left(\sum_{k=2}^{p} c(\rho_k) \right) \wedge \delta_{Z_1}. \end{aligned} \tag{41}$$

Exactly the same computations lead to

$$dd^c U_2 = dd^c \Upsilon^2_{\lambda=0} = \left(\sum_{k=2}^{p} c(\rho_k) \right) \wedge \delta_{Z_1}. \tag{42}$$

Altogether we have

$$\begin{aligned} c_p dd^c S_0 = dd^c \Upsilon_{\lambda=0} &= dd^c U_0 - dd^c U_1 - dd^c U_2 \\ &= \sum_{k=2}^{p} c(\rho_k)^2 \wedge \delta_{Z_{1,k}} - 2 \left(\sum_{k=2}^{p} c(\rho_k) \right) \wedge \delta_{Z_1}. \end{aligned} \tag{43}$$

We have now computed every term in (36) and collecting them together yields

$$dd^c \Gamma_0 + \delta_Z = \left(c(\rho_1) - \sum_{k=2}^{p} c(\rho_k) \right) \wedge \delta_{Z_1} + \sum_{k=2}^{p} c(\rho_k)^2 \wedge \delta_{Z_{1,k}},$$

which is exactly (28). This concludes the proof of Lemma 3.

□

Proposition 4. *Let $\mathcal{D}_k$, $1 \le k \le p$, effective divisors on X, defined by global sections $s_1, \dots, s_p$, and intersecting properly. Let $\rho_1, \dots, \rho_p$ be C^∞ hermitian metrics on the line bundles $[\mathcal{D}_k]$, $1 \le k \le p$. One can construct a $(p-1, p-1)$-current valued meromorphic map $\lambda \mapsto G_\lambda$ in the complex plane with a simple pole at $\lambda = 0$, of the form*

$$G_\lambda = \frac{1}{\lambda^{p-2}} \mathcal{Q}(\lambda, \|s_1\|_{\rho_1}^{2\lambda}, \dots, \|s_p\|_{\rho_p}^{2\lambda}, c(\rho_1), \dots, c(\rho_p)),$$

(where $\mathcal{Q}$ is a polynomial, $c(\rho_k)$ is the first Chern form of the line bundle $[\mathcal{D}_k]$ equipped with the hermitian metric ρ_k, the multiplication between Chern forms being the exterior product, and $\|s_k\|_{\rho_k}^2 := |s_k|^2/\rho_k$) such that the coefficient G_0 of λ^0 in the Laurent development of G_λ about the origin satisfies the Green equation

$$dd^c G_0 + \delta_Z = \bigwedge_{k=1}^{p} c(\rho_k)\,, \tag{44}$$

where Z is the intersection cycle of the divisors.

Proof. The proof is by induction. For $p = 1$ one chooses a C^∞ metric ρ_1 on the line bundle $[\mathcal{D}_1]$, and a global section s_1 of the line bundle, then let

$$G_\lambda = -\frac{1}{\lambda} \left(\frac{|s_1|^2}{\rho_1} \right)^\lambda .$$

As a consequence of Lemma 1, one can write the analytic continuation of G_λ about the origin as

$$-\frac{1}{\lambda} + G_0 + \lambda H_\lambda$$

and

$$dd^c G_0 + \delta_{\mathcal{D}_1} = dd^c \log \rho_1 = c(\rho_1).$$

This is the Lelong-Poincaré equation, see also [GH].

Assume that the conclusion of the Proposition holds for collections of q divisors, $q < p$. Therefore, one can find a $(p-2, p-2)$-current valued map $\tilde{G}_\lambda$ with a simple pole at the origin and such that the corresponding coefficient $\tilde{G}_0$ satisfies

$$dd^c \tilde{G}_0 + \delta_{Z_1} = \bigwedge_{j=2}^{p} c(\rho_j) \tag{45}$$

where $Z_1 = \mathcal{D}_2 \cdots \mathcal{D}_p$. Similarly, when $p \ge 3$, one can find for any $2 \le k \le p$, a $(p-3, p-3)$-current valued map G_λ^k with a simple pole at the origin and such that the corresponding coefficient G_0^k satisfies

$$dd^c G_0^k + \delta_{Z_{1,k}} = \bigwedge_{\substack{j=2 \\ j \ne k}}^{p} c(\rho_j)\,, \tag{46}$$

where

$$Z_{1,k} = \prod_{\substack{j=2 \\ j\neq k}}^{p} \mathcal{D}_j \, .$$

We consider the current-valued map defined in Lemma 3, namely

$$\Gamma_\lambda = \frac{(-1)^{p(p+1)/2}}{(2\pi i)^{p-1}} \lambda^{p-2} \left(\frac{|s_1 \cdots s_p|^2}{\rho_1 \cdots \rho_p} \right)^{\lambda} \bigwedge_{j=2}^{p} \bar{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j}$$

and consider the $(p-1, p-1)$-current valued map

$$G_\lambda = \Gamma_\lambda + \left(c(\rho_1) - \sum_{k=2}^{p} c(\rho_k) \right) \wedge \tilde{G}_\lambda + \sum_{k=2}^{p} c(\rho_k)^2 \wedge G_\lambda^k \, . \tag{47}$$

It is clear that G_λ has a simple pole at the origin and, from the fact that all the Chern forms are d and d^c closed, we have

$$dd^c G_0 = dd^c \Gamma_0 + \left(c(\rho_1) - \sum_{k=2}^{p} c(\rho_k) \right) \wedge dd^c \tilde{G}_0 + \sum_{k=2}^{p} c(\rho_k)^2 \wedge dd^c G_0^k \, .$$

Applying Lemma 3, (45), and (46), we conclude that G_0 satisfies the Green equation (44).

□

Remark. The current G_0 that we have just defined is C^∞ outside the union of the supports of the divisors.

At least under additional hypotheses, one can adapt the previous construction to obtain a positive current G_0. In the following lemma we use the same notation as in the proof of Lemma 3.

Lemma 5. *Assume the conditions of Proposition 4 hold and let $K \subseteq X$ be compact. There exists a positive constant $C = C(K)$ such that the current Γ_0^C defined as the coefficient of λ^0 in the Laurent development about the origin of*

$$\Gamma_\lambda^C = c_p \lambda^{p-2} \left(C \frac{|s_1 \cdots s_p|^2}{\rho_1 \cdots \rho_p} \right)^{\lambda} \bigwedge_{j=2}^{p} \bar{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j}$$

is a positive current on K.

Proof. We choose $C', C'' > 0$ such that on the compact set K we have

$$\frac{C'|s_1|^2}{\rho_1} < 1 \text{ and } \frac{C''|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} < 1 \, .$$

We let $C = C'C''$ and introduce the meromorphic current-valued map

$$\Phi_\lambda = c_p \lambda^{p-2} \left(C'' \frac{|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} \right)^\lambda \bigwedge_{j=2}^{p} \overline{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j} \tag{48}$$

Consider now the difference
(49)

$$\Gamma_\lambda^C - \Phi_\lambda = c_p \lambda^{p-1} \left(C'' \frac{|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} \right)^\lambda \left(\frac{\left(\frac{C'|s_1|^2}{\rho_1} \right)^\lambda - 1}{\lambda} \right) \bigwedge_{j=2}^{p} \overline{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j}$$

From (16) we infer that

$$\Phi_\lambda = -\frac{\delta_{Z_1}}{\lambda} + \Phi_0 + O(\lambda)\,, \tag{50}$$

and, hence, the function in (49) is holomorphic at $\lambda = 0$. Moreover, for $\lambda > 0$, the differential form

$$c_p \lambda^{p-1} \left(C'' \frac{|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} \right)^\lambda \log \left(\frac{C'|s_1|^2}{\rho_1} \right) \bigwedge_{j=2}^{p} \overline{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j} \tag{51}$$

is integrable and positive. The fact that it is integrable can be seen using resolution of singularities as it was done in [BGVY] to prove (16), only logarithmic derivatives of the new local coordinates w_j and of $\overline{w}_j$ times a logarithmic term appear as singularities. The positivity is a consequence of the fact that the logarithm in (51) is negative due to the choice of C' and the remaining differential form is negative due to the form of the expression and the value

$$c_p = \frac{(-1)^{p(p+1)/2}}{(2\pi i)^{p-1}}.$$

We conclude the value at $\lambda = 0$ of (49) is a positive current on K, in other words

$$\Gamma_0^C - \Phi_0 \geq 0$$

on K. To conclude the proof of the lemma it is sufficient to show that $\Phi_0 \geq 0$. For that purpose, consider for $\lambda > 0$ the differential form

$$\lambda \Phi_\lambda = c_p \lambda^{p-1} \left(C'' \frac{|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} \right)^\lambda \bigwedge_{j=2}^{p} \overline{\partial} \log \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=2}^{p} \partial \log \frac{|s_j|^2}{\rho_j}$$

which, even when multiplied by

$$\left| \log \left(C'' \frac{|s_2 \cdots s_p|^2}{\rho_2 \cdots \rho_p} \right) \right| ,$$

is integrable by the same reasons given about (51). It would now suffice to show that for any positive test form φ with support in K and any $\lambda_0 > 0$ the derivative of the map

$$\lambda \mapsto \lambda \int \Phi_\lambda \wedge \varphi$$

evaluated at λ_0 is non-negative. This derivative can be computed using Lebesgue's theorem on differentiation of integrals with respect to parameters, due to the integrability of the formal derivative, which was discussed above. The positivity is a consequence of the choice of C'' since the logarithm term in the derivative is negative and the differential form that remains (after removing the logarithm) is also negative. The same argument was used earlier. Thus, $\Phi_0 \geq 0$ on K and so $\Gamma_0^C \geq 0$ also.

□

Proposition 6. *Let X be a compact Kählerian manifold with a Kähler form ω. Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ be global effective divisors on X, which intersect properly. Let $\rho_1, \ldots, \rho_p$ be C^∞ hermitian metrics on the line bundles $[\mathcal{D}_k]$, $1 \leq k \leq p$, such that these line bundles, equipped with such metrics, are positive. There is a $(p-1, p-1)$-current valued meromorphic map G_λ, with a simple pole at the origin, and such that the coefficient G_0 of λ^0 in its Laurent development about $\lambda = 0$ is a positive current, smooth ouside the union of the supports $|\mathcal{D}_j|$, which is a solution of the equation*

$$dd^c G_0 + \delta_Z = \bigwedge_{k=1}^{p} c(\rho_k)$$

where Z is the intersection cycle and $c(\rho_k)$ the first Chern form of the hermitian line bundle $([\mathcal{D}_k], \rho_k)$.

Proof. If $m_1, \ldots, m_p$ are positive integers and $s_1, \ldots, s_p$ global sections of the divisors $\mathcal{D}_j$, then $s_1^{m_1}, \ldots, s_p^{m_p}$ are global sections of the divisors $m_j \mathcal{D}_j$. Let Z^m be the corresponding intersection cycle. Using these sections to compute locally the integration current via formula (25), we see that

$$\delta_{Z^m} = m_1 \cdots m_p \delta_Z \tag{52}$$

Furthermore, $\rho_k^{m_k}$ is a C^∞ hermitian metric on the line bundle $[m_k \mathcal{D}_k]$. The first Chern form of this hermitian line bundle is $c(\rho_k^{m_k}) = m_k c(\rho_k)$. Since all hermitian bundles $([\mathcal{D}_k], \rho_k)$ are positive, we can choose now the m_j so that for any j, $1 \leq j \leq p-1$, one has

$$c(\rho_j^{m_j}) = m_j c(\rho_j) \geq \sum_{k=j+1}^{p} m_k c(\rho_k) = \sum_{k=j+1}^{p} c(\rho_k^{m_k}). \tag{53}$$

We will first construct a current-valued map $\tilde{G}_\lambda$ such that

$$dd^c \tilde{G}_0 + \delta_{Z^m} = \bigwedge_{k=1}^{p} c(\rho_k^{m_k}),$$

$\tilde{G}_0 \geq 0$, and $\tilde{G}_0$ is smooth outside $\bigcup |\mathcal{D}_k|$. Once this is done, we will take

$$G_\lambda = \frac{1}{m_1 \cdots m_p} \tilde{G}_\lambda$$

This will work because of the identity (52). The construction of $\tilde{G}_\lambda$ is done by an iterative procedure that is an adaptation of the one used in the proof of Proposition 4. Let us start with the distribution valued map

$$\lambda \mapsto -\frac{1}{\lambda} \left(\frac{C_p |s_p^{m_p}|^2}{\rho_p^{m_p}} \right)^\lambda ,$$

where C_p is a strictly positive constant such that

$$\frac{C_p |s_p|^2}{\rho_p} < 1$$

on the compact manifold X. Let $1 \leq q \leq p-1$. Let $Z^{m,q}$ be the cycle

$$Z^{m,q} := \prod_{l=q}^{p} m_l \mathcal{D}_l \, .$$

Assume that we have already constructed a current-valued map $\tilde{G}_\lambda^{(q)}$, and, when $q < p-1$, also current-valued maps $\tilde{G}_\lambda^{(q,k)}$, $q+1 \leq k \leq p$, all with simple poles at the origin, such that the currents $\tilde{G}_0^{(q)}$ and $\tilde{G}_0^{(q,k)}$ are positive currents on X, smooth outside $\bigcup_{k=q}^{p} |\mathcal{D}_k|$, satisfying the Green equations

$$dd^c \tilde{G}_0^{(q)} + \delta_{\tilde{Z}_q^m} = \bigwedge_{l=q+1}^{p} c(\rho_l^{m_l}) \tag{54}$$

$$dd^c \tilde{G}_0^{(q,k)} + \delta_{\tilde{Z}_{q,k}^m} = \bigwedge_{\substack{l=q+1 \\ l \neq k}}^{p} c(\rho_l^{m_l}) \, , \tag{55}$$

where

$$\tilde{Z}_q^m \, : = \prod_{l=q+1}^{p} m_l \mathcal{D}_l$$

$$\tilde{Z}_{q,k}^m \, : = \prod_{\substack{l=q+1 \\ l \neq k}}^{p} m_l \mathcal{D}_l \, .$$

We know from Lemma 5 that, for some convenient constant $C = C_q$, the current-valued map

$$\Gamma_\lambda^{C,q} = c_{p,q} \lambda^{p-q-1} \left(C \frac{|s_q^{m_q} \cdots s_p^{m_p}|^2}{\rho_q^{m_q} \cdots \rho_p^{m_p}} \right)^\lambda \bigwedge_{j=q+1}^{p} \overline{\partial} \log \frac{|s_j^{m_j}|^2}{\rho_j^{m_j}} \wedge \bigwedge_{j=q+1}^{p} \partial \log \frac{|s_j^{m_j}|^2}{\rho_j^{m_j}} \, ,$$

$$c_{p,q} = \frac{(-1)^{\frac{(p-q+1)(p-q+2)}{2}}}{(2\pi i)^{p-q}}\,,$$

has a simple pole at the origin and is such that $\Gamma_0^{C,q}$ is a positive current, smooth outside $\bigcup_{k=q}^{p} |\mathcal{D}_k|$. Furthermore, since C^λ does not contribute to the dd^c, we have, as a consequence of Lemma 3,

$$dd^c\Gamma_0^{C,q} + \delta_{Z^{m,q}} = \left(c(\rho_q^{m_q}) - \sum_{k=q+1}^{p} c(\rho_k^{m_k}) \right) \wedge \delta_{\tilde{Z}_q^m} + \sum_{k=q+1}^{p} c(\rho_k^{m_k})^2 \wedge \delta_{\tilde{Z}_{q,k}^m}\,.$$

Thanks to the identities (54) and (55), we see, as in the proof of proposition 4, that the map

$$\tilde{G}_\lambda^q = \Gamma_\lambda^{C,q} + \left(c(\rho_q^{m_q}) - \sum_{k=q+1}^{p} c(\rho_k^{m_k}) \right) \wedge \tilde{G}_\lambda^{(q)} + \sum_{k=q+1}^{p} c(\rho_k^{m_k})^2 \wedge \tilde{G}_\lambda^{(q,k)}$$

has a simple pole at the origin. Moreover, $dd^c\tilde{G}_0^q$ is a positive current, smooth outside $\bigcup_{k=q}^{p} |\mathcal{D}_k|$, and solution of

$$dd^c\tilde{G}_0^q + \delta_{Z^{m,q}} = \bigwedge_{k=q}^{p} c(\rho_k^{m_k})\,. \tag{56}$$

We continue this process until we get to $q = 1$. At this stage, the map $\lambda \mapsto \tilde{G}_\lambda^1$ is a meromorphic current-valued map with a simple pole at the origin, such that $dd^c\tilde{G}_0^1$ is a positive current, smooth outside $\bigcup |\mathcal{D}_k|$, and solution of (56) with $q = 1$. This is the map $\tilde{G}_\lambda$ we need, and the proposition is proved.

□

As an example of this proposition, let $X = \mathbf{P}^n(\mathbf{C})$ and $Q_1, \ldots, Q_p$ be p homogeneous polynomials in $n+1$ variables, we consider the metrics in homogeneous coordinates

$$\rho_j(x) = ||x||^{2D_j} \quad \text{with} \quad D_j = deg(Q_j)\,.$$

These are clearly C^∞ metrics on the line bundles $[\mathcal{D}_j]$ associated to the divisors $\operatorname{div} Q_j$. We have

$$c(\rho_j) = D_j dd^c \log ||x||^2 = D_j dd^c\omega\,.$$

Therefore, the current-valued map G_λ constructed in the Proposition 6 satisfies

$$dd^c G_0 + \delta_Z = D_1 \cdots D_p \omega^p = H(\delta_Z)$$

where, as before, H represents the harmonic projection, and $D = D_1 \cdots D_p$ is the degree of the cycle Z (Bézout's theorem).

4. About a formula of H. Levine. In [Le] H. Levine introduced an explicit formula which solves the Green equation in $\mathbf{P}^n(\mathbf{C})$ for the cycle $\Pi = \{x_0 = \ldots = x_{p-1} = 0\}$. Let,

$$\alpha(x) := dd^c \log \left(\sum_{j=0}^{p-1} |x_j|^2 \right)$$

then the globally defined current

$$L(x) = -\log \left(\frac{\sum_{j=0}^{p-1} |x_j|^2}{||x||^2} \right) \left(\sum_{k=0}^{p-1} \alpha(x)^k \wedge \omega(x)^{p-1-k} \right)$$

is integrable and it is a solution of the equation

$$dd^c L + \delta_\Pi = \omega^p$$

as we have already pointed out in the Introduction. It is immediate to see that the current-valued map

$$L_\lambda = -\frac{1}{\lambda} \left(\frac{\sum_{j=0}^{p-1} |x_j|^2}{||x||^2} \right)^\lambda \left(\sum_{k=0}^{p-1} \alpha(x)^k \wedge \omega(x)^{p-1-k} \right)$$

has a simple pole at the origin and the coefficient of λ^0 in its Laurent development about the origin is exactly L.

The same construction works if Π is replaced by a cycle $Z = \{Q_1 = \ldots = Q_p = 0\}$ such that $dQ_1 \wedge \cdots \wedge dQ_p \neq 0$ on $|Z|$ and the polynomials Q_j have the same degree D. Namely,the Green current is the integrable current

$$\Gamma = -\log \left(\frac{\sum_{j=1}^{p} |Q_j|^2}{||x||^{2D}} \right) \left(\sum_{k=0}^{p-1} \left(dd^c \log \left(\sum_{j=1}^{p} |Q_j|^2 \right) \right)^k \wedge (D\omega)^{p-1-k} \right)$$

which can be obtained from the Laurent development about $\lambda = 0$ of the current-valued map

$$\Gamma_\lambda = -\frac{1}{\lambda} \left(\frac{\sum_{j=1}^{p} |Q_j|^2}{||x||^{2D}} \right)^\lambda \left(\sum_{k=0}^{p-1} \left(dd^c \log \left(\sum_{j=1}^{p} |Q_j|^2 \right) \right)^k \wedge (D\omega)^{p-1-k} \right) . \tag{57}$$

In this case, Γ satisfies the equation

$$dd^c \Gamma + \delta_Z = D^p \omega^p . \tag{58}$$

We will see later that, even though Γ_λ can be defined as a meromorphic map with a simple pole at the origin, when Z has singularities, it is not clear that the coefficient of λ^0 in the Laurent development of (57) about $\lambda = 0$ satisfies the Green equation (58). Nevertheless, in the case of $\mathbf{P}^n(\mathbf{C})$, one can overcome this difficulty and construct by analytic continuation methods, a current that is smooth outside $|Z|$ and satisfies the Green equation (5), when Z is defined as a complete intersection by homogeneous polynomials of respective degrees $D_1, \ldots, D_p$. We need first the following lemma.

Lemma 7. *Let $Q_1, \ldots, Q_p$ be homogeneous polynomials of the same degree D in $n+1$ variables defining a complete intersection cycle Z in $\mathbf{P}^n(\mathbf{C})$. The (p,p)-current valued map I_λ globally defined in homogeneous coordinates by*

$$I_\lambda = \frac{i}{2\pi}\lambda \left(\frac{\sum_{j=1}^p |Q_j|^2}{||x||^{2D}}\right)^\lambda \partial \log \left(\frac{\sum_{j=1}^p |Q_j|^2}{||x||^{2D}}\right) \wedge \overline{\partial} \log \left(\frac{\sum_{j=1}^p |Q_j|^2}{||x||^{2D}}\right) \wedge \wedge \left(dd^c \log \left(\sum_{j=1}^p |Q_j|^2\right)\right)^{p-1} \tag{59}$$

is holomorphic in the half-plane $\{\mathrm{Re}\lambda > -\epsilon\}$, $(\epsilon > 0)$, and its value at $\lambda = 0$ is δ_Z.

Proof. Outside $|Z|$ we can compute

$$\partial \log \left(\sum_{j=1}^p |Q_j|^2\right) = \sum_{j=1}^p \frac{\overline{Q_j}}{||Q||^2}\partial Q_j = \sum_{j=1}^p \psi_j \partial Q_j \,,$$

$$\overline{\partial} \log ||Q||^2 = \sum_{j=1}^p \overline{\psi_j}\, \overline{\partial} Q_j \,,$$

$$\overline{\partial}\partial \log ||Q||^2 = \sum_{j=1}^p \overline{\partial}\psi_j \wedge \partial Q_j \,,$$

with the obvious meaning for ψ_j and $||Q||^2$. Thus we have, with the notation used in (15)-(16) and performing the same computations as in [BGVY, p. 83],

$$\overline{\partial} \log ||Q||^2 \wedge \partial \log ||Q||^2 \wedge (\overline{\partial}\partial \log ||Q||^2)^{p-1} = (-1)^{\frac{p(p-1)}{2}} (p-1)!\, ||Q||^{-2p}\, \overline{\partial Q} \wedge \partial Q \,.$$

Note that I_λ can be written as

$$I_\lambda = \frac{\lambda}{(2\pi i)^p} \left(\frac{||Q||^2}{||x||^{2D}}\right)^\lambda (\overline{\partial} \log \frac{||Q||^2}{||x||^{2D}}) \wedge (\partial \log \frac{||Q||^2}{||x||^{2D}}) \wedge (\overline{\partial}\partial \log ||Q||^2)^{p-1}$$

Hence, we can rewrite

$$\begin{aligned} I_\lambda =& \frac{\lambda (-1)^{\frac{p(p-1)}{2}} (p-1)!}{(2\pi i)^p} ||Q||^{2(\lambda - p)} ||x||^{-2\lambda D} \overline{\partial Q} \wedge \partial Q \\ &- \frac{D\lambda}{(2\pi i)^p} \left(\frac{||Q||^2}{||x||^{2D}}\right)^\lambda \overline{\partial} \log ||Q||^2 \wedge \partial \log ||x||^2 \wedge (\overline{\partial}\partial \log ||Q||^2)^{p-1} \\ &+ \frac{D\lambda}{(2\pi i)^p} \left(\frac{||Q||^2}{||x||^{2D}}\right)^\lambda \partial \log ||Q||^2 \wedge \overline{\partial} \log ||x||^2 \wedge (\overline{\partial}\partial \log ||Q||^2)^{p-1} \\ &+ \frac{D^2\lambda}{(2\pi i)^p} \left(\frac{||Q||^2}{||x||^{2D}}\right)^\lambda \overline{\partial} \log ||x||^2 \wedge \partial \log ||x||^2 \wedge (\overline{\partial}\partial \log ||Q||^2)^{p-1} \,. \end{aligned} \tag{60}$$

Every term in (60) is defined locally, but the sum defines a global current-valued map. From (20) we conclude that the first term in (60) has an analytic continuation beyond the origin as a holomorphic function and its value at $\lambda = 0$ is the integration current δ_Z. The remaining terms are combinations of expressions of the form, either

$$\lambda \left(\frac{||Q||^2}{||x||^{2D}} \right)^{\lambda} \frac{\overline{\partial Q} \wedge Q_k \, \theta_1}{||Q||^{2p}} \tag{61}$$

or

$$\lambda \left(\frac{||Q||^2}{||x||^{2D}} \right)^{\lambda} \frac{\partial Q \wedge \overline{Q}_k \, \theta_2}{||Q||^{2p}} \,, \tag{61'}$$

for some smooth forms θ_j. Using (19), the last two expressions define holomorphic functions near the origin and their value at the origin is zero. This is due to the fact that, from (19), the residue current appears in the value at $\lambda = 0$ of (61), the residue current is annihilated by the ideal generated by the Q_j in the space of differential forms [BGVY, Theorem 3.18]. The same reasoning, this time applied to the conjugate of the residue current (and the $\overline{Q_j}$), leads to the vanishing of (61′) at the origin. Therefore, in a half-plane $\mathrm{Re}\lambda > -\epsilon$ ($\epsilon > 0$),

$$I_\lambda = \delta_Z + \lambda J_\lambda \,.$$

This concludes the proof of the lemma.

□

In the above lemma, we used extensively the fact that all polynomials defining the cycle had the same degree. In fact, we have a more general result, valid on any analytic manifold X. Since we will not use this result later, we will just sketch its proof (which is similar to the proof of Proposition 5.21 in [BGVY]).

Proposition 8. *Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$, p effective divisors on an n-dimensional analytic manifold. Suppose that these divisors are defined by global sections $s_1, \ldots, s_p$ and that they intersect properly on X along the cycle Z. Let $\rho_1, \ldots, \rho_p$ be C^∞ metrics on the line bundles $[\mathcal{D}_1], \ldots, [\mathcal{D}_p]$. Then the globally defined (p,p) current-valued map*

$$J_\lambda := \frac{(-1)^{p(p-1)/2}(p-1)!\lambda}{(2\pi i)^p} \left(\sum_{j=1}^{p} \frac{|s_j|^2}{\rho_j} \right)^{\lambda - p} \bigwedge_{j=1}^{p} \overline{\partial} \frac{|s_j|^2}{\rho_j} \wedge \bigwedge_{j=1}^{p} \partial \log \frac{|s_j|^2}{\rho_j} \tag{62}$$

is holomorphic in half-plane $\mathrm{Re}\lambda > -\epsilon$ containing the origin and its value at that point is δ_Z.

Proof. The result is a local, therefore it is enough to prove it when X is an open subset of $\mathbf{C}^n$. As in the proof [BGVY, Proposition 5.21], we proceed by induction on the codimension $n - p$. Let us do it first for $p = n$, we can assume $|Z| = 0$. Let φ be a test function, holomorphic in the closed ball $\overline{B}(0,r)$. A variation of the usual proof of the Bochner-Martinelli formula shows that for any smooth map σ from a neighborhood U of $|z|=r$ into $\mathbf{C}^n$ such that

$$\sum_{j=1}^{n} s_j\sigma_j \neq 0 \quad \text{in}\, U\,,$$

then, the local residue of $\varphi(z)\,dz$ at $z=0$ equals

$$< \overline{\partial}\frac{1}{s}, \varphi\, dz >_0 = \frac{(-1)^{n(n-1)/2}(n-1)!}{(2\pi i)^n}\int_{|z|=r} \frac{\sum_{k=1}^{n}(-1)^{k-1}\sigma_k \bigwedge_{j\neq k}\overline{\partial}\sigma_j\wedge\varphi\, dz}{\left(\sum_{j=1}^{n} s_j\sigma_j\right)^n}\,.$$

In particular, we can let

$$\sigma_k = \frac{\overline{s_k}}{\rho_k}$$

and, setting

$$||s||_\rho^2 = \sum_{k=1}^{n}\frac{|s_k|^2}{\rho_k}\,,$$

we have

$$(63)\qquad < \overline{\partial}\frac{1}{s}, \varphi\, dz >_0 = \frac{(-1)^{n(n-1)/2}(n-1)!}{(2\pi i)^n}\int_{|z|=r} \frac{\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{s_k}}{\rho_k}\bigwedge_{j\neq k}\overline{\partial}\,\frac{\overline{s_j}}{\rho_j}\wedge\varphi\, dz}{||s||_\rho^{2n}}.$$

This expression can also be understood as the value at $\lambda=0$ of the entire function

$$(64)\qquad \vartheta(\lambda) = \frac{(-1)^{n(n-1)/2}(n-1)!}{(2\pi i)^n}\int_{|z|=r} ||s||_\rho^{2(\lambda-n)}\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{s_k}}{\rho_k}\bigwedge_{j\neq k}\overline{\partial}\,\frac{\overline{s_j}}{\rho_j}\wedge\varphi\, dz.$$

Using the Stokes theorem we have

$$(65)\qquad < \overline{\partial}\frac{1}{s}, \varphi\, dz >_0 = \left(\frac{(-1)^{n(n-1)/2}(n-1)!\,\lambda}{(2\pi i)^n}\int_{\mathbf{C}^n} ||s||_\rho^{2(\lambda-n)}\bigwedge_{k=1}^{n}\overline{\partial}\,\frac{\overline{s_k}}{\rho_k}\wedge\varphi\, dz\right)_{\lambda=0}$$

The function of λ on the right hand side of the last formula can be shown to be entire by using its previous representation (64) and the fact that the integral over the set $|z|>r$ is clearly an entire function of λ. Therefore, from (13) we conclude that the integration current δ_Z acting on the test function φ is just the value at $\lambda=0$ of the entire function

$$\lambda \mapsto \frac{(-1)^{n(n-1)/2}(n-1)!\,\lambda}{(2\pi i)^n}\int_{\mathbf{C}^n} ||s||_\rho^{2(\lambda-n)}\bigwedge_{k=1}^{n}\overline{\partial}\,\frac{\overline{s_k}}{\rho_k}\wedge\varphi\, ds\,.$$

We now remark that

$$\bigwedge_{k=1}^{n}\overline{\partial}\,\frac{\overline{s_k}}{\rho_k}\wedge ds = \bigwedge_{k=1}^{n}\overline{\partial}\,\frac{|s_k|^2}{\rho_k}\wedge\bigwedge_{k=1}^{n}\frac{ds_k}{s_k}\,,$$

that we rewrite

$$\bigwedge_{k=1}^{n} \overline{\partial}\,\frac{\overline{s_k}}{\rho_k} \wedge ds = \bigwedge_{k=1}^{n} \overline{\partial}\,\frac{|s_k|^2}{\rho_k} \wedge \bigwedge_{k=1}^{n} \overline{\partial}\log\frac{|s_k|^2}{\rho_k} + \Omega\,.$$

It is immediate to remark that the distribution-valued map

$$||s||_{\rho}^{2(\lambda-n)}\Omega$$

can be continued as a holomorphic function in a neighborhood of the origin. Furthermore, one can see from (65) that its value at the origin is a linear combination of terms of the form

$$< \overline{\partial}\frac{1}{s}, s_k\theta >$$

or their conjugates, where θ is a smooth form. These terms vanish because of the properties of the residue current mentioned in the first section. Hence, we have

(66)

$$< \delta_Z, \varphi >= \left(\frac{(-1)^{n(n-1)/2}(n-1)!\,\lambda}{(2\pi i)^n} \int \|s\|_{\rho}^{2(\lambda-n)}\varphi \bigwedge_{k=1}^{n} \overline{\partial}\,\frac{|s_k|^2}{\rho_k} \wedge \bigwedge_{k=1}^{n} \overline{\partial}\log\frac{|s_k|^2}{\rho_k} \right)_{\lambda=0}$$

It is not hard to check that the distribution-valued map

$$\frac{(-1)^{n(n-1)/2}(n-1)!\,\lambda}{(2\pi i)^n} \|s\|_{\rho}^{2(\lambda-n)} \bigwedge_{k=1}^{n} \overline{\partial}\,\frac{|s_k|^2}{\rho_k} \wedge \bigwedge_{k=1}^{n} \overline{\partial}\log\frac{|s_k|^2}{\rho_k}$$

can be analytically continued as a holomorphic map in $\mathrm{Re}\lambda > -\epsilon$ and whose value at the origin is annihilated (as a distribution) by the functions $\overline{z}_k$. This can be seen using resolution of singularities, exactly as in the proof of Theorem 3.25 in [BGVY]. Therefore, the proposition holds for $p = n$ (since any test function can be written near the origin as the sum of a holomorphic function and some element in the ideal generated by the $\overline{z}_k$.)

In order to complete the proof of the proposition for arbitrary p, we need first to prove by induction on $n-p$ that the current-valued map

$$\eta_\lambda = \frac{(-1)^{p(p-1)/2}(p-1)!\,\lambda}{(2\pi i)^p} \int_{\mathbf{C}^n} ||s||_{\rho}^{2(\lambda-p)} \bigwedge_{k=1}^{p} \overline{\partial}\,\frac{\overline{s_k}}{\rho_k}$$

is holomorphic in a half-plane $\mathrm{Re}\lambda > -\epsilon$ containing the origin, its value at zero being the residue current $\overline{\partial}(1/s)$. The proof of this fact is exactly that of Proposition 5.21 in [BGVY]. We refer the reader to it. Once this is done, one can show, exactly as in the case of $p = n$, that the value at the origin of $\eta_\lambda \wedge ds$ (that is, δ_Z) does not change if one replaces

$$\bigwedge_{k=1}^{p} \overline{\partial}\,\frac{\overline{s_k}}{\rho_k} \wedge ds$$

by

$$\bigwedge_{k=1}^{p} \overline{\partial}\,\frac{|s_k|^2}{\rho_k} \wedge \bigwedge_{k=1}^{p} \overline{\partial}\log\frac{|s_k|^2}{\rho_k}\,.$$

This follows from the fact that the residue current just obtained as $\eta_{|\lambda=0}$ is annihilated by the ideal generated by the s_k, the same is true for the conjugate current, with respect to the ideal generated by the $\overline{s_k}$. Since the new expression $\tilde{\eta}_\lambda \wedge ds$ thus obtained is exactly the J_λ of the statement, the proposition follows.

□

Consider a codimension p cycle Z in $\mathbf{P}^n(\mathbf{C})$, which is defined by homogeneous polynomials Q_j of degrees $D_1, \ldots, D_p$. We now proceed to construct, by the analytic continuation method, a normalized Green current, smooth outside the support of the cycle. First, we remark that we can assume that all the degrees are equal, otherwise, let $D = l_1 D_1 = \cdots = l_p D_p$ be the least common multiple of the D_j, $\ell = l_1 \cdots l_p$, and consider the analytic cycle Z' defined by the $Q_j^{l_j}$. We have already seen in (52) that $\delta_{Z'} = \ell \delta_Z$. Suppose that G' is a normalized Green current (for the cycle Z') obtained by means of analytic continuation, smooth outside $|Z|$. Then the current

$$G = \frac{1}{\ell} G'$$

is a normalized current with the required properties for the cycle Z. We will assume from now on that all Q_j have he same degree D. As we mentioned previously, the current we constructed inspired by Levine's idea does not solve our problem. This can be seen as follows. Let

$$\begin{aligned} \alpha :=& dd^c \log \|Q\|^2 \\ \gamma :=& \frac{\|Q\|^2}{\|x\|^{2D}} \end{aligned}$$

With this notation, the current-valued map in (57) is

$$\Gamma_\lambda = -\frac{1}{\lambda} \gamma^\lambda \left(\sum_{k=0}^{p-1} \alpha^k \wedge (D\omega)^{p-1-k} \right)$$

An immediate computation shows that, for $\text{Re}\lambda >> 0$,

$$dd^c \Gamma_\lambda = -\gamma^\lambda \left(\alpha - D\omega + \frac{i}{2\pi} \lambda \frac{\partial \gamma}{\gamma} \wedge \frac{\overline{\partial} \gamma}{\gamma} \right) \wedge \left(\sum_{k=0}^{p-1} \alpha^k \wedge (D\omega)^{p-1-k} \right),$$

that is, since $\alpha^p = 0$,

$$\begin{aligned} dd^c \Gamma_\lambda = \gamma^\lambda D^p \omega^p - & \frac{i}{2\pi} \lambda \gamma^\lambda \frac{\partial \gamma}{\gamma} \wedge \frac{\overline{\partial} \gamma}{\gamma} \wedge \alpha^{p-1} \\ & - \frac{i}{2\pi} \lambda \gamma^\lambda \frac{\partial \gamma}{\gamma} \wedge \frac{\overline{\partial} \gamma}{\gamma} \wedge \left(\sum_{k=0}^{p-2} \alpha^k \wedge (D\omega)^{p-1-k} \right). \end{aligned} \tag{67}$$

It is immediate to show (using resolution of singularities as in the proof of Theorem 3.25 in [BGVY]) that $dd^c\Gamma_\lambda$ is holomorphic in a half plane $\mathrm{Re}\lambda > -\epsilon$. The value at the origin of the sum of the two first terms in (67) equals, by Lemma 7,

$$D^p\omega^p - \delta_Z \,.$$

Unfortunately, apart from the smooth case we already mentioned (note we are not in this situation here since the Q_j are powers of the original ones), the other term seems to give a non zero contribution to the value of $dd^c\Gamma_\lambda$ at the origin.

On the other hand, as it follows from [GK], the Levine idea provides the construction of a Green current which solves our equation

$$dd^cG = D^p\omega^p - \delta_Z \,.$$

The current G is defined as

$$L_Q(x) = -\log\left(\frac{\|Q\|^2}{||x||^{2D}}\right)\left(\sum_{k=0}^{p-1}(dd^c\log\|Q\|^2)^k\wedge\omega(x)^{p-1-k}\right)$$

(where the product is defined as in Monge-Ampère theory.) As we mentionned it just before, it does not seem clear-though we think it is true- that such a current can be obtained as the "value" at $\lambda = 0$ of the zeta-function

$$\lambda \mapsto -\frac{1}{\lambda}\frac{\|Q\|^2}{\|x\|^{2D}}^{\lambda}\sum_{k=0}^{p-1}(dd^c\log\|Q\|^2)^k\wedge\omega^{p-1-k}$$

(which has a simple pole at $\lambda = 0$). This is the reason why the Levine idea, which appears as the more natural method to construct normalized Green currents with the required properties, does not provide a solution for our problem (get a Green current from a zeta-function related to functional equations for the Q_j^λ) in an obvious way. In order to get around this difficulty, we inspired ourselves from the argument used in [Vo] and in [BGS] (Lemma 1.2.2 and section 6.1) and consider first the case of the diagonal in $\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})$, later we go back to consider more general cycles Z.

Consider the (n,n)-current valued map in $\mathbf{P}^{2n+1}(\mathbf{C})$ which is globally defined in the homogeneous coordinates $(x_0,\ldots,x_n,y_0\ldots,y_n)$ by

$$(68)\quad L_\lambda = \frac{-1}{\lambda}\left(\frac{||x-y||^2}{||x||^2+||y||^2}\right)^\lambda\left(\sum_{k=0}^{n}\left(dd^c\log||x-y||^2\right)^k\wedge(dd^c\log(||x||^2+||y||^2))^{n-k}\right)$$

which is the Levine form for the subspace $x = y$ in $\mathbf{P}^{2n+1}(\mathbf{C})$. We introduce now the C^∞ map

$$\begin{aligned}\pi : (\mathbf{C}^{n+1})^*\times(\mathbf{C}^{n+1})^*\times(\mathbf{C}^2)^* &\longrightarrow (\mathbf{C}^{n+2})^*\\ (x,y,(\beta_0,\beta_1)) &\mapsto (\beta_0x,\beta_1y)\,.\end{aligned}$$

Let us fix λ, $\mathrm{Re}\lambda >> 0$. While the pullback $\pi^*(L_\lambda)$ does not define a current on $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$, for each x, y fixed it is well-defined on $\mathbf{P}^1(\mathbf{C})$. Therefore, we can consider this pullback as a (n,n)-current on $(\mathbf{C}^n)^* \times (\mathbf{C}^n)^* \times \mathbf{P}^1(\mathbf{C})$. Now, we can define a $(n-1, n-1)$-current on $(\mathbf{C}^n)^* \times (\mathbf{C}^n)^*$ by

$$\Upsilon_\lambda(x,y) = \int_{\beta \in \mathbf{P}^1(\mathbf{C})} \pi^*(L_\lambda)(x,y,\beta)\,. \tag{69}$$

Since, we are averaging over $\mathbf{P}^1(\mathbf{C})$, the differential form $\Upsilon_\lambda(x,y)$ is now well defined on $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, and it is holomorphically dependent on λ for $\mathrm{Re}\lambda >> 0$. We already know a (p,p)-current valued holomorphic function on $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, namely the map given in bihomogeneous coordinates (x,y) by

$$\begin{aligned} I_\lambda(y) = \frac{i}{2\pi}\lambda \left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)^\lambda \partial \log \left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right) \wedge \bar{\partial} \log \left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right) \wedge \\ \wedge \left(dd^c \log \left(\sum_{j=1}^p |Q_j(y)|^2\right)\right)^{p-1} \end{aligned} \tag{70}$$

In fact, it depends only on y. (Compare with (59).)

Proposition 9. *The $(p-1,p-1)$-current valued map G_λ on $\mathbf{P}^n(\mathbf{C})$ defined for $\mathrm{Re}\lambda > 0$ and $\mathrm{Re}\lambda^2 >> 0$ by*

$$< G_\lambda, \psi >= \int_{\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})} \psi(x) \wedge I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x,y)$$

can be analytically continued to the complex plane as a meromorphic map with a simple pole at $\lambda = 0$. The coefficient G_0 of λ^0 in the Laurent development about the origin is a current which is smooth outside $|Z|$, and satisfies the equation

$$dd^c G_0 + \delta_Z = D^p \omega^p$$

Proof. We are going to show that for any test form ψ the function $\lambda \mapsto < G_\lambda, \psi >$ can be analytically continued as a meromorphic function with a simple pole at the origin and we will then compute locally $dd^c G_0$. We can assume, for example, that on $supp(\psi)$ we have $x_0 \neq 0$. Therefore we can rewrite $< G_\lambda, \psi >$ as

$$< G_\lambda, \psi >= \int_{\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})} \psi(x) \wedge I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x/x_0, y)\,.$$

Using a partition of unity, to show that the analytic continuation exists and to compute the the action of $dd^c G_0$ it is enough to study

$$< \varpi_\lambda, \psi >= \int_{\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \theta(y) I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x/x_0, y)$$

for a test function θ of small support. We will assume that $y_0 \neq 0$ on $supp(\theta)$. Thus we can rewrite

$$< \varpi_\lambda, \psi >= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \theta(y) I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x/x_0, y/y_0)$$

which can be also written as

$$(71) \qquad < \varpi_\lambda, \psi >= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \theta(y) I_{\lambda^2}(y) \wedge \pi^*(L_\lambda)(x/x_0, y/y_0, \beta)\,.$$

Now all the functions involving singularities are non-negative real analytic functions of all the variables x, y, β, and one can apply Atiyah's theorem to show that the analytic continuation in λ exits as a meromorphic function. The crucial point now is that the functions $Q_j(y)$, $1 \leq j \leq p$, together with the functions

$$\phi_k(x, y, \beta) = \beta_0 \frac{x_k}{x_0} - \beta_1 \frac{y_k}{y_0}\,, \quad k = 0, \ldots, n$$

define a complete intersection, i.e., a cycle of codimension $n + 1 + p$ in $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. One can rewrite $\pi^*(L_\lambda)$ in terms of the functions ϕ_k and the non-vanishing smooth function $\varsigma = ||\beta_0 x/x_0||^2 + ||\beta_1 y/y_0||^2$,

$$(72) \qquad \pi^*(L_\lambda) = \frac{-1}{\lambda} \left(\frac{||\phi||^2}{\varsigma}\right)^\lambda \left(\sum_{k=0}^{n} \left(dd^c \log ||\phi||^2\right)^k \wedge (dd^c \log \varsigma)^{n-k}\right).$$

Now we use a resolution of singularities $Y \xrightarrow{\kappa} \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$, as in the proof of [BGVY, Theorem 3.25], so that in local coordinates w all the functions $Q_1, \ldots, Q_p$, $\phi_0, \ldots, \phi_n$ can be written as

$$\begin{aligned} \kappa^*(Q_j) =& u_j \cdot w^{a_j} \quad (j = 1, \ldots, p) \quad \text{and} \quad a_j \in \mathbf{N}^{2n+1} \\ \kappa^*(\phi_k) =& v_k \cdot w^{b_k} \quad (k = 0, \ldots, n) \quad \text{and} \quad b_k \in \mathbf{N}^{2n+1} \end{aligned}$$

u_j, v_k non-vanishing holomorphic functions. So we are led to study the integrand of (71) in the new coordinates w, and, after using a partition of unity, we are in a local chart U of X. Once we are in this situation, one can construct a toric manifold Y' and a proper map $Y' \xrightarrow{\kappa'} U$, defined by monoidal transformations, so that in local coordinates w' on Y' one has

$$\begin{aligned} \kappa'^* \circ \kappa^*(Q_j) =& u'_j \cdot w'^{a'_j} \quad (j = 1, \ldots, p) \quad \text{and} \quad a'_j \in \mathbf{N}^{2n+1} \\ \kappa'^* \circ \kappa^*(\phi_k) =& v'_k \cdot w'^{b'_k} \quad (k = 0, \ldots, n) \quad \text{and} \quad b'_k \in \mathbf{N}^{2n+1} \end{aligned}$$

with the additional property that all the monomials $w'^{a'_j}, 1 \leq j \leq p$, are multiples of a distinguished one, m, taken to be one of them. Once we have this setup, we use a partition of unity in Y' and we are led to study the integral in a local chart U'. Finally, we construct a new toric manifold $T \xrightarrow{\kappa''} U'$, such that in the local coordinates t, the corresponding second set of monomials $t^{b''_k} = \kappa''^*(w'^{b'_k})$ contains also distinguished monomial m_2. Note that the first set of monomials $t^{a''_j} = \kappa''^*(w'^{a'_j})$ still contains a distinguished monomial $m_1 = \kappa''^* m$. To simplify the notation let us denote $\tau = \kappa \circ \kappa' \circ \kappa''$. From now on, we are reduced to study all our problems about analytic continuation on a local chart in T. In such a chart we have

$$\begin{aligned}\sum_{j=1}^{p} |\tau^*(Q_j)|^2 =& |m_1|^2 \upsilon_1 \\ \sum_{k=0}^{n} |\tau^*(\phi_k)|^2 =& |m_2|^2 \upsilon_2\end{aligned} \tag{73}$$

where the two functions υ_i are real analytic functions, non-vanishing in the local chart. Therefore, the differential forms which appear, respectively, in the expression of $\tau^*\pi^*(I_{\lambda^2})$ (see (70)) and $\tau^*\pi^*(L_\lambda)$ (see (72)), that is,

$$\begin{aligned}\alpha_1 :=& \tau^* \left(\left(dd^c \log \left(\sum_{j=1}^{p} |Q_j(y)|^2 \right) \right)^{p-1} \right) \\ \alpha_2 :=& \tau^* \left(\sum_{k=0}^{n} \left(dd^c \log ||\phi||^2 \right)^k \wedge (dd^c \log \varsigma)^{n-k} \right)\end{aligned}$$

are smooth forms in the chart, since $dd^c \log |m_i|^2 = 0$. As a consequence, one can write $\tau^*\pi^*(L_\lambda)$ as

$$\frac{-1}{\lambda} \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)} \right)^{\lambda} \alpha_2$$

and, similarly,

$$\tau^*\pi^*(I_{\lambda^2}) = \frac{i}{2\pi} \lambda^2 \left(\frac{\tau^*(||Q||^2)}{\tau^*(||y||^{2D})} \right)^{\lambda^2} \left(\frac{\partial m_1}{m_1} - \varphi_1 \right) \wedge \left(\frac{\overline{\partial m_1}}{\overline{m_1}} - \varphi_2 \right) \wedge \alpha_1$$

where φ_1, φ_2 are smooth forms. Thus, (71) is a finite sum of integrals of the type

$$\lambda \int \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)} \right)^{\lambda} \left(\frac{\tau^*(||Q||^2)}{\tau^*(||y||^{2D})} \right)^{\lambda^2} \left(\frac{\partial m_1}{m_1} - \varphi_1 \right) \wedge \left(\frac{\overline{\partial m_1}}{\overline{m_1}} - \varphi_2 \right) \wedge \alpha \wedge \xi \tau^*\pi^*(\psi)$$

Here ξ a test form and α is a smooth form, up to multiplicative constant $\alpha_1 \wedge \alpha_2$. Since, this last expression is itself a sum of integrals where the only vanishing denominators are of the form $t_h \bar{t}_l$, where t_h and t_l divide the monomial m_1. We do one integration by parts in order to eliminate the singularity due to $\overline{t_h}$. This introduces a division by a factor of the form $n_1\lambda^2 + n_2\lambda$, $n_1, n_2 \in \mathbf{N}$ and $n_1 \neq 0$. Since we have already a factor λ in the last expression, this proves that the function G_λ has at most a simple pole at the origin.

Now, we start with the computation of dd^cG_0. What follows is inspired on the proof of [BGVY, Proposition 5.21], but significantly harder. Now, we have, from Stokes theorem, for $\mathrm{Re}\lambda^2 >> 0$,

$$< G_\lambda, dd^c(\psi) >=< H_{\lambda^2,\lambda}, \psi > + \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge dd^c(I_{\lambda^2})(y) \wedge \Upsilon_\lambda(x,y)$$

$$+ \frac{i}{2\pi} \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \left(\partial I_{\lambda^2}(y) \wedge \overline{\partial}\Upsilon_\lambda(x,y) - \overline{\partial} I_{\lambda^2} \wedge \partial\Upsilon_\lambda(x,y)\right) \tag{74}$$

where for a $(n-p, n-p)$ test form ψ on $\mathbf{P}^n(\mathbf{C})$, the function of two complex variables λ_1, λ_2, defined when $\mathrm{Re}\lambda_1 >> 0$, $\mathrm{Re}\lambda_2 >> 0$ as

$$< H_{\lambda_1,\lambda_2}, \psi >:= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge I_{\lambda_1}(y) \wedge dd^c(\Upsilon_{\lambda_2}(x,y))$$

We first show that $< H_{\lambda_1,\lambda_2}, \psi >$ can be analytically continued, as a holomorphic function of two variables, to a product of halfplanes $\{\mathrm{Re}\lambda_1 > -\epsilon_1\} \times \{\mathrm{Re}\lambda_2 > -\epsilon_2\}$ containing the origin. As before, we can localize the problem near a point where $x_0y_0 \neq 0$ and consider the analytic continuation of the function of two variables

$$< \tilde{\varpi}_{\lambda_1,\lambda_2}, \psi >:= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \theta(y) I_{\lambda_1}(y) \wedge dd^c\pi^*(L_{\lambda_2})(x/x_0, y/y_0, \beta)\,. \tag{75}$$

Now we can verify for $\mathrm{Re}\lambda >> 0$ that

$$\overline{\partial}\left[\frac{i}{2\pi}\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)^\lambda \partial\log\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right) \wedge \left(dd^c\log\left(\sum_{j=1}^p |Q_j(y)|^2\right)\right)^{p-1}\right]$$

$$= -I_\lambda - \left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right)^\lambda dd^c\log\left(\frac{\sum_{j=1}^p |Q_j(y)|^2}{||y||^{2D}}\right) \wedge \left(dd^c\log\left(\sum_{j=1}^p |Q_j(y)|^2\right)\right)^{p-1}$$

$$= -I_\lambda + \tilde{I}_\lambda$$

The last line defines $\tilde{I}_\lambda$. It is also convenient to denote by K_λ the expression between brackets in the first line. Thus we have for the smooth function θ,

$$\theta I_\lambda = \theta\tilde{I}_\lambda + \overline{\partial}\theta \wedge K_\lambda - \overline{\partial}(\theta K_\lambda)$$

and so, we can replace in (75) the form $\theta(y)I_{\lambda_1}(y)$ by the last expression and obtain

$$< \tilde{\varpi}_{\lambda_1,\lambda_2},\psi >= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \theta(y)\tilde{I}_{\lambda_1}(y) \wedge dd^c\pi^*(L_{\lambda_2})(x/x_0, y/y_0, \beta)$$
$$+ \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \overline{\partial}\theta(y) \wedge K_{\lambda_1}(y) \wedge dd^c\pi^*(L_{\lambda_2})(x/x_0, y/y_0, \beta)$$
$$- \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \overline{\partial}(\theta K_{\lambda_1}(y)) \wedge dd^c\pi^*(L_{\lambda_2})(x/x_0, y/y_0, \beta)$$

In the third integral, we can now apply Stokes' theorem and see that this term (including the sign) becomes

$$\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \theta(y)\overline{\partial}\psi(x)\wedge K_{\lambda_1}(y)\wedge dd^c\pi^*(L_{\lambda_2})(x/x_0,y/y_0,\beta)$$

We can now group together the last two terms of the earlier formula and rewrite the complete function of λ_1,λ_2 as

$$\begin{aligned} <\tilde{\varpi}_{\lambda_1,\lambda_2},\psi> &= \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x)\theta(y)\wedge \tilde{I}_{\lambda_1}(y)\wedge dd^c\pi^*(L_{\lambda_2})(x/x_0,y/y_0,\beta) \\ &+ \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \overline{\partial}[\psi(x)\theta(y)]\wedge K_{\lambda_1}(y)\wedge dd^c\pi^*(L_{\lambda_2})(x/x_0,y/y_0,\beta) \end{aligned} \tag{76}$$

Let us now return to the question of the analyticity in the two variables. By using successive resolutions of singularities as done earlier, we reduce ourselves to the situation where, up to product by non vanishing holomorphic functions, all functions $\tau^*\pi^*(Q_j)$, $\tau^*\pi^*\phi_k$ are monomials; we have this way two lists of monomials in the local coordinates t. Our resolution of singularities is such that we can assume that among these two lists, there are two distinguished monomials (one for each list) m_1,m_2 such that in particular (73) holds. Since

$$\tau^*\pi^*(dd^c[L_{\lambda_2}(x/x_0,y/y_0,\beta)]) = dd^c[\tau^*\pi^*(L_{\lambda_2}(x/x_0,y/y_0,\beta)]\,,$$

it follows from the computations in (67) that one has
(77)

$$\tau^*\pi^*(dd^c[L_{\lambda_2}(\frac{x}{x_0},\frac{y}{y_0},\beta)]) = \left(\frac{\tau^*\|\phi\|^2}{\varsigma}\right)^{\lambda_2}\left(\tilde{\alpha}_1+\lambda_2\left(\frac{\partial m_2}{m_2}-\tilde{\varphi}_1\right)\wedge\left(\frac{\overline{\partial m_2}}{\overline{m_2}}-\tilde{\varphi}_2\right)\wedge\tilde{\alpha}_2\right)$$

where the $\tilde{\alpha}_j$ and the $\tilde{\varphi}_j$ are smooth forms. Due to its expression, the form $\tau^*\pi^*(\tilde{I}_{\lambda_1})$ can be written as

$$\tau^*\pi^*(\tilde{I}_{\lambda_1}) = \left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1}\alpha_3 \tag{78}$$

where α_3 is a smooth form. Similarly, one can compute $\tau^*\pi^*(K_{\lambda_1})$ and get for this term an expression of the form

$$\tau^*\pi^*(K_{\lambda_1}) = \left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1}\left(\frac{\partial m_1}{m_1}-\varphi_3\right)\wedge\alpha_4\,, \tag{79}$$

where φ_3 and α_4 are smooth forms. We conclude that the function $<\tilde{\varpi}_{\lambda_1,\lambda_2},\psi>$ is a linear combination of four kinds of terms

$$\lambda_2\int\left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1}\left(\frac{\tau^*\|\phi\|^2}{\varsigma}\right)^{\lambda_2}\frac{\sigma}{t_ht_k\bar{t}_l}\wedge\xi\overline{\partial}(\tau^*\pi^*(\theta\psi)) \tag{i}$$

(ii)
$$\int \left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1} \left(\frac{\tau^*\|\phi\|^2}{\varsigma}\right)^{\lambda_2} \frac{\sigma}{t_k} \wedge \xi\overline{\partial}(\tau^*\pi^*(\theta\psi))$$

(iii)
$$\lambda_2 \int \left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1} \left(\frac{\tau^*\|\phi\|^2}{\varsigma}\right)^{\lambda_2} \frac{\sigma}{t_k\bar{t}_l} \wedge \xi\tau^*\pi^*(\theta\psi)$$

(iv)
$$\int \left(\frac{\|\tau^*(Q_j)\|^2}{\tau^*(\|y\|^2)}\right)^{\lambda_1} \left(\frac{\tau^*\|\phi\|^2}{\varsigma}\right)^{\lambda_2} \sigma \wedge \xi\tau^*\pi^*(\theta\psi)$$

where σ is a smooth form (dependent on the functions $\tau^*(Q_j)$ and $\tau^*(\phi_k)$), t_h, t_k divide the product m_1m_2, t_l divides m_2, and ξ is test function. The fact that expressions of the form (ii) or (iv) are holomorphic functions in (λ_1,λ_2) in $\{\mathrm{Re}\lambda_1 > -\epsilon, \mathrm{Re}\lambda_2 > -\epsilon\}$ is obvious since the functions v_i that appear in (73) are assumed to be non vanishing on the support of the test function ξ. For the two other expressions (i) and (iii), the situation is a bit more delicate. What we do is essentially to eliminate the $\bar{t}_l$ in the denominator with the help of one integration by parts. To do that, we profit from the existence of the coefficient λ_2 in front of the expression. The only problem is to take care that the coordinate t_l does not divide also the monomial m_1. Here the fact that the system $(Q_1,\dots,Q_p,\phi_0,\dots,\phi_n)$ defines a complete intersection plays an essential role. In fact, one can show, as in the proof of Theorem 2 of [BY1] that, under such a hypothesis, terms of the form (i) or (iii) contain $\bar{t}_l$ as a fictitious singularity. Hence we are done, and we have completely proved the analyticity of $(\lambda_1,\lambda_2) \mapsto < H_{\lambda_1,\lambda_2},\psi >$ in some domain of the form $\{\mathrm{Re}\lambda_1 > -\epsilon, \mathrm{Re}\lambda_2 > -\epsilon\}$.

We now compute the value at the origin of the function $(\lambda_1,\lambda_2) \mapsto < H_{\lambda_1,\lambda_2},\psi >$. To do that, we first compute $< H_{\lambda_1,0},\psi >$ for $\mathrm{Re}\lambda_1 >> 0$. Once is done, we will let λ_1 tend to 0. Since the function of two variables $(\lambda_1,\lambda_2) \mapsto < H_{\lambda_1,\lambda_2},\psi >$ is holomorphic in a product of half planes $\{\mathrm{Re}\lambda_1 > -\epsilon, \mathrm{Re}\lambda_2 > -\epsilon\}$, we will recover that way its value at the origin. Let us start with the computation of $< H_{\lambda_1,0},\psi >$ for $\mathrm{Re}\lambda_1 >> 0$. We use the fact that the set defined in homogeneous coordinates (x,y,β) as $\{(x,y,\beta),\ \beta_0 x = \beta_1 y\}$ is a smooth manifold Δ (defined as a complete intersection) in $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. Let us recall that $< H_{\lambda_1,\lambda_2},\psi >$ is the sum of a finite number of terms of the type (75), obtained using localizing functions $\theta(y)$ defining a partition of unity. This implies (as seen in (58)) that locally (let us say in the open set $x_0y_0 \neq 0$), the current Γ_0, defined as the coefficient of λ_2^0 in the Laurent development of

$$\lambda_2 \mapsto \pi^*(L_{\lambda_2})(x/x_0, y/y_0, \beta)$$

about the origin (as a meromorphic current-valued map of λ_2) satisfies

(80)
$$dd^c\Gamma_0 + \delta_\Delta = [dd^c \log(\|\beta_0 x/x_0\|^2 + \|\beta_1 y/y_0\|^2)]^2)]^{n+1}\,.$$

From (80) we get, for $\mathrm{Re}\lambda_1 >> 0$,

$$< H_{\lambda_1,0},\psi > + \int_\Delta \psi(x) \wedge I_{\lambda_1}(y)$$

(81)
$$= \int_{(x,y)\in\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge I_{\lambda_1}(y) \wedge \left(\int_{\beta\in\mathbf{P}^1(\mathbf{C})} \Omega(\beta_0 x, \beta_1 y)^{n+1}\right)$$

where

$$\Omega(x,y) = dd^c \log(\|x\|^2 + \|y\|^2)$$

is the harmonic form (in $\mathbf{P}^{2n+1}(\mathbf{C})$) defining the Fubini-Study metric in $\mathbf{P}^{2n+1}(\mathbf{C})$. Now, in (81), we can use the analytic continuation (as a function of λ_1) and compute its value at $\lambda_1 = 0$. From the definition of Δ,

$$\int_\Delta \psi(x) \wedge I_{\lambda_1}(y) = \int_{\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge I_{\lambda_1}(x)\,.$$

We know from Lemma 7 that the value at $\lambda_1 = 0$ of this expression makes sense; it is equal to $< \delta_Z, \psi >$. Finally, we obtain the following formula

$$< H_{0,0}\,, \psi > + < \delta_Z, \psi >= \int_{x\in\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \left(\int_{\beta\in\mathbf{P}^1(\mathbf{C})} \int_{y\in Z} \Omega(\beta_0 x, \beta_1 y)^{n+1} \right)$$

We are left to compute the smooth differential form

$$x \mapsto \int_{\beta\in\mathbf{P}^1(\mathbf{C})} \int_{y\in Z} \Omega(\beta_0 x, \beta_1 y)^{n+1}\,. \tag{82}$$

The easy way to do this computation is to show first that this form is harmonic in $\mathbf{P}^n(\mathbf{C})$. This follows from the obvious fact that, for $y \in (\mathbf{C}^{n+1})^*$ and $\beta \in (\mathbf{C}^2)^*$ fixed, the function $x \mapsto \log(\|\beta_0 x\|^2 + \|\beta_1 y\|^2)$ is invariant under the action of the unitary group $U(n+1)$. Thus, the differential form (82) has the same invariance. On the other hand, any differential form in $\mathbf{P}^n(\mathbf{C})$ invariant under the action of $U(n+1)$ is d and d^* closed (cf. [He, Exercise 1, p. 191]), thus harmonic. From degree considerations, we conclude that (82) is a multiple of ω^p. Thus, we have

$$< H_{0,0}, \psi > + < \delta_Z, \psi >= c \int_{\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \omega^p(x) \tag{83}$$

for some constant c.

We need now to show that the remaining expressions in (74) define holomorphic functions of λ near the origin and to compute their values at $\lambda = 0$. For this purpose, we need a few preliminary computations.

$$\overline{\partial} I_{\lambda^2} = \lambda^2 \left(\frac{||Q||^2}{||y||^{2D}} \right)^{\lambda^2} \overline{\partial} \log \left(\frac{||Q||^2}{||y||^{2D}} \right) \wedge dd^c \log \frac{||Q||^2}{||y||^{2D}} \wedge \left(dd^c \log ||Q||^2 \right)^{p-1} \tag{84}$$

$$\partial I_{\lambda^2} = -\lambda^2 \left(\frac{||Q||^2}{||y||^{2D}} \right)^{\lambda^2} \partial \log \left(\frac{||Q||^2}{||y||^{2D}} \right) \wedge dd^c \log \frac{||Q||^2}{||y||^{2D}} \wedge \left(dd^c \log ||Q||^2 \right)^{p-1} \tag{85}$$

$$dd^c I_{\lambda^2} = \lambda^4 \frac{i}{2\pi}\left(\frac{||Q||^2}{||y||^{2D}}\right)^{\lambda^2} \partial \log\left(\frac{||Q||^2}{||y||^{2D}}\right) \wedge \overline{\partial} \log\left(\frac{||Q||^2}{||y||^{2D}}\right) \wedge$$

$$\wedge\, dd^c \log \frac{||Q||^2}{||y||^{2D}} \wedge \left(dd^c \log ||Q||^2\right)^{p-1} + R_\lambda$$

$$= S_\lambda + R_\lambda\,, \tag{86}$$

where

$$R_\lambda := \lambda^2 \left(\frac{||Q||^2}{||y||^{2D}}\right)^{\lambda^2} \wedge \left(dd^c \log \frac{||Q||^2}{||y||^{2D}}\right)^2 \wedge \left(dd^c \log ||Q||^2\right)^{p-1}\,. \tag{87}$$

Moreover, we have also

$$\overline{\partial}\pi^*(L_\lambda) = -\left(\frac{||\phi||^2}{\varsigma}\right)^{\lambda} \overline{\partial} \log\left(\frac{||\phi||^2}{\varsigma}\right) \wedge \left(\sum_{k=0}^{n} \left(dd^c \log ||\phi||^2\right)^k \wedge (dd^c \log \varsigma)^{n-k}\right) \tag{88}$$

and

$$\partial\pi^*(L_\lambda) = -\left(\frac{||\phi||^2}{\varsigma}\right)^{\lambda} \partial \log\left(\frac{||\phi||^2}{\varsigma}\right) \wedge \left(\sum_{k=0}^{n} \left(dd^c \log ||\phi||^2\right)^k \wedge (dd^c \log \varsigma)^{n-k}\right) \tag{89}$$

We now proceed to show that, for any $(n-p, n-p)$ test form ψ, the meromorphic function

$$\lambda \longrightarrow \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge dd^c(I_{\lambda^2})(y) \wedge \Upsilon_\lambda(x,y)$$

$$+ \frac{i}{2\pi} \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \left(\partial I_{\lambda^2}(y) \wedge \overline{\partial}\Upsilon_\lambda(x,y) - \overline{\partial} I_{\lambda^2} \wedge \partial\Upsilon_\lambda(x,y)\right) \tag{90}$$

can be continued as a meromorphic function of λ which has $\lambda = 0$ as a zero. We now use the resolution of singularities we used before and write out in local coordinates the pullback of all these forms. As a consequence of (73) we have

$$\tau^*(\overline{\partial} I_{\lambda^2}) = \lambda^2 \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{\overline{\partial m_1}}{\overline{m_1}} - \chi_1\right) \wedge \gamma_1 \tag{84$'$}$$

$$\tau^*(\partial I_{\lambda^2}) = \lambda^2 \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{\partial m_1}{m_1} - \chi_2\right) \wedge \gamma_2 \tag{85$'$}$$

$$\tau^*(S_\lambda) = \lambda^4 \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{\overline{\partial m_1}}{\overline{m_1}} - \chi_3\right) \wedge \left(\frac{\partial m_1}{m_1} - \chi_4\right) \wedge \gamma_3 \tag{86$'$}$$

$$\tau^*(R_\lambda) = \lambda^2 \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \gamma_4 \tag{87$'$}$$

$$\overline{\partial}\tau^*\pi^*(L_\lambda) = \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)}\right)^{\lambda} \left(\frac{\overline{\partial m_2}}{\overline{m_2}} - \tilde{\chi}_1\right) \wedge \tilde{\gamma}_1 \tag{88$'$}$$

$$\partial\tau^*\pi^*(L_\lambda) = \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)}\right)^{\lambda} \left(\frac{\partial m_2}{m_2} - \tilde{\chi}_2\right) \wedge \tilde{\gamma}_2 \tag{89$'$}$$

where all the $\gamma_j, \tilde{\gamma}_j, \chi_j, \tilde{\chi}_j$ are smooth forms. Let us now consider the cross-terms in (90), for example

$$\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge \partial I_{\lambda^2}(y) \wedge \overline{\partial}\Upsilon_\lambda(x,y)$$

In local coordinates it contributes a finite sum of integrals of the form

$$\lambda^2 \int \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)}\right)^{\lambda} \left(\frac{\partial m_1}{m_1} - \chi_2\right) \wedge \left(\frac{\overline{\partial m_2}}{\overline{m}_2} - \tilde{\chi}_1\right) \wedge \xi$$

where ξ is a smooth form with compact support in the local chart. If we expand the logarithmic derivatives of the monomials in the integrand we see that the only non-integrable expressions are those which contain in the denominator $|t_h|^2$, for t_h dividing both m_1 and m_2. We need to eliminate, for example, $\overline{t_h}$ by an integration by parts, so that what remains is integrable when $\lambda = 0$. To perform this integration by parts, we divide by $n_1\lambda^2 + n_2\lambda$, $n_1, n_2 \in \mathbf{N}$, $n_2 > 0$ because t_h divides m_2. Since there is a factor λ^2 in front of the integral, the function of λ we obtain vanishes when $\lambda = 0$. The other cross-term vanishes at $\lambda = 0$ for the same reason. We have two terms left to study, namely,

$$\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge R_\lambda(y) \wedge \Upsilon_\lambda(x,y)\,, \tag{91}$$

$$\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \psi(x) \wedge S_\lambda(y) \wedge \Upsilon_\lambda(x,y)\,. \tag{92}$$

Using the identity (87′) and the expression already used for $\tau^*\pi^*(L_\lambda)$, we see that in local coordinates the integral (91) is a linear combination of terms of the form

$$\lambda \int \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)}\right)^{\lambda} \xi$$

where ξ is a smooth test form. These terms are holomorphic near $\lambda = 0$ and vanish there. The term (92) can be written, together with of (86′), as a linear combination of terms like

$$\lambda^3 \int \left(\frac{\tau^*||Q||^2}{\tau^*||y||^{2D}}\right)^{\lambda^2} \left(\frac{||\tau^*(\phi)||^2}{\tau^*(\varsigma)}\right)^{\lambda} \left(\frac{\overline{\partial m_1}}{\overline{m}_1} - \chi_3\right) \wedge \left(\frac{\partial m_1}{m_1} - \chi_4\right) \xi$$

Expanding the logarithmic derivatives, one sees that the non-integrable terms have denominators of the form $|t_h|^2$, with t_h dividing m_1. We eliminate this singularity by making $\overline{t_h}$ disappear with one integration by parts, which implies division by $n_1\lambda^2 + n_2\lambda$, with $n_1 > 0$. In the worst case appears when $n_2 = 0$, but the factor λ^3 takes care of this. We are left with at least a factor λ, thus the function vanishes for $\lambda = 0$. In other words, (90) defines a holomorphic function $\lambda \mapsto < W_\lambda, \psi >$ vanishing at $\lambda = 0$.

Now, we recall from (74) that

$$< G_\lambda, dd^c\psi > = < dd^c G_\lambda, \psi > = < H_{\lambda^2,\lambda}, \psi > + < W_\lambda, \psi > \ .$$

So we have

$$< dd^c G_0, \psi > = < H_{0,0}, \psi >$$

and therefore, from (83),

$$dd^c G_0 + \delta_Z = c\omega^p \, .$$

To compute c we take the harmonic projection of both sides, so that

$$c\omega^p = H(\delta_Z) = \text{degree}(Z)\omega^p = D^p\omega^p \, .$$

This concludes the proof that G_0 satisfies the Green equation.

It remains to show that the current G_0 is smooth outside $|Z|$. Consider a point $x^0 \in \mathbf{P}^n(\mathbf{C})\backslash|Z|$ and let ψ be a test form with support in a neighborhood of x^0 and disjoint from $|Z|$. We can assume, without loss of generality, that the coordinate x_0 doesn't vanish on *supp*(ψ). Recalling the way G_0 was defined we also need to introduce a partition of unity $\theta_i(y)$ of $\mathbf{P}^n(\mathbf{C})$ whose elements are of one of the two forms, either the support is disjoint from $|Z|$ or it is disjoint from $\{x^0\}$, in any case, their support is assumed to be contained in a chart $\{y_j \neq 0\}$. Now we consider the "value" at $\lambda = 0$ of (71) with $\theta = \theta_i$. That is, we consider an expression of the form

$$(93) \qquad \left(\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge \theta(y) I_{\lambda^2}(y) \wedge \pi^*(L_\lambda)(x/x_0, y/y_0, \beta) \right)_{\Big|\lambda=0} .$$

We will suppose, for instance, that the support of θ is included in $\{y_0 \neq 0\}$. Suppose first that the support of θ is disjoint from $|Z|$. In this case, the form $\theta(y)I_{\lambda^2}$ can be written as $\lambda^2 A(y,\lambda)$, where A is an entire function of λ and a smooth form in y. Moreover, for $\mathrm{Re}\lambda > -\epsilon$, the differential form in x, y, β

$$B(x,y,\beta,\lambda) := -\left(\frac{||\phi||^2}{\varsigma}\right)^\lambda \pi^* \left(\sum_{k=0}^{n} \left(dd^c \log ||\phi||^2\right)^k \wedge (dd^c \log \varsigma)^{n-k} \right) ,$$

is integrable. This is immediate using resolution of singularities as done before, in fact, it is a consequence of (73). Since, moreover, the integral in (93) is given by

$$\lambda \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x) \wedge A(y,\lambda) \wedge B(x,y,\beta,\lambda) \, ,$$

it is well defined for $\lambda = 0$ and its value is zero. Thus, there is no contribution to G_0 when the support of θ is disjoint from $|Z|$. Consider now the remaining possibility, that is, the support of θ does not contain x^0. In this case, for x close to x^0 (we will assume this remains true in some neighborhood of the support of ψ), the differential form appearing in $\psi(x)\wedge\theta(y)\pi^*(L_\lambda)$ is non singular. Since the analytic continuation of I_{λ^2} near the origin is

$$I_{\lambda^2} = \delta_Z + \lambda^2 T + \cdots$$

we see immediately that

$$\left(\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})} \psi(x)\wedge\theta(y)I_\lambda(y)\wedge\pi^*(L_\lambda)(x/x_0, y/y_0,\beta)\right)_{\big|\lambda=0} =$$

$$\int_{\mathbf{P}^n(\mathbf{C})} \psi(x)\wedge\int_{Z\times\mathbf{P}^1(\mathbf{C})} \theta(y)\log\left(\frac{\|\Phi(x,y,\beta)\|^2}{\varsigma(x,y,\beta)}\right) B(x,y,\beta)\,. \tag{94}$$

The right hand side of (94) is a smooth function of x as this can be seen by applying again Lebesgue's differentiation theorem. This proves that outside $|Z|$ G_0 is a smooth current.

□

Remarks.

1. Instead of λ^2,λ in the definition of G_λ in Proposition 9, we can take λ^p (corresponding to I) and λ^q (corresponding to Υ) with integers $p>q>0$. This defines a new current G'_0 that coincides with G_0 outside $|Z|$ and has the same dd^c everywhere. The choice $p\le q$ does not provide a solution of the Green equation.

2. We can compare our construction to that of Gillet-Soulé [BGS, Section 6.1]. Since the description we gave of G_0 in the local charts involves multiplication of logarithm of coordinates by integration currents, this current may not be of log-type in the sense of Gillet-Soulé. Note that the current Γ_0 constructed in (57), following the idea of Levine, is smooth outside the support of Z and it has log-type. Unfortunately, in the non-smooth case it does not seem clear at once that it solves the Green equation. Our current G_0 is smooth in $\mathbf{P}^n(\mathbf{C})\setminus|Z|$, which is enough to use it for the computation of heights, as we will see in the next section. For this reason, we are not interested in the local behaviour of this current near $|Z|$, but in the way we can compute them just as values at the origin of zeta functions. It can also be shown, as in Lemma 5, that for some convenient choice of positive constants C_1, C_2, the map $C_1^{\lambda^2} C_2^{\lambda}\, G_\lambda$ defines a positive Green current at $\lambda=0$. Thus, all the properties required by Gillet-Soulé, except for the log-type, are fulfilled. Our construction differs from that of Gillet-Soulé since in our case, resolution of singularities appears only as an auxiliary tool and the final expression of the current G_0 is global. Moreover, we express the Green current as the value at the origin of a zeta function involving the generators of the ideal defining the cycle. Of course, we are restricted to the complete intersection case, which is not the case in the Gillet-Soulé approach. On the other hand, we do not need to assume that the cycle Z is irreducible as they do (in order to define the product of the integration current on $Z\times\mathbf{P}^n(\mathbf{C})$ with a Green current for the diagonal in $\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})$.). The action of the current G is obtained as a combination of the Laurent coefficients in the development at $\lambda_1=\lambda_2=0$ of expressions of the form *(i)* to *(iv)*. The pullbacks of such coefficients on the final desingularization are combinations of currents γ of the form

$$(ln|t_{j_1}|^2)^p ln(|t_{j_2}|^2)^q \mathrm{PV}\frac{1}{t_{j_3}t_{j_4}}\omega,$$

where $p,q \in \mathbf{N}$, $j_1,\ldots,j_4 \in \{1,\ldots,n\}$, PV denotes the principal value and ω a smooth form. The action of the pullback of G on a test form ψ can be expressed as a linear combination of terms of the form

$$(ln|t_{j_1}|^2)^p ln(|t_{j_2}|^2)^q \partial_{j_3,j_4}(\omega \wedge \psi),$$

where ∂_{j_3,j_4} is the operator transforming the coefficients of the test form ψ in their partial derivatives or order 2 with respect to t_{j_3}, t_{j_4} and ω is a smooth form. The multiplication of such expressions is well defined in the sense of currents.

3. Demailly has done also remarkable work on the relation between product of currents and intersection theory, obtaining a number of important algebraic results using complex analytic methods. There are two very clear surveys of this work in [De1] and [De2], and we refer the reader to them as well as to one of his original papers [De3] for a clear exposition of his techniques.

5. Zeta functions and logarithmic heights. In this section we consider p homogeneous polynomials $Q_1,\ldots,Q_p$ with integral coefficients defining a complete intersection variety in $\mathbf{P}^n(\mathbf{C})$. Let $\mathcal{Z}$ be the corresponding arithmetic cycle and $Z = \mathcal{Z}(\mathbf{C})$. . Let us assume that the set

$$\{x \in \mathbf{P}^n(\mathbf{C}) : \ x_0 = \cdots = x_{n-p} = Q_1 = \cdots = Q_p = 0\} = \emptyset\,,$$

so that if we denote by $\mathit{\Pi}$ the arithmetic cycle

$$\{x = (x', x'') : \ x' := (x_0,\ldots,x_{n-p}) = 0\}$$

then $\mathit{\Pi}\cdot\mathcal{Z}$ is an $n+1$ codimensional cycle in $\mathbf{P}^n$, that is,

$$\mathit{\Pi}\cdot\mathcal{Z} = \sum_{\tau \text{ prime}} n_\tau$$

We recall that if G_Z is a normalized Green current of log-type, then one can define the height of $\mathcal{Z}$ as

$$h(\mathcal{Z}) = \sum_{\tau \text{ prime}} n_\tau \log\tau + \frac{deg(Z)}{2}\sum_{k=p}^{n}\sum_{j=1}^{k}\frac{1}{j} + \frac{1}{2}\int_{\Pi} G_Z \tag{95}$$

Let us assume, for the time being, that all the Q_j have the same degree D. We know the current G defined in Proposition 9 (and denoted G_0 there) as the "value" at $\lambda = 0$ of the function

$$G_\lambda : \lambda \mapsto \int_{\mathbf{P}^n(\mathbf{C})} I_{\lambda^2}(y)\wedge\Upsilon_\lambda(x,y)$$

satisfies the Green equation

$$dd^cG + \delta_Z = D^p\omega^p$$

Let γ_Z the real number defined as the "value" at $\lambda = 0$ of

$$\int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} \omega(x)^{n-p+1} I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x,y)$$

and, for $\mathrm{Re}\lambda^2 >> 0$, $\mathrm{Re}\lambda > 0$, let

$$(x'', y) \mapsto \Omega_\lambda(x'', y)$$

be the restriction of the smooth differential form $I_{\lambda^2}(y) \wedge \Upsilon_\lambda(x,y)$ to $\Pi \times \mathbf{P}^n(\mathbf{C})$. It is immediate to verify (via Atiyah's theorem) that the map just defined has an analytic continuation as $(n+p-1, n+p-1)$-current valued meromorphic map. We have the following proposition.

Proposition 10. *The logarithmic height $h(\mathcal{Z})$ equals the "value" at $\lambda = 0$ of the map*

$$\lambda \mapsto \sum_{\tau \text{ prime}} n_\tau \log \tau + \frac{D^p}{2} \sum_{k=p}^{n} \sum_{j=1}^{k} \frac{1}{j} - \frac{\gamma_Z}{2} + \frac{1}{2} \int_{\Pi \times \mathbf{P}^n(\mathbf{C})} \Omega_\lambda(x'', y) \tag{96}$$

Proof. We consider the current

$$T = G - G_Z - \gamma_Z \omega^{p-1}\,.$$

This current is orthogonal to the harmonic forms. In fact, G_Z is already orthogonal to them by definition and $\gamma_Z\omega^{p-1}$ is the harmonic projection of G. Furthermore, the current T satisfies $dd^cT = 0$ and it is smooth outside $|Z|$. Thus, using the dd^c-Lemma (see [GS1, Theorem 1.2.1], [GH, p. 149]), there exist two currents T_1, T_2, which are smooth outside $|Z|$ such that

$$\begin{aligned} \partial T =& \partial\overline{\partial} T_1 \\ \overline{\partial} T =& \overline{\partial}\partial T_2 \end{aligned}$$

so that the current

$$\tilde{T} := T - \overline{\partial} T_1 - \partial T_2$$

is d-closed. As a consequence of the Hodge decomposition, one can write

$$\tilde{T} = H(\tilde{T}) + dd^*(\mathcal{G}_{p-1,p-1}\tilde{T})$$

where $\mathcal{G}_{p-1,p-1}$ is the Green operator the Laplacian on $(p-1,p-1)$ forms. Due to the properties of the Green operator, the current $\mathcal{G}_{p-1,p-1}\tilde{T}$ is smooth outside $|Z|$. Let T_3 be d^* applied to this last current. It is, of course, also smooth outside $|Z|$. Then we have

$$T = \overline{\partial}(T_1 + T_3) + \partial(T_2 + T_3) + H(\tilde{T})$$

Since $H(T) = 0$, it follows from this identity that $H(\tilde{T}) = 0$. Therefore, we have

$$T = \overline{\partial} U + \partial V \tag{97}$$

where U and V are currents smooth outside $|Z|$. Since $|Z|$ does not intersect Π, we can restrict (97) to Π and write

$$T_{|\Pi} = \overline{\partial}(U_{|\Pi}) + \partial(V_{|\Pi})\,. \tag{98}$$

Clearly, since $|Z|$ is disjoint from Π, we have, from formula (94)

$$\left(\int_{\Pi \times \mathbf{P}^n(\mathbf{C})} \Omega_\lambda(x'', y) \right)_{|\lambda=0} = \int_{\Pi} G(x)$$

Since the integral on Π of the restriction of T is zero by Stokes' theorem and (98), we have

$$\left(\int_{\Pi \times \mathbf{P}^n(\mathbf{C})} \Omega_\lambda(x'', y) \right)_{|\lambda=0} = \int_{\Pi} G_Z(x) + \gamma_Z \int_{\Pi} \omega^{p-1} = \int_{\Pi} G_Z(x) + \gamma_Z\,. \tag{99}$$

We can now substitute (99) in the formula (95) and we get the statement of the proposition.

□

Remark. In case the polynomials Q_j have different degrees D_j, following the previous section we construct the current-valued functions $I_{\lambda^2}, \Upsilon_\lambda$ of Proposition 9, associated to the polynomials $Q_1^{l_1}, \ldots, Q_p^{l_p}$ of common degree $D = l_1 D_1 = \ldots = l_p D_p$, the least common multiple of the degrees D_j, and denote $\ell = l_1 \cdots l_p$. The corresponding analytic cycle will be denoted by Z'. Let Ω_λ be the corresponding restriction to $\Pi \times \mathbf{P}^n(\mathbf{C})$ and $\gamma = \gamma_{Z'}$. Then the logarithmic height of $\mathcal{Z}$ is the "value" at $\lambda = 0$ of the map

$$\lambda \mapsto \sum_{\tau \text{ prime}} n_\tau \log \tau + \frac{D_1 \cdots D_p}{2} \sum_{k=p}^{n} \sum_{j=1}^{k} \frac{1}{j} - \frac{\gamma}{2\ell} + \frac{1}{2\ell} \int_{\Pi \times \mathbf{P}^n(\mathbf{C})} \Omega_\lambda(x'', y)\,.$$

It follows from Proposition 10 and the remark above that the value of the logarithmic height of a complete intersection cycle in $\mathbf{P}^n$ (that is, a cycle $\mathcal{Z}$ such that $Z = \mathcal{Z}(\mathbf{C})$ is defined as a complete intersection in $\mathbf{P}^n(\mathbf{C})$ by homogeneous polynomials $Q_1, \ldots, Q_p$ with integer coefficients) can be recovered as the value of some coefficient in the Laurent development at $\lambda = 0$ of some zeta function. Despite the fact that there seems to be no hope to get a closed expression for such a zeta function in general, one can expect such a function satisfies some holonomicity properties (in the sense of [WZ]). In order to illustrate this with a concrete example, we will consider the case of quadratic hypersurfaces in $\mathbf{P}^n$.

Proposition 11. *Let Q be an homogeneous polynomial in $n+1$ variables with integer coefficients and ζ_Q the zeta function defined by (12). The re exists a non zero difference operator with coefficients in $\mathbf{Z}[s]$, $\mathcal{P}(s) = \sum_{\alpha=0}^{N} p_\alpha(s)\Delta^{N-\alpha}$, such that*

$$\mathcal{P}[\zeta_Q](2s) := \sum_{\alpha=0}^{N} p_\alpha(s)\zeta_Q(2(s+N-\alpha)) \equiv 0 \tag{100}$$

the identity (100) being understood as an identity between meromorphic functions. Moreover, when $Q(X)$ is of the form

$$Q(X) = Q_{0,m}(X) = \sum_{j=0}^{m} X_j^2, \;\; 0 \leq m \leq n$$

or when $n \geq 2m+1$ and

$$Q(X) = \sum_{k=0}^{m} \lambda_k (X_{2k}^2 + X_{2k+1}^2)$$

where the λ_k are non zero integers, there is a closed (and explicit) formula for the function ζ_Q.

Proof. As seen in the introduction, we have

$$\frac{\zeta_Q(2s)\Gamma(n+1+s)}{n!} = \frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} exp(-\|z\|^2)|Q(z)|^{2s} dm(z)\,.$$

Since any product of two holonomic functions in the sense of [WZ] remains holonomic, it is enough to prove the existence of a non zero difference operator wit h coefficients in $\mathbf{Z}[s]$, $\widetilde{\mathcal{P}} = \sum_{\beta=0}^{M} \tilde{p}_\beta(s)\Delta^{N-\beta}$ such that

$$\widetilde{\mathcal{P}}[F_Q](s) := \sum_{\alpha=0}^{M} \tilde{p}_\beta(s) F_Q(s+M-\alpha) \equiv 0 \tag{101}$$

where F_Q is the meromorphic function

$$F_Q(s) := \frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} exp(-\|z\|^2)|Q(z)|^{2s} dm(z)\,.$$

Moreover, since it is immediate to notice that for some convenient integer K, the function

$$s \mapsto K^{-2s}\zeta_Q(2s) = \frac{n!}{\Gamma(n+1+s)} F_Q(s)$$

is bounded in the half plane $\operatorname{Re} z > 0$, it will be enough (from Carlson's theorem [Bo]) to show that some identity (101) is valid for all integers $k \in \mathbf{N}$.

Let us write $Q(X) = X^tAX$, where A is a symetric matrix with integer coefficients. Let us write $A = U^tDU$, where U is an orthogonal real matrix and D a diagonal matrix with real coefficients $\lambda_0, \dots, \lambda_n$. Note that any symmetric polynomial in the λ_j is in $\mathbf{Q}$ (since the λ_j are the eigenvalues of A). Now, for any positive integer k, we have

$$\begin{aligned}
F_q(k) &= \frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} e^{-\|z\|^2} \Big| \sum_{j=0}^{n} \lambda_j z_j^2 \Big|^{2k} dm(z) \\
&= \sum_{\substack{a_0+\dots+a_n=k \\ b_0+\dots+b_n=k \\ a_i,b_i\in\mathbf{N}}} \binom{k}{a_0,\dots,a_n} \binom{k}{b_0,\dots,b_n} \left(\frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} e^{-\|z\|^2} \prod_{j=0}^{n} \lambda_j^{a_j+b_j} z_j^{2a_j} \overline{z}_j^{2b_j} dm(z) \right) \\
&= \sum_{\substack{a_0+\dots+a_n=k \\ a_i\in\mathbf{N}}} \binom{k}{a_0,\dots,a_n}^2 \prod_{j=0}^{n} (2a_j)! \lambda_j^{2a_j} = (k!)^2 \sum_{\substack{a_0+\dots+a_n=k \\ a_i\in\mathbf{N}}} \prod_{j=0}^{n} \binom{2a_j}{a_j} \lambda_j^{2a_j} \\
&= (k!)^2 C^{2k} \Big\langle X^k, \prod_{j=0}^{n} \left(1 - 4 \left(\frac{\lambda_j}{C} \right)^2 X \right)^{-1/2} \Big\rangle ,
\end{aligned}$$

where C is some positive integer such that $2\lambda_j/C < 1$ for $j = 0, \dots, n$ and $< X^k, f(X) >$ denotes the coefficient of X^k in the Taylor expansion of f about $X = 0$. Consider now, for $u_0, \dots, u_n$ in $]-1, 1[$, the function

$$t \in]-1,1[\mapsto \Phi_u(t) := \prod_{j=0}^{n} (1 - u_j X)^{-1/2} = \sum_{k=0}^{\infty} \Phi_{u,k} t^k .$$

On has, in $]-1, 1[$,

$$\frac{\Phi'_u(t)}{\Phi_u(t)} = \frac{1}{2} \left(\sum_{j=0}^{n} \frac{u_j}{1 - u_j t} \right) = \frac{\Psi_u(t)}{2 \prod_{j=0}^{n} (1 - u_j t)} \tag{102}$$

where Ψ_u is a polynomial whose coefficients are symmetric polynomials in $u_0, \dots, u_n$. If we let

$$2 \prod_{j=0}^{n} (1 - u_j t) = \sum_{l=0}^{n+1} \sigma_{u,l} t^l \qquad \Psi_u(t) = \sum_{l=0}^{n} \tau_{u,l} t^l$$

we have, for any $k \in \mathbf{N}$, $k \geq n$,

$$\sum_{l=0}^{n} \tau_{u,l} \Phi_{u,k-l} = \sum_{l=0}^{n} \sigma_{u,l} (k + 1 - l) \Phi_{u,k+1-l} . \tag{103}$$

Since, for any positive integer k, we have

$$F_Q(k) = (k!)^2 C^{2k} \Phi_{u,k} ,$$

where $u_j = 4\lambda_j^2/C$, there is a difference operator with coefficients in $\mathbf{Z}[s]$,

$$\widetilde{\mathcal{P}} = \sum_{\beta=0}^{M} \tilde{p}_\beta(s)\Delta^{N-\beta},$$

such that the identity (101) holds for any k sufficiently large, and therefore for any $s \in \mathbf{C}$ if the identity is understood as an identity between meromorphic functions of s. The fact that the coefficients are in $\mathbf{Z}[s]$ follows from the fact that all coefficients $\sigma_{u,l}$, $\tau_{u,l}$ in (103) are symetric polynomials in $\lambda_0, \ldots, \lambda_n$.

The explicit formula for ζ_Q when all λ_j are equal to 1 up to m was discovered by Cassaigne and Maillot [CaM]. Let us derive it here in a slightly different way. From Carlson's theorem (as explained in [CaM]), it is enough to get a closed formula for $\zeta_Q(k)$, where k is a positive integer. From the fact that

$$\Phi_{1,\ldots,1,0,\ldots,0}(t) := (1-t)^{-\frac{m+1}{2}} = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{m+1}{2}+k)}{\Gamma(\frac{m+1}{2})\Gamma(k+1)} t^k\,,$$

we get that if $Q_{0,m}(X) = \sum_{j=0}^{m} X_j^2$, $0 \le m \le n$, then

$$F_{Q_{0,m}}(k) = \frac{\Gamma(k+1)\Gamma(\frac{m+1}{2}+k)4^k}{\Gamma(\frac{m+1}{2})}$$

from which it follows, if one uses the duplication formula for the Γ function ([GR, 8.335, p.938], that, for any s (the identity beeing an identity between meromorphic functions),

$$\zeta_{Q_{0,m}}(s) = \frac{n!\Gamma(m/2)\Gamma(\frac{s}{2}+1)\Gamma(s+m)}{\Gamma(n+1+s)\Gamma(m)\Gamma(\frac{m+s}{2})}$$

which is the result in [CassMa]. Let us now look at the second example, when $n \ge 2m+1$ and

$$Q(X) = \sum_{k=0}^{m} \lambda_k (X_{2k}^2 + X_{2k+1}^2)\,.$$

We may suppose the $\lambda_k \ge 0$. Consider the rational function

$$R(t) = \frac{1}{\prod_{k=0}^{m}(1-4\lambda_k^2 t)}$$

and its decomposition

$$R(t) = \sum_{j=1}^{q} \sum_{l=1}^{m_q} \frac{\alpha_{j,l}}{(1-4\lambda_j^2 t)^l} \tag{104}$$

where $\lambda_1, \dots, \lambda_q$ are the distinct elements in the sequence $\lambda_0, \dots, \lambda_m$ and $m_1, \dots, m_q$ the number of times they are repeated ($m_1 + \dots + m_q = m + 1$). We have in this case, for any $k \in \mathbf{N}^*$,

$$F_Q(2k) = \Gamma(k+1) \sum_{j=1}^{q} \sum_{l=1}^{m_q} \frac{\alpha_{j,l} (2\lambda_j)^{2k}\, \Gamma(l+k)}{\Gamma(l)}$$

from which we can deduce (using again the duplication formula) the following expression for $\zeta_Q(s)$,

$$\zeta_Q(s) = \frac{n!\Gamma\left(\frac{s}{2}+1\right)}{\Gamma(n+1+s)} \left(\sum_{j=1}^{q} \sum_{l=1}^{m_q} \alpha_{j,l} |\lambda_j|^s \frac{\Gamma(2l-1+s)\Gamma(l-\frac{1}{2})}{\Gamma(l+\frac{s-1}{2})\Gamma(2l+1)} \right) .$$

The proposition is completely proved.

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