# Residual kernels with singularities on coordinate planes

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**Abstract**

A finite collection of planes $\{E_\nu\}$ in $\mathbb{C}^d$ is called an atomic family if the top de Rham cohomology group of its complement is generated by a single element. A closed differential form generating this group is called a residual kernel for the atomic family. We construct new residual kernels in the case when $E_\nu$ are coordinate planes such that the complement $\mathbb{C}^d \setminus \bigcup E_\nu$ admits a toric action with the orbit space being homeomorphic to a compact projective toric variety. They generalize the well-known Bochner-Martinelli and Sorani differential forms. The kernels obtained are used to estabilish a new formula of integral representations for functions holomorphic in Reinhardt polyhedra.

## Introduction

The multidimensional residue theory as well as the theory of integral representations for holomorphic functions are grounded on several model differential forms called *kernels*. These are the Cauchy kernel in $\mathbb{C}^d$ whose singular set consists of all coordinate hyperplanes and the Bochner-Martinelli kernel with singularity at the origin, which is the zero-dimensional coordinate subspace of $\mathbb{C}^d$. Observe that these two model kernels have been the source of other fundamental kernels and concepts of residue by means of homological procedures.

For instance, starting with the Bochner-Martinelli formula in the domain $G \subset \mathbb{C}^d$, J. Leray was able to deduce a new integral representation formula (calling it the Cauchy-Fantappiè formula) by lifting the boundary $\partial G$ into a complex quartic in $\mathbb{C}^{2d}$ preserving its homology class [18]. Later this idea was developed in the works of W. Koppelman [15] and others [1, 19], resulting in new representation formulas for $\overline{\partial}$-closed forms.

Another fundamental integral formula in complex analysis, namely, the Weil formula, turns out to be related to the Cauchy formula by the transformation

[0] A. Shchuplev was supported by a grant of the President of the Russian Federation (project no. NSh-1212.2003.1). A. Yger and A. Tsikh were partially supported by the French Ministry of Education, through the P.A.S.T program. A. Tsikh thanks also RFBR (project no. 02-01-00167) for partial support.

law for the Grothendieck residue [24]. Moreover, the connection between those two formulas demonstrates how to produce various algebraic concepts of residue, starting from the residue for a Laurent series as the coefficient $c_{-I}$. Lastly, let us point at the Sorani integral kernels [23] regular in $(\mathbb{C}^p)_* \times (\mathbb{C}_*)^{d-p}$. These forms appear as intermediate differential forms between the Cauchy and Bochner-Martinelli kernels in the proof of the de Rham (or Dolbeault) isomorphism based on the Mayer-Vietoris cohomology principle [12].

Notice that the Cauchy and Bochner-Martinelli kernels together with a number of the Sorani kernels possess two common properties: firstly, their singular sets are the unions of coordinate planes, and secondly, the top cohomology group of the complement to a singular set is generated by a single element. The aim of the present paper is to construct new kernels with singularities on such unions of coordinate planes.

So, let $\{E_\nu\}$ be a non empty finite collection of planes (of arbitrary dimensions) in $\mathbb{C}^d$. Let us introduce the concept of atomicity for such a family and illustrate with some (positive and negative) examples.

**Definition.** *The family $\{E_\nu\}$ is said to be atomic if the top de Rham cohomology group $H^k(\mathbb{C}^d \setminus \bigcup_\nu E_\nu)$ is generated by one element:*

$$H^k\left(\mathbb{C}^d \setminus \bigcup_\nu E_\nu\right) \simeq \begin{cases} \mathbb{C} \ \textit{if} \ \ k = k_0, \\ 0 \ \textit{whenever} \ \ k > k_0. \end{cases}$$

*A generating form of the group $H^{k_0}$ is called a residual kernel for the atomic family $\{E_\nu\}$.*

**Examples:**

1) the family of all coordinate hyperplanes $\mathcal{E} = \{E_\nu\}_{1\le\nu\le d}$, where

$$E_\nu := \{\zeta \in \mathbb{C}^d : \zeta_j = 0\}$$

is atomic; in this case the complement $\mathbb{C}^d \setminus \bigcup_\nu E_\nu$ is the $d$-dimensional complex torus $(\mathbb{C}_*)^d = \mathbb{T}^d$, which is homotopically equivalent to the real $d$-dimensional torus, so for this example $k_0 = d$; thus, the Cauchy form

$$\eta_C = \frac{d\zeta_1}{\zeta_1} \wedge \cdots \wedge \frac{d\zeta_d}{\zeta_d}$$

is a kernel for the family of all coordinate hyperplanes.

2) the family $\mathcal{E} = \big\{\{0\}\big\}$ consisting of the single 0-dimensional coordinate plane $\{\zeta = 0\}$ is also atomic, since $\mathbb{C}^d \setminus \{0\}$ is homotopically equivalent to the $(2d-1)$-dimensional real sphere $\mathbb{S}^{2d-1}$, in this example $k_0 = 2d-1$; so, the Bochner-Martinelli form

$$\eta_{BM} = \frac{\sum_{k=1}^d (-1)^{k-1} \overline{\zeta}_k d\overline{\zeta}[k] \wedge d\zeta}{||\zeta||^{2d}}$$

is a kernel for $\mathcal{E} = \big\{\{0\}\big\}$.

3) the family $\mathcal{E} = \{l_1, l_2, l_3\}$ of coordinate lines

$$l_1 := \{\zeta_2 = \zeta_3 = 0\}, l_2 := \{\zeta_1 = \zeta_3 = 0\}, l_3 := \{\zeta_1 = \zeta_2 = 0\}$$

in $\mathbb{C}^3$ is not atomic. In order to make this observation clear let us note that by de Rham's theorem it is enough to show that the top homology group $H_k(\mathbb{C}^3 \setminus \bigcup_j l_j)$ with coefficients in $\mathbb{Z}$ cannot be generated by a single element. By the Alexander-Pontryagin duality (see [2]) for any $k \in \mathbb{N}$ one has

$$H_k(\mathbb{C}^3 \setminus \bigcup_j l_j) = \widetilde{H}_{6-k-1}(\overline{l_1 \cup l_2 \cup l_3}),$$

where $\overline{l_1 \cup l_2 \cup l_3}$ denotes the closure of the lines $l_j$ in the compactification $\mathbb{S}^6$ of $\mathbb{C}^3$ and $\widetilde{H}_k$ denote the reduced homology groups; and since $\overline{l_1 \cup l_2 \cup l_3}$ consists of three 2-dimensional spheres with two common points (namely 0 and $\infty$), it is easy to see that

$$\widetilde{H}_{6-k-1}(\overline{l_1 \cup l_2 \cup l_3}) \simeq \begin{cases} \mathbb{Z}^2 \text{ if } k = 4, \\ 0 \text{ whenever } k > 4, \end{cases}$$

which proves $\{l_1, l_2, l_3\}$ is not an atomic family.

We shall proceed in the following order. First, we shall show that a class of atomic families of cordinate planes is delivered by the exceptional sets $Z(\Sigma)$ associated with fans (conical polyhedra) $\Sigma$ in $\mathbb{R}^n$ (Proposition 1). These sets appear in the representation of toric varieties $X_\Sigma$ [4] as quotients $(\mathbb{C}^d \setminus Z(\Sigma))\big/G$, which generalize the well-known representation $\mathbb{C}^{n+1} \setminus \{0\}\big/\sim$ of the complex projective space $\mathbb{P}_n$.

The construction of kernels for $Z(\Sigma)$ is given in Theorem 2. The prototype of our kernels is the Bochner-Martinelli form $\eta_{BM}$, which combines the Fubini-Study volume form on $\mathbb{P}_n$ and the one-dimensional Cauchy kernel [11, 20], i.e. up to a costant non-zero factor

$$\begin{aligned}\eta_{BM} &= \omega(\zeta) \wedge \frac{d\zeta_{n+1}}{\zeta_{n+1}} = \\ &= \frac{1}{n!}\left(\mathrm{dd^c} \ln \left(1 + \left|\frac{\zeta_1}{\zeta_{n+1}}\right|^2 + \cdots + \left|\frac{\zeta_n}{\zeta_{n+1}}\right|^2\right)\right)^n \wedge \frac{d\zeta_{n+1}}{\zeta_{n+1}}.\end{aligned}$$

In general case, our kernel is the exterior product of a volume form of $X_\Sigma$ (in homogeneous coordinates) and the $r$-dimensional Cauchy kernel. The key element of the proof of this result is Theorem 1 on the multidimensional perspective. According to this theorem, a toric variety $X_\Sigma$ can be attached to the affine space $\mathbb{C}^d$ at infinity as a subvariety of codimension $r$ in such a way that the orbit $G \cdot \zeta$ of each point $\zeta \in \mathbb{C}^d \setminus Z(\Sigma)$ intersects $X_\Sigma$ in a unique point. Such an embedding of $X_\Sigma$ allows us to present a non-trivial cycle of maximal dimension in $\mathbb{C}^d \setminus Z(\Sigma)$ and integrate the kernel over it. This cycle turns out to be homologous to $\mu^{-1}(\rho)$ where $\mu$ is the moment map and $\rho$ is taken from the Kähler cone of $X_\Sigma$.

Section 4 is devoted to the construction of specific volume forms $\omega$ on projective toric varieties induced by the Fubini-Study metrics. These forms are relatively simple, and the volumes of toric varieties with respect to such forms are easily computed.

In the final section, we formulate and prove an integral representation formula for holomorphic functions in special Reinhardt domains.

It should also be mentioned that the results were partially announced in [25] and [22].

# 1. The atomicity coming from toricity

We shall give a sufficient combinatorial condition for the formulated atomicity property to hold. This combinatorial condition allows us to equip the complement of an atomic family with a group action that provides a special representation of a toric variety [9] in terms of homogeneous coordinates.

In further detail, every compact $n$-dimensional toric variety $X = X_\Sigma$ is defined by a complete fan $\Sigma$ in $\mathbb{R}^n$. By a fan $\Sigma$ is meant a set of rational strongly convex polyhedral cones $\sigma$ such that

(1) each face of a cone in $\Sigma$ is also a cone in $\Sigma$;

(2) the intersection of two cones in $\Sigma$ is a face of each.

It is said that a fan is *complete* if its support $|\Sigma|$ covers the whole space $\mathbb{R}^n$. Let $v_1, \dots, v_d \in \mathbb{Z}^n \subset \mathbb{R}^n$ be primitive generators of one-dimensional cones in $\Sigma$. A cone $\sigma$ of $\Sigma$ is called *simple* if its generators form a part of a basis for the lattice $\mathbb{Z}^n$. Let us denote also by $\Sigma(k)$ the set of all $k$-dimensional cones in $\Sigma$. Following the construction in [4], to each one-dimensional cone of $\Sigma$ (or to each $v_j$, $j = 1, \dots, d$) we assign a complex variable $\zeta_j$, $j = 1, \dots, d$, so we obtain the space $\mathbb{C}^d_{\zeta_1,\dots,\zeta_d}$ of *homogeneous coordinates* for the toric variety $X_\Sigma$ corresponding to the fan $\Sigma$. For any cone $\sigma \in \Sigma$, let $\zeta_{\hat\sigma}$ be the monomial

$$\zeta_{\hat\sigma} := \prod_{v_j \notin \sigma} \zeta_j$$

and let $Z(\Sigma)$ be the set

$$Z(\Sigma) := \{\zeta \in \mathbb{C}^d : \zeta_{\hat\sigma} = 0 \ \forall \sigma \in \Sigma(n)\}. \tag{1}$$

The set $Z(\Sigma)$ assumes the role of the origin $\{0\}$ in the representation of the projective space as the quotient of $\mathbb{C}^d \setminus \{0\}$ by the diagonal action of the complex torus $\mathbb{T}$. To define a group action in the analogous construction of a toric variety $X_\Sigma$, one utilises the Chow group $A_{n-1}(X_\Sigma)$ of Weil divisors modulo rational equivalence on $X_\Sigma$. The Chow group is given by the exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\ \alpha\ } \mathbb{Z}^d \xrightarrow{\ \beta\ } A_{n-1}(X_\Sigma) \longrightarrow 0, \tag{2}$$

where $\alpha$ sends $m \in \mathbb{Z}^n$ to $(\langle m, v_1\rangle, \dots, \langle m, v_d\rangle)$. Then any element $g$ of the group $G$ defined to be

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{T})$$

acts on $\zeta \in \mathbb{C}^d \setminus Z(\Sigma)$ according to $g \cdot \zeta = (g([D_i])\zeta_i)$ with $[D_i]$, $i = 1, \dots, d$ being the classes of the basis elements of $\mathbb{Z}^d$ in $A_{n-1}(X_\Sigma)$. It follows from [4] that the toric variety $X_\Sigma$ can be represented as a geometrical quotient

$$(\mathbb{C}^d \setminus Z(\Sigma)) \Big/ G.$$

**Remark 1.** Let us say some words on the structure of $G$ and homogeneous functions (with respect to $G$) on $\mathbb{C}^d \setminus Z(\Sigma)$. When $A_{n-1}(X_\Sigma)$ has no torsion, the group $G$ is given by the $r$-parametric surface

$$G = \{g = (\lambda_1^{a_{11}} \dots \lambda_r^{a_{r1}}, \dots, \lambda_1^{a_{1d}} \dots \lambda_r^{a_{rd}}) \colon \lambda_i \in \mathbb{T}\} \subset \mathbb{T}^d,$$

where $r = d - n$ and the group action $g \cdot \zeta$ is the usual product in $\mathbb{T}^d$. The exponents $a_{ij}$ are chosen in a way that

$$\begin{cases} a_{11}v_1 + \dots + a_{1d}v_d = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}v_1 + \dots + a_{rd}v_d = 0 \end{cases} \tag{3}$$

is the lattice (basis) of relations between the generators $v_i$ of $\Sigma$.

The degree of the monomial $\zeta^a$ from $\mathbb{C}[\zeta_1, \dots, \zeta_d]$ is defined to be the class $\beta(a) \in A_{n-1}(X_\Sigma)$ of the exponent $a \in \mathbb{Z}^d$ (see (2)). Therefore two monomials $\zeta^a$ and $\zeta^b$ have the same degree if and only if $\alpha - \beta$ is equal to $\alpha(m)$ for some $m \in \mathbb{Z}^n$.

For the simplicial fans there is an equivalent construction of the set $Z(\Sigma)$ [3]. A subset of generators $\{v_i,\, i \in I\}$ as well as the index set $I \subset \{1, \dots, d\}$ is called a *primitive collection* if the vectors $v_i$ do not generate any cone of $\Sigma$ but every proper subset of them does so. All primitive collections $I$ constitute a set that we shall denote by $\mathcal{P}$. Then by [3] the set $Z(\Sigma)$ coincides with the union of coordinate planes

$$Z(\Sigma) = \bigcup_{I \in \mathcal{P}} E_I,$$

where $E_I = \{\zeta_{i_1} = \dots = \zeta_{i_k} = 0\}$ and the union is taken over all primitive collections.

For collections of coordinate planes coming from simplicial complete fans the following assertion holds.

**Proposition 1.** *The collection $\{E_I\}_{i \in \mathcal{P}}$ is an atomic family.*

*Proof.* Since $\Sigma$ is a complete fan, the set $\mathbb{C}^d \setminus Z(\Sigma)$ is a bundle over the *compact* $n$-dimensional complex simplicial toric variety $X_\Sigma$, with fibers being homeomorphic to several copies of the torus $\mathbb{T}^r$ [4]. In other words, $\mathbb{C}^d \setminus Z(\Sigma)$ is homotopically equivalent to some oriented compact real analytic cycle of real dimension $2n + r = d + n$ that one can interpret as a bundle over connected compact oriented manifold $X_\Sigma$ with several disjoint copies of the real torus $\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{r \text{ times}}$ as a fiber. $\square$

**Example 1.** The family of all codimension 2 planes in $\mathbb{C}^d$

$$E_{ij} = \{\zeta_i = \zeta_j = 0\}, \quad 1 < |i - j| < d$$

is atomic. Indeed, let $\Sigma$ be any fan in $\mathbb{R}^2$ with $d$ generators. Let us number them in the counterclockwise order, then every pair $\{i, j\}$, $1 \le i < j \le d$ becomes a primitive collection unless $v_i$ and $v_j$ are contiguous.

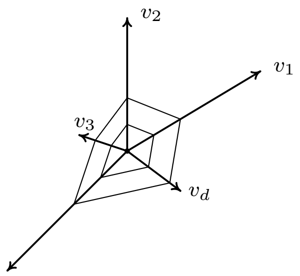


Fig.1. A fan in $\mathbb{R}^2$.

**Remark 2.** It would be interesting to deduce Proposition 1 from the result of Goresky-MacPherson [10] (see also [7]), which could help to estabilish a general criterion for the atomicity property to hold.

# 2. Multidimensional perspective

The Renaissance (linear or one-dimensional) perspective has given rise to the notion of projective space $\mathbb{P}_n$ as the set of all lines $\{l\}$ in an affine space $\mathbb{A}^{n+1}$ passing through the center point. Moreover $\mathbb{P}_n$ can be attached 'at the infinity' to the space $\mathbb{A}^{n+1}$ to form $\mathbb{P}_{n+1} = \mathbb{A}^{n+1} \sqcup \mathbb{P}_n$ in such a way that the closure $\bar{l}$ in $\mathbb{P}_{n+1}$ of every line intersects the attached $\mathbb{P}_n$ in a single point corresponding to $l$. In this section we shall prove an analogous result for toric varieties. Note that with the exception of the weighted projective spaces, all toric varieties are defined as spaces of $r$-dimensional orbits where $r \ge 2$. This allows us to interprete this result as the *multidimensional perspective*.

Assume that the fan $\Sigma \in \mathbb{R}^n$ contains at least one simple $n$-dimensional cone. With this assumption made, we prove the following

**Theorem 1.** *Let $\Sigma$ be a simplicial complete fan in $\mathbb{R}^n$ with $d$ generators. There exists a $d$-dimensional simplicial and compact toric variety*

$$X_{\widetilde{\Sigma}} = \mathbb{C}^d \sqcup (\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r)$$

*with 'infinite' toric hypersurfaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ such that its 'skeleton' $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ is isomorphic to $X_\Sigma$. Moreover, for every $\zeta \in \mathbb{C}^d \setminus Z(\Sigma) \subset X_{\widetilde{\Sigma}}$ the closure $\overline{G \cdot \zeta}$ of its orbit in $X_{\widetilde{\Sigma}}$ intersects 'the skeleton of infinity' in a unique point corresponding to the class of $\zeta$ under the isomorphism.*

To illustrate this assertion one can consider an embedding of $\mathbb{P}_1 \times \mathbb{P}_1$ into

$$\mathbb{P}_2 \times \mathbb{P}_2 = \mathbb{C}^4 \sqcup (\mathcal{X}_1 \cup \mathcal{X}_2)$$

as the intersection of two infinite hypersurfaces $\mathcal{X}_1 = \mathbb{P}_1^\infty \times \mathbb{P}_2$ and $\mathcal{X}_2 = \mathbb{P}_2 \times \mathbb{P}_1^\infty$. The fan $\Sigma$ that corresponds to $\mathbb{P}_1 \times \mathbb{P}_1$has four generators (see Fig.2), and the set $Z(\Sigma) = E_{13} \cup E_{24}$ with $E_{13} = \{\zeta_1 = \zeta_3 = 0\}$ and $E_{24} = \{\zeta_2 = \zeta_4 = 0\}$ in $\mathbb{C}^4$. The relative arrangement of these objects is depicted on Fig. 3 where the orbit $G \cdot \zeta$ is specified by the shadowed area and 'the skeleton of infinity' $X_\Sigma = \mathcal{X}_1 \cap \mathcal{X}_2$ attached to $\mathbb{C}^4$ is the ridge.

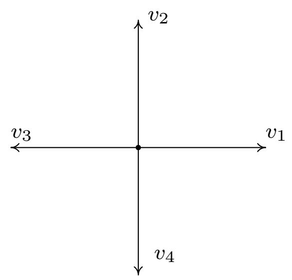


Fig.2.

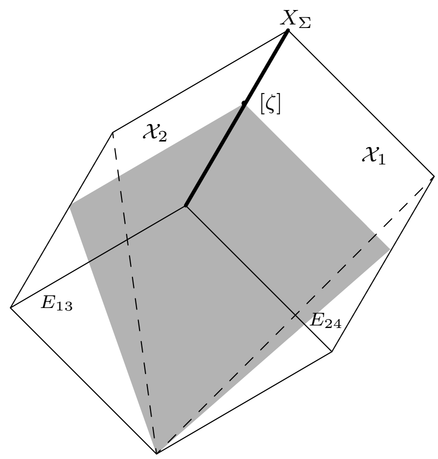


Fig.3.

*Proof.* Let us construct the fan $\widetilde{\Sigma}$ in $\mathbb{R}^d$ (and by that the corresponding toric variety $X_{\widetilde{\Sigma}}$) starting from the fan $\Sigma$.

Without loss of generality, we may assume that the simple cone in $\Sigma$, mentioned in the theorem, is generated by the canonical basis $v_1 = e_1, \ldots, v_n = e_n$ of $\mathbb{Z}^n \subset \mathbb{R}^n$. Consider in $\mathbb{Z}^d = \mathbb{Z}^{n+r}$ the following $d + r$ primitive vectors:

$$\begin{aligned} \widetilde{e}_1 = (e_1, 0''), \quad \ldots \quad &, \widetilde{e}_n = (e_n, 0''), \\ \widetilde{e}_{n+1} = (0', e_1''), \quad \ldots \quad &, \widetilde{e}_{n+r} = (0', e_r''), \\ \widetilde{v}_{n+1} = (v_{n+1}, -e_1''), \quad \ldots \quad &, \widetilde{v}_{n+r} = (v_{n+r}, -e_r''), \end{aligned} \tag{4}$$

where $(e''_1, \ldots, e''_r)$ denotes the canonical basis of $\mathbb{R}^r$ and $0'$, $0''$ the neutral elements in $\mathbb{Z}^n$ and $\mathbb{Z}^r$, respectively. These $n+2r$ distinct vectors span $\mathbb{R}^d = \mathbb{R}^{n+r}$ as a vector space; they are going to form the set of 1-dimensional cones of our $d$-dimensional fan $\widetilde{\Sigma}$. We need then to describe how to organize the cones of the fan $\widetilde{\Sigma}$, starting with this collection of generators.

Let $I := \{1, ..., n\}$ and $J := \{n+1, ..., n+r\}$. The prescription rules for the organization of $d$-dimesional cones suggest the following three steps:

- choose any $n$-dimensional cone $\sigma = \langle v_{m_1}, ..., v_{m_n} \rangle$ in the fan $\Sigma$ and divide the set of indices $\{m_1, ..., m_n\}$ into

$$K := \{m_1, ..., m_n\} \cap I, \quad L := \{m_1, ..., m_n\} \cap J;$$

- divide (in some arbitrary way) the complement $J \setminus L$ into two disjoint subsets $Q$ and $S$, so that one gets an ordered partition $\{Q, S\}$ of $J \setminus L$;
- consider the $d$-dimensional cone $\widetilde{\sigma} = \langle \tilde{e}_K, \tilde{e}_L, \tilde{e}_Q, \tilde{v}_S, \tilde{v}_L \rangle$.

Define now the fan $\widetilde{\Sigma}$ as the collection of all such $d$-dimensional cones $\widetilde{\sigma}$ together with all their faces. The following proposition holds.

**Proposition 2.** *The collection $\widetilde{\Sigma}$ is a complete simplicial fan in $\mathbb{R}^{n+r}$; furthermore, if $\Sigma$ is simple so is $\widetilde{\Sigma}$.*

*Proof of the proposition.* Let us note first that the cardinals $|K|$, $|L|$, $|Q|$, $|S|$ satisfy

$$|K| + |L| = n \ , \ |L| + |Q| + |S| = r,$$

and all $d$ generators of a cone $\widetilde{\sigma}$ are linearly independent, so every such cone is $d$-dimensional and simplicial.

The proof reduces to the verification of three points.

**1.** *$\widetilde{\Sigma}$ is a fan.*

To prove this, it is enough to show that if one of the generators of $\widetilde{\Sigma}$ (taken from the list (4)) is not an edge in some $d$-dimensional cone $\widetilde{\sigma}$, then it does not belong to this cone. Let us take such a cone $\widetilde{\sigma}$ and order all the index sets obtained. Write the coordinates of all the cone's generators (with respect to the basis $(\widetilde{e}_1, \ldots, \widetilde{e}_{n+r})$ into the rows of the $(n+r) \times (n+r)$-matrix

$$
\begin{array}{c|c|c|c|c|}
 & \overbrace{\qquad\qquad}^{n} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{r}} \\
\hline
\widetilde{e}_K & e_K & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} \\
\hline
\widetilde{e}_L & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{matrix} \\
\hline
\widetilde{e}_Q & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} \\
\hline
\widetilde{v}_S & v_S & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} -1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -1 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} \\
\hline
\widetilde{v}_L & v_L & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} -1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -1 \end{matrix} \\
\hline
 & I & Q & S & L
\end{array}
$$

The generators that are not used as edges in the construction of the cone are:

$$
\begin{array}{ll}
(a) & \widetilde{e}_{I\setminus K}; \\
(b) & \widetilde{e}_S; \\
(c) & \widetilde{v}_Q.
\end{array}
$$

Consider cases a), b), c) separately.

a) Let $i \in I \setminus K$. If we assume that $\widetilde{e}_i \in \widetilde{\sigma}$ then in its representation as a non-negative linear combination of generators of $\widetilde{\sigma}$ the coefficients at $\widetilde{v}_S$ must be zeroes. It follows from the fact that the $S$-coordinates of $\widetilde{e}_i$ are zero, but at the same time in the $S$-column of the matrix for $\widetilde{\sigma}$ non-zero elements stand only on the corresponding diagonal of the block $\widetilde{v}_S$-row and they are negative. Hence, the vector $e_i$ must belong to the cone $\langle e_K, v_L\rangle$ of the fan $\Sigma$ that contradicts the definition of a fan and the assumption that $i \in I \setminus K$.

b) If $\widetilde{e}_s \in \widetilde{\sigma}$ then in its representation as a non-negative linear combination of generators of $\widetilde{\sigma}$ the coefficients at $v_s$ must be $-1$, which is impossible.

c) This case is analogous to the previous one, it is enough to consider the $Q$-column.

**2.** *If $\Sigma$ is primitive then so is $\widetilde{\Sigma}$.*

Add $\widetilde{e}_L$-rows to the $\widetilde{v}_L$-rows of the matrix for $\widetilde{\sigma}$. Then it is easy to see that the determinant

$$\det \widetilde{\sigma} = \det \begin{pmatrix} e_K \\ v_L \end{pmatrix} = \det \sigma$$

is equal to $\pm 1$ in the case $\Sigma$ is primitive.

**3.** *Completeness of $\Sigma$ implies completeness of $\widetilde{\Sigma}$.*

In order to prove this it is enough to show that each $(d-1)$-dimensional face of any $d$-dimensional cone $\widetilde{\sigma}$ of $\widetilde{\Sigma}$ is a face of some other $d$-dimensional cone of $\widetilde{\Sigma}$.

There are five types of $(d-1)$-dimensional faces for an arbitrary $d$-cone $\widetilde{\sigma}$ as follows

$$\begin{aligned}
\tau_k &= \langle \tilde{e}_{K\setminus k},\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ k \in K; \\
\tau_l &= \langle \tilde{e}_K,\, \tilde{e}_{L\setminus l},\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ l \in L; \\
\tau_q &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_{Q\setminus q},\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ q \in Q; \\
\tau_s &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_{S\setminus s},\, \tilde{v}_L \rangle,\ s \in S; \\
\tau'_l &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_{L\setminus l} \rangle,\ l \in L.
\end{aligned}$$

Consider each of these types separately.

1) *faces of the type $\tau_k$.* The sets of indices $K \subset I$ and $L \subset J$ are so definied that $\sigma = \langle e_K,\, v_L \rangle$ is an $n$-dimensional cone of $\Sigma$. The cone $\langle e_{K\setminus k},\, v_L \rangle$ is then a face of $\sigma$ and since the fan $\Sigma$ is complete, this cone must be a face of some other $n$-dimensional cone of the fan. We distinguish two cases

i) $\langle e_{K\setminus k},\, v_L \rangle \in \Sigma$ is a face of a cone of type $\langle e_{(K\setminus k)\cup i},\, v_L \rangle \in \Sigma$, $k \in K$, $i \in I$;

ii) $\langle e_{K\setminus k},\, v_L \rangle \in \Sigma$ is a face of a cone of type $\langle e_{K\setminus k},\, v_{L\cup j} \rangle \in \Sigma$, $k \in K$, $j \in J \setminus L$.

In the first case the cone $\tau_k$ is a face of the cone

$$\langle \widetilde{e}_{(K\setminus k)\cup i},\, \widetilde{e}_L,\, \widetilde{e}_Q,\, \widetilde{v}_S,\, \widetilde{v}_L \rangle.$$

In the second case it is a face of the cone

$$\langle \widetilde{e}_{K\setminus k},\, \widetilde{e}_{L\cup j},\, \widetilde{e}_{Q\setminus j},\, \widetilde{v}_S,\, \widetilde{v}_{L\cup j} \rangle \text{ if } j \in Q,$$

(this face is obtained by the elimination of $\widetilde{v}_j$ and renaming of vectors $\widetilde{e}_{L\cup j}$ and $\widetilde{e}_{Q\setminus j}$ by $\widetilde{e}_L$ and $\widetilde{e}_Q$) and of the cone

$$\langle \widetilde{e}_{K\setminus k},\, \widetilde{e}_{L\cup j},\, \widetilde{e}_Q,\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{L\cup j} \rangle \text{ if } j \in S$$

(by the same reasoning).

Obviously, each of these three $d$-dimensional cones does not coincide with $\widetilde{\sigma}$ and belongs to $\widetilde{\Sigma}$, for instance, the last cone belongs to $\widetilde{\Sigma}$, because by the assumption $\langle e_{K\setminus k},\, v_{L\cup j} \rangle$ is a cone of $\Sigma$ and $\{Q,\, S\setminus j\}$ is a partition of $J\setminus(L\cup j)$ (recall that $(Q,\, S)$ is a partition of $J \setminus L$ by the definition of $\widetilde{\sigma}$).

2) *faces of the type $\tau_l$.* There are two cases:

iii) $\langle e_K,\, v_{L\setminus l}\rangle \in \Sigma$ is a face of a cone of type $\langle e_{K\cup i},\, v_{L\setminus l}\rangle \in \Sigma$, $l \in L$, $i \in I\setminus K$.

iv) $\langle e_K,\, v_{L\setminus l}\rangle \in \Sigma$ is a face of a cone of type $\langle e_K,\, v_{(L\setminus l)\cup j}\rangle \in \Sigma$, $l \in L$, $j \in J \setminus K$.

In case iii) the cone $\tau_l$ is a face of the cone

$$\langle \widetilde{e}_{K\cup i},\, \widetilde{e}_{L\setminus l},\, \widetilde{e}_Q,\, \widetilde{v}_{S\cup l},\, \widetilde{v}_{L\setminus l}\rangle.$$

In the latter it is a face of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{Q\setminus j},\, \widetilde{v}_S,\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j \in Q$$

and of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_Q,\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j \in S.$$

All these three $d$-dimensional cones do not coincide with $\widetilde{\sigma}$.
3) _faces of the type $\tau_q$._ The cone $\tau_q$ is a face of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_L,\, \widetilde{e}_{Q\setminus q},\, \widetilde{v}_{S\cup q},\, \widetilde{v}_L\rangle,$$

which differs from $\widetilde{\sigma}$.
4) _faces of the type $\tau_s$._ The cone $\tau_s$ is a face of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_L,\, \widetilde{e}_{Q\cup s},\, \widetilde{v}_{S\setminus s},\, \widetilde{v}_L\rangle,$$

which is different from $\widetilde{\sigma}$ as well.
5) _faces of the type $\tau_l'$._ The proof is similiar to the case 2) of faces of the type $\tau_l$. Two possibilities, iii) and iv), are conceivable. In case iii) the cone $\tau_l'$ is a face of the cone

$$\langle \widetilde{e}_{K\cup i},\, \widetilde{e}_{L\setminus l},\, \widetilde{e}_{Q\cup l},\, \widetilde{v}_S,\, \widetilde{v}_{L\setminus l}\rangle,$$

and in case iv) it is a face of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{(Q\cup l)\setminus j},\, \widetilde{v}_S,\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j \in Q$$

and of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{Q\cup l},\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j \in S.$$

$\square$

Now we can finish the proof of theorem. Compare the actions of the groups $G$ and $\widetilde{G}$ on $\mathbb{C}^d \setminus Z(\Sigma)$ and $\mathbb{C}^{d+r} \setminus Z(\widetilde{\Sigma})$, respectively. Note that every relation

$$\mu_1 v_1 + \cdots + \mu_d v_d = 0$$

between the generators $v_1, \ldots, v_d$ of the fan $\Sigma$ defines the relation

$$\mu_1 \widetilde{v}_1 + \cdots + \mu_d \widetilde{v}_d + \mu_{n+1}\widetilde{v}_{d+1} + \cdots + \mu_{n+r}\widetilde{v}_{d+r} = 0$$

between the generators

$$\widetilde{v}_i = \widetilde{e}_i,\ i = 1, \ldots, n,\ \widetilde{v}_{d+j} = \widetilde{e}_{n+j},\ j = 1, \ldots, r,\ \widetilde{v}_k,\ k = n+1, \ldots, d$$

of the fan $\widetilde{\Sigma}$ (see (2.1)). So, if a basis of relations between $v_1, \ldots, v_d$ consists of the vectors

$$\begin{aligned}&\mu^1 = (\mu_{11}, \dots, \mu_{1d}) = p^1 \oplus q^1 \in \mathbb{Z}^n \oplus \mathbb{Z}^r_{\geq},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\mu^r = (\mu_{r1}, \dots, \mu_{rd}) = p^r \oplus q^r \in \mathbb{Z}^n \oplus \mathbb{Z}^r_{\geq},\end{aligned}$$

then the vectors

$$\begin{aligned}&\widetilde{\mu}^1 = p^1 \oplus q^1 \oplus q^1,\\ &\dots\dots\dots\dots\dots\dots\\ &\widetilde{\mu}^r = p^r \oplus q^r \oplus q^r\end{aligned}$$

can be chosen so that they constitute a basis of relations between $\widetilde{v}_1, \dots, \widetilde{v}_{d+r}$.

Note that $q^1, \dots, q^r$ can be chosen such that they form an identity matrix. Assign a complex variable to each of generators of the fan $\widetilde{\Sigma}$ in the following way

$$\begin{aligned}\widetilde{e}_j &\longleftrightarrow \xi_j,\ j = 1, \dots, d,\\ &\dots\dots\dots\dots\dots\dots\\ \widetilde{v}_{n+j} &\longleftrightarrow \xi_{d+j},\ j = 1, \dots, r.\end{aligned}$$

Therefore the group actions occuring in the definitions of the varieties $X_\Sigma$ and $X_{\widetilde{\Sigma}}$ are as follows:

$$G = (\lambda^p,\, \lambda^q) = (\lambda_1^{p_1^1} \dots \lambda_r^{p_1^r}, \dots, \lambda_1^{p_n^1} \dots \lambda_r^{p_n^r},\, \lambda_1^{q_1^1} \dots \lambda_r^{q_1^r}, \dots, \lambda_1^{q_r^1} \dots \lambda_r^{q_r^r}),$$

$$\widetilde{G} = (\lambda^p,\, \lambda^q,\, \lambda^q).$$

Consider now the coordinate chart $\widetilde{U} \simeq \mathbb{C}^d$ of the variety $X_{\widetilde{\Sigma}}$ corresponding to the cone $\langle \widetilde{e}_I, \widetilde{e}_J\rangle$ with local coordinates $\zeta = (\zeta_1, \dots, \zeta_d)$. In the homogeneous coordinates $\xi$ of $X_{\widetilde{\Sigma}}$ this chart is defined by the condition $\xi_{d+1} \neq 0, \dots, \xi_{d+r} \neq 0$, and every class in $\widetilde{U}$ has a representative of the kind $(a, 1, \dots, 1)$ in $\mathbb{C}^{d+r} \setminus Z(\widetilde{\Sigma})$.

Let $a \in \mathbb{C}^d \setminus Z(\Sigma) \subset \widetilde{U} \subset X_{\widetilde{\Sigma}}$ then for every fixed $g = (\lambda^p, \lambda^q) \in G$ and the corresponding element $\widetilde{g} = (\lambda^p, \lambda^q, \lambda^q) \in \widetilde{G}$ one has

$$g \cdot a = [(g \cdot a,\, 1, \dots, 1)]_{\widetilde{G}} = [\widetilde{g} \cdot (a,\, \lambda^{-q})]_{\widetilde{G}} = [(a,\, \lambda^{-q})]_{\widetilde{G}},$$

where $[\cdot]$ denotes the class of an element by the action of the group in the lower index. The fact that $\lambda^{-q} = (\lambda_1^{-1}, \dots, \lambda_r^{-1})$ implies that

$$\lim_{\substack{\lambda_1 \to \infty\\ \dots\\ \lambda_r \to \infty}} g \cdot a = \lim_{\substack{\lambda_1 \to \infty\\ \dots\\ \lambda_r \to \infty}} [(a,\, \lambda^{-q}])]_{\widetilde{G}} = [(a,\, 0, \dots, 0)]_{\widetilde{G}}.$$

Denoting

$$\mathcal{X}_j = \{\xi_{d+j} = 0\},\ j = 1, \dots, r$$

we conclude that the closure of the orbit $G \cdot a$ of a point $a \in \mathbb{C}^d \setminus Z(\Sigma)$ intersects

$$\mathcal{X}_1 \cap \dots \cap \mathcal{X}_r = \{\xi_{d+1} = \dots = \xi_{d+r} = 0\}$$

in a unique point $[(a, 0, \dots, 0)]_{\widetilde{G}} = ([a]_G, 0, \dots, 0)$ corresponding to the class $[a]_G \in X_\Sigma$. On the other hand, every point $a \in \mathbb{C}^d \setminus Z(\Sigma)$ canonically corresponds a unique point $(a, 0, \dots, 0) \in \mathbb{C}^{d+r} \setminus Z(\widetilde{\Sigma})$ such that there is an isomorphism

$$X_\Sigma \simeq \mathcal{X}_1 \cap \dots \cap \mathcal{X}_r.$$

This completes the proof of Theorem 1. □

## 3. The construction of residual kernels

Let us describe residual kernels associated to complete projective toric varieties, by which we mean kernels with singularities on $Z(\Sigma)$. We assume that the fan $\Sigma$ has $d$ primitive generators and at least one simple $n$-dimensional cone.

The construction of kernels that we present involves volume form of the toric variety. In general, such a variety is not smooth and by a *volume form* on a compact $n$-dimensional toric variety $X_\Sigma$ we mean a differential $(n, n)$-form $\omega$ that is regular in smooth points of $X_\Sigma$, positive and integrable in $\mathbb{T}^n \subset X_\Sigma$. In the homogeneous coordinates $\zeta$ of $X_\Sigma$ is given by

$$\omega(\zeta) = g(\zeta, \overline{\zeta})\overline{E(\zeta)} \wedge E(\zeta),$$

where $E(\zeta)$ denotes the Euler form [5]

$$E(\zeta) = \sum_{|I|=n} \det(v_I)\zeta_{\hat{I}}\, d\zeta_I$$

and $g(\zeta, \overline{\zeta})$ is a $C^\infty$-function in $\mathbb{C}^d \setminus Z(\Sigma)$ of appropriate degree [4] such that it makes the form $\omega$ to be $G$-invariant.

Denoting the set $\mathbb{C}^d \setminus \{\zeta_{\hat{\sigma}} = 0\}$ by $U_\sigma$ and taking into account (1), we have the canonical covering of the complement to $Z(\Sigma)$

$$\mathbb{C}^d \setminus Z(\Sigma) = \bigcup_{\sigma \in \Sigma(n)} U_\sigma.$$

In accordance with notation $\zeta_{\hat{\sigma}}$, for every $n$-dimensional cone $\sigma$ we introduce the differential $r$-form

$$\frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} = \bigwedge_{v_j \notin \sigma} \frac{d\zeta_j}{\zeta_j}$$

regular in $U_\sigma$.

**Theorem 2.** *Let $\omega(\zeta)$ be a volume form of $X_\Sigma$ given in homogeneous coordinates. Then the list of forms*

$$\eta_\sigma(\zeta) = \omega(\zeta) \wedge \frac{1}{\det \sigma} \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} \tag{5}$$

*defined in $U_\sigma$ for every $\sigma \in \Sigma(n)$ constitutes a globally defined, closed and regular $(d, n)$-form in $\mathbb{C}^d \setminus Z(\Sigma)$ being a kernel for $Z(\Sigma)$.*

*Proof.* The exterior product $E(\zeta)\wedge\frac{d\zeta_{\hat\sigma}}{\zeta_{\hat\sigma}}$ consists of a single term corresponding to the index set of generators of $\sigma$:

$$E(\zeta)\wedge\frac{d\zeta_{\hat\sigma}}{\zeta_{\hat\sigma}}=\det\sigma\, d\zeta.$$

Consequently, the form $\eta_\sigma(\zeta)$ in $U_\sigma$ equals

$$\eta_\sigma(\zeta)=\omega(\zeta)\wedge\frac{1}{\det\sigma}\frac{d\zeta_{\hat\sigma}}{\zeta_{\hat\sigma}}=g(\zeta,\,\overline{\zeta})\overline{E(\zeta)}d\zeta,$$

which shows that $\eta_\sigma$ does not depend on the choice of the cone $\sigma$ and the list $\{\eta_\sigma\}$, $\sigma\in\Sigma(n)$ constitutes a global form regular in the union of all $U_\sigma$, which is $\mathbb{C}^d\setminus Z(\Sigma)$.

The differential of $\eta(\zeta)$ in some $U_\sigma$ is obviously equal to $\overline{\partial}\omega(\zeta)\wedge\frac{1}{\det\sigma}\frac{d\zeta_{\hat\sigma}}{\zeta_{\hat\sigma}}$. Taking into account that $\omega(\zeta)$ comes from an $(n,\,n)$-form on an $n$-dimensional variety, which implies that its differential is zero, we conclude that the form $\eta(\zeta)$ is closed.

Recall that the set $\mathbb{C}^d\setminus Z(\Sigma)$ is homotopically equivalent to an oriented compact real analytic cycle $\gamma$ of real dimension $d+n$ (see Theorem 1 or[4]). By Proposition 1, this cycle generates the top homology group of the complement to $Z(\Sigma)$. Hence, the obtained differential $(d,\,n)$-form $\eta(\zeta)$ has the right degree and is a kernel, provided that it is not cohomologically trivial.

To compute the integral

$$\int_\gamma \eta(\zeta)$$

we regard the space $\mathbb{C}^d$, where the form $\eta(\zeta)$ lives, as the chart $\widetilde{U}$ in the toric variety $X_{\widetilde\Sigma}$. This chart is the affine toric variety $X_{\widetilde\sigma}$ that corresponds to the cone $\widetilde\sigma=\langle\widetilde{e}_I,\,\widetilde{e}_J\rangle=\mathbb{R}^d_+$ generated by the first $d=n+r$ vectors from the list (4). According to Theorem 1, the variety $X_\Sigma$ is a complete intersection of toric hypersurfaces $\mathcal{X}_1\cap\cdots\cap\mathcal{X}_r$ in $X_{\widetilde\Sigma}$. Let us show that this form has poles of the first order along all those hypersurfaces $\mathcal{X}_j$, $j=1,\ldots,\,r$. For that we look at the form $\eta(\zeta)$ in another coordinate chart of $X_{\widetilde\Sigma}$ whose intersection with 'the skeleton' is dense in it. There is a number of such standard charts in the variety, we choose the one that corresponds to the simple cone $\langle\widetilde{e}_I,\,\widetilde{v}_J\rangle\in\widetilde\Sigma$. This chart $\widetilde{V}$ is homeomorphic to $\mathbb{C}^d$ and the formulas

$$\begin{gathered}w_1=\zeta_1\zeta_{n+1}^{v^1_{n+1}}\ldots\zeta_d^{v^1_d},\\ \ldots\ldots\ldots\ldots\ldots\\ w_n=\zeta_n\zeta_{n+1}^{v^n_{n+1}}\ldots\zeta_d^{v^n_d},\\ w_{n+1}=\tfrac{1}{\zeta_{n+1}},\ldots,\,w_d=\tfrac{1}{\zeta_d}\end{gathered}$$

relate its local coordinates $w=(w_1,\ldots,\,w_d)$ to the local coordinates $\zeta$ in $\widetilde{U}$ [13, section 27.9] or [8]. Notice that the first $n$ equalities are nothing but the relations between local and homogeneous coordinates in the variety $X_\Sigma$ and easy calculation shows that in local coordinates $w$

$$\eta = (-1)^r \omega(w_1, \dots, w_n) \wedge \frac{dw_{n+1}}{w_{n+1}} \wedge \dots \wedge \frac{dw_d}{w_d}. \tag{6}$$

On the other hand, the relation between the local coordinates $w$ and the homogeneous coordinates $\xi$ of $X_{\widetilde{\Sigma}}$

$$\begin{gathered} w_1 w_{n+1}^{v_{n+1}^1} \dots w_d^{v_d^1} = \xi_1 \xi_{d+1}^{v_{n+1}^1} \dots \xi_{d+r}^{v_d^1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ w_n w_{n+1}^{v_{n+1}^n} \dots w_d^{v_d^n} = \xi_n \xi_{d+1}^{v_{n+1}^n} \dots \xi_{d+r}^{v_d^n}, \\ \tfrac{1}{w_{n+1}} = \tfrac{\xi_{n+1}}{\xi_{d+1}}, \dots, \tfrac{1}{w_d} = \tfrac{\xi_d}{\xi_{d+r}}, \end{gathered}$$

shows that a toric hypersurface $\mathcal{X}_j = \{\xi_{d+j} = 0\}$ in $\widetilde{W}$ is given by the equation $w_{n+j} = 0$. Thus, a global differential form on $X_{\widetilde{\Sigma}}$ given by $\eta(\zeta)$ in $\widetilde{U}$ has poles of the first order along 'the skeleton of infinity'.

Let us show how to choose the cycle $\gamma$. We follow the construction of the Leray coboundary for cycles in subvariety $S$ of the complex codimension $r$ in the variety $X$ [18, 1]. In our case we consider the subvariety $S = \mathcal{X}_1 \cap \dots \cap \mathcal{X}_r$ of $X = X_{\widetilde{\Sigma}}$ as in Theorem 1. By this theorem the closure of the orbit $G \cdot \zeta$ in $X_{\widetilde{\Sigma}}$ of any $\zeta \in \mathbb{C}^d \setminus Z(\Sigma)$ intersects 'the skeleton'

$$\mathcal{X}_1 \cap \dots \cap \mathcal{X}_r \simeq X_\Sigma$$

in a unique point $[\zeta]_G$. Let us take in every orbit a real $r$-dimensional torus such that their union over all $[\zeta]$ forms a continuous family of tori in $\mathbb{C}^d \setminus Z(\Sigma)$, which we shall take as $\gamma$.

Then by (6) and Fubini's theorem

$$\int_\gamma \eta(\zeta) = \int_{X_\Sigma} \omega(w) \int_{\mathbb{S}^1} \frac{dw_{n+1}}{w_{n+1}} \dots \int_{\mathbb{S}^1} \frac{dw_d}{w_d} = (2\pi i)^r \mathrm{Vol}_\omega(X_\Sigma)$$

with the volume $\mathrm{Vol}_\omega(X_\Sigma) = \int\limits_{X_\Sigma} \omega.$ □

In the next section we shall point out a natural class of volume forms on $X_\Sigma$ and, correspondingly, the class of kernels $\eta$, for which the volume $\mathrm{Vol}_\omega(X_\Sigma)$ is easily computed. At the moment let us consider another class of kernels and volume forms examined in the special case in [17].

**Example 2.**

Let $\Sigma$ be a fan in $\mathbb{R}^n$. Assume that its generators are vertices of a reflexive polytope $P$. In this case the dual polytope $P^\vee = \{x \in \mathbb{R}^n \langle y, x\rangle \geq -1 \, \forall y \in P\}$ is also integer.

Let $\{u_j\}$ be the vertex set of $P^\vee$. Then the differential form

$$\eta = \frac{\overline{E(\zeta)} \wedge d\zeta}{\sum_j c_j |\zeta^{\alpha(u_j)+I}|^2}, \quad c_j > 0$$

is a kernel for $Z(\Sigma)$. Indeed, let $u_j$ be dual to the cone $\sigma \in \Sigma$. According to the definition of $\alpha$ in (2) the coordinates of $\alpha(u_j)$ with indices that correspond to vectors generating $\sigma$ are equal to $-1$ and are non-negative for other indices (since $u_j$ is integer). Whence, taking into account (1), the form $\eta$ has singularity only along $Z(\Sigma)$. The form $\eta$ has degree zero, since $E(\zeta)$, $d\zeta$, and $\zeta^I$ have the same degree and the degree of $\zeta^{\alpha(u_j)}$ is zero (see Remark 1). The corresponding volume form in homogeneous coordinates is

$$\omega = \frac{\overline{E(\zeta)} \wedge E(\zeta)}{\sum_j c_j |\zeta^{\alpha(u_j)+I}|^2}.$$

Analogously, for reasons of homogeneity, a vector $k = (k_1, \ldots, k_d)$ of natural divisors of $d$ defines the differential $(d,\, d-1)$-form

$$\eta_{[k]} = \frac{\sum_{l=1}^d (-1)^l k_l \zeta_l d\overline{\zeta}[l] \wedge d\zeta}{\sum\limits_{\langle s,\, k\rangle = d} c_s |\zeta|^{2s}},$$

where $c_s \geq 0$ and $c_s > 0$ for $s = (0, \ldots, \frac{d}{k_j}, \ldots, 0)$. This form is a kernel in $\mathbb{C}^d \setminus \{0\}$ that generalizes the Bochner-Martinelli kernel, which one obtains choosing all $k_j = 1$.

A representative $\gamma$ of the only non-trivial class of the top homology group can be given by an analytic expression. This is the point where the fact that $X_\Sigma$ is projective plays a key role. Indeed, in this case the results [6, Theorem 4.1] and [14, Section 7.4] suggests taking $\gamma = \mu^{-1}(\rho)$ where $\mu$ is the moment map

$$\mu \colon \mathbb{C}^d \longrightarrow \mathbb{R}^r_+$$

associated with the action of the maximal compact subgroup of $G$ on the complement of $Z(\Sigma)$. A point $\rho$ is taken from the interior of the Kähler cone $K_\Sigma$ of $X_\Sigma$.

Let us give the accurate descriptions of $\mu$ and $K_\Sigma$. The moment map $\mu$ is the composition $\beta_{\mathbb{R}} \circ m$ of two mappings where $m(\zeta) = (|\zeta_1|^2, \ldots, |\zeta_d|^2)$ and $\beta_{\mathbb{R}}$ comes from the exact sequence

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^d \xrightarrow{\beta_{\mathbb{R}}} A_{n-1}(X_\Sigma) \otimes \mathbb{R} \longrightarrow 0$$

obtained by tensoring (2) with $\mathbb{R}$. Note that $A_{n-1}(X_\Sigma) \otimes \mathbb{R} \simeq \mathbb{R}^r$ since the fan $X_\Sigma$ is projective and simplicial. Having a lattice of relations (3), one immediately obtains the explicit formulas

$$\mu_i(\zeta) = a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2,\ i = 1, \ldots, r.$$

Observe that the coefficients $a_{ij}$ in the lattice can be chosen all non-negative. Let $|\zeta_k|^2$ occur with negative sign. Since the fan is complete, the vector $-v_k$ belongs to some cone in $\Sigma$ and equals a linear combination of its generators with non-negative coefficients. Multiplying this equality by appropriate positive factors and adding the results to the initial relations we eliminate all $|\zeta_k|^2$ with negative signs.

There is a procedure for defining of the Kähler cone $K_\Sigma$ (see [3]). Let $I \in \mathcal{P}$ be a primitive collection for the fan $\Sigma$. As the fan is complete, the sum $\sum_{i\in I} v_i$ belongs to some cone of $\Sigma$ generated by $\{v_j\}$, $j \in J$, and therefore is a linear combination of them

$$\sum_{i\in I} v_i = \sum_{j\in J} c_j v_j$$

with all $c_j$ being positive rational numbers. Since the row-vectors in (3) $a_1, \ldots, a_r$ form a basis of relations between generators, this relation can be rewritten as

$$\sum_{i\in I} v_i - \sum_{j\in J} c_j v_j = t_1^I(a_{11}v_1 + \cdots + a_{1d}v_d) + \cdots + t_r^I(a_{r1}v_1 + \cdots + a_{rd}v_d).$$

Then

$$K_\Sigma = \{\rho \in \mathbb{R}^r \colon l_I(\rho) := t_1^I\rho_1 + \cdots + t_r^I\rho_r > 0 \text{ for all } I \in \mathcal{P}\}. \tag{7}$$

# 4. The Fubini-Study volume form of $X_\Sigma$

There is a natural choice of the volume form in the construction of the kernel. Let $\Delta$ be an $n$-dimensional simple (each vertex belongs to exactly $n$ edges) integral polytope $\Delta \subset \mathbb{R}^n$. Given $\Delta$, there is a complete simplicial toric variety $X_\Sigma$ associated to the fan $\Sigma$ dual to $\Delta$. The variety $X_\Sigma$ constructed in this way admits a closed embedding into the projective space as follows.

Let $P(z) = \sum\limits_{\alpha\in\Delta\cap\mathbb{Z}^n} c_\alpha z^\alpha$ be a Laurent polynomial in the torus $\mathbb{T}^n$ with all non-negative coefficients $c_\alpha$ such that its Newton polytope $N_P$ coincides with $\Delta$. Put elements of $\Delta \cap \mathbb{Z}^n$ in the order $\alpha_1 \ldots, \alpha_N$ and define an embedding of the torus $f\colon \mathbb{T}^n \longrightarrow \mathbb{P}_{N-1}$ by

$$(z_1, \ldots, z_n) \longmapsto (\sqrt{c_{\alpha_1}}z^{\alpha_1} : \ \ldots : \sqrt{c_{\alpha_N}}z^{\alpha_N}).$$

The closure $\overline{f(\mathbb{T}^n)}$ is then the image of $X_\Sigma$, which may have singularities, but observe that the $f(\mathbb{T}^n) \subset f(X_\Sigma)$ is always smooth.

We introduce a differential $(n, n)$-form $\omega$ on the torus $\mathbb{T}^n$ as the pullback image of the Fubini-Study volume form $\omega_{FS}^n$:

$$\omega = \frac{1}{n!} f^*(\omega_{FS}^n) = \frac{1}{n!}\left(\mathrm{dd^c} \ln P(|z_1|^2, \ldots, |z_n|^2)\right)^n.$$

This is a well-defined positive in the torus $\mathbb{T}^n \subset X_\Sigma$ differential $(n, n)$-form; it may vanish or be not defined in some points of the variety, however that does not affect the value of the integral

$$\int\limits_{\mathrm{reg} X_\Sigma} \omega = \int\limits_{\mathbb{T}^n} \omega.$$

The following simple proposition gives the exact value of the volume.

**Proposition 3.**

$$\mathrm{Vol}(X_\Sigma) = \pi^n \mathrm{Vol}(\Delta).$$

*Proof.* The obvious change of variables in the integral gives

$$\int\limits_{\mathbb{T}^n} \omega = \frac{1}{n!} \int\limits_{f(\mathbb{T}^n)} \omega_{FS}^n.$$

The volume of this algebraic subvariety of projective space with respect to the Fubini-Study metric equals

$$\frac{\pi^n}{n!} \deg(f).$$

It only remains to compute the degree of the embedding $f$. Let the homogeneous linear forms $l_j(\xi)$, $j = 1, \ldots, n$ define a generic plane of codimension $n$ in $\mathbb{P}_{N-1}$. Then the degree of $f$ equals the number of solutions to the system of equations $l_j(\xi)\Big|_{f(\mathbb{T}^n)}$ with the same Newton polytope. The number of solutions to such a system is given by Kushnirenko's theorem [16] and equals $n!\mathrm{Vol}(\Delta)$, and the assertion follows. □

Notice also that integrating $\omega$ with respect to angular coordinates in this equality provides a new proof of Passare's formula, which gives the volume of a polytope as an integral of rational form over the positive orthant [21].

**Example 3.** The volume form of $\mathbb{P}_1 \times \mathbb{P}_1$.

The product of two copies of the Riemann sphere (projective line) is a toric variety associated with the two-dimensional complete fan $\Sigma$ on Fig.3(a). Let $P$ be a polynomial $P(z_1, z_2) = 1+z_1+z_2+az_1z_2$ where the coefficient $a$ is positive. Its Newton polytope $N_P$ is the unit square in $\mathbb{R}^2$ (Fig.4) and is obviously dual to the fan $\Sigma$ (Fig. 3).

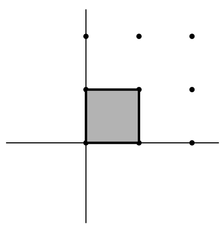

Fig.4. The Newton polytope of $P(z)$.

Following the construction, we define a differential form on $\mathbb{T}^2 \subset \mathbb{P}_1 \times \mathbb{P}_1$ as the pull-back of $\omega_{FS}^2$ under the mapping $f\colon (z_1, z_2) \mapsto (1\colon z_1\colon z_2\colon \sqrt{a}z_1z_2)$

$$\omega = \frac{1}{2!} f^*(\omega_{FS}^2) = \left(\frac{i}{2}\right)^2 \frac{1 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2}{(1 + |z_1|^2 + |z_2|^2 + a|z_1|^2|z_2|^2)^3} \, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

The volume of $\mathbb{P}_1 \times \mathbb{P}_1$ with respect to this measure is equal to $\pi^2$. Note that the volume form does not coincide with the product of two volume forms on copies of $\mathbb{P}_1$ (it happens only if $a = 1$), although it gives the same volume.

## 5. Integral representation with residual kernels

In this section we shall prove integral representation formulas with the kernels obtained. The first step in this direction is the following proposition. We keep our notation where $K_\Sigma$ is the Kähler cone (see (7)) and $\eta$ is the residual kernel (5) defined for the fan $\Sigma$.

**Proposition 4.** *Let $\rho \in K_\Sigma$ and $U_\rho$ be a complete Reinhardt domain*

$$\begin{cases} a_{11}|\zeta_1|^2 + \cdots + a_{1d}|\zeta_d|^2 < \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1|^2 + \cdots + a_{rd}|\zeta_d|^2 < \rho_r, \end{cases}$$

*with distinguished boundary $\gamma = \mu^{-1}(\rho)$. Then for every $f \in \mathcal{O}(U_\rho) \cap C(\overline{U}_\rho)$*

$$f(0) = \frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_\gamma f(\zeta)\eta(\zeta).$$

*Proof.* Let $V \subset \mathbb{C}^d$ be a polydisc centered at the origin and relatively compact in $U_\rho$.The function $f$ admits a Taylor series expansion $\sum\limits_\beta a_\beta \zeta^\beta$ about the origin that converges on compact subsets of $V$. Choose $\rho' \in K_\Sigma$ such that $\gamma' = \mu^{-1}(\rho')$ compactly lies in $V$. Then by Stokes' theorem the cycle $\gamma$ can be replaced by $\gamma'$ where the series converges absolutely and uniformly. Let us show that

$$\int\limits_{\gamma'} \zeta^\beta \eta(\zeta) = 0 \text{ if } \beta \neq 0.$$

Notice that the following change of variables

$$\begin{cases} \zeta_1 \mapsto e^{i(a_{11}t_1 + \cdots + a_{r1}t_r)}\zeta_1, \\ \dots\dots\dots\dots\dots\dots \\ \zeta_d \mapsto e^{i(a_{1d}t_1 + \cdots + a_{rd}t_r)}\zeta_d \end{cases}$$

with all $t_j$ being real, preserves the integration set and the kernel, as the latter is homogeneous with respect to this action; but the integrand gets the coefficient $e^{i((a_{11}t_1+\cdots+a_{r1}t_r)\beta_1+\ldots(a_{1d}t_1+\cdots+a_{rd}t_r)\beta_d)}$. The rank of the matrix $A = (a_{ij})$ is $r$, so the image of the linear mapping given by $A$ is $\mathbb{R}^r$. Therefore, for any $\beta \neq 0$ one can choose $t = (t_1, \dots, t_r)$ such that the coefficient is not equal to 1, so the integral must equal 0.

The statement follows now from Theorem 2. □

Recall that the Kähler cone of $X_\Sigma$ is defined by the system of linear inequalities $l_I(\rho) > 0$. For a fixed $\rho$, define a domain $D$ of $\mathbb{C}^d$ by the system

$$|\zeta_{i_1}|^2 + \cdots + |\zeta_{i_k}|^2 < t_1^I \rho_1 + \cdots + t_r^I \rho_r \tag{8}$$

for all primitive collections $I \in \mathcal{P}$ of $\Sigma$.

**Proposition 5.** *The domain $D$ is a subdomain of $U_\rho$.*

*Proof.* Note that the rational vectors $t^I = (t_1^I, \ldots, t_r^I)$ are the interior normal vectors to the faces of the Kähler cone. Therefore they generate the dual cone $b_1 t^{I_1} + \cdots + b_s t^{I_s}$ where $b_j \in \mathbb{R}^r$, $b_j \geq 0$. Since the Kähler cone is not empty and contained in the positive orthant $\mathbb{R}^r_+$, the dual cone is also non-empty and contains the positive orthant. This means that every basis vector $e_i$ of $\mathbb{R}^r$ can be expressed as a linear combination of $\{t^I\}$ with non-negative rational coefficients. Hense we can sum the inequalities (8) multiplied by these coefficients to get $\rho_i$ on the right side and

$$a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2 + b_1 \left( \sum_{j \in J_1} c_j |\zeta_j|^2 \right) + \cdots + b_s \left( \sum_{j \in J_s} c_j |\zeta_j|^2 \right)$$

on the left with the same inequality sign. So, $a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2 < \rho_i$ and the proposition is proved. □

Now we extend the representation of the function at the origin (Proposition 4) to the representation in a domain. The formula we shall obtain combines the properties of the Bochner-Ono formula [1] (as it represents values of a function in a subdomain) as well as of the one of Sorani [23] (as one integrates over the distinguished boundary).

**Theorem 3.** *Let $f \in \mathcal{O}(U_\rho) \cap C(\overline{U}_\rho)$. Then for every $z \in D \subset U_\rho$*

$$f(z) = \frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_\gamma f(\zeta) \eta(\zeta - z).$$

*Proof.* Let $z$ be in $D$. Consider the homotopy $\Gamma(t)$ of the cycle $\mu^{-1}(\rho) = \gamma$

$$\begin{cases} a_{11}|\zeta_1 - tz_1|^2 + \cdots + a_{1d}|\zeta_d - tz_d|^2 = R_1(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r1}|\zeta_1 - tz_1|^2 + \cdots + a_{rd}|\zeta_d - tz_d|^2 = R_r(t). \end{cases}$$

with the vector-function $R(t) = (R_1(t), \ldots, R_r(t))$ given by

$$R(t) = (1-t)^2 \rho + t\varepsilon,\ t \in [0,\ 1],$$

where $\varepsilon$ is a point from the Kähler cone $K_\Sigma$ of $X_\Sigma$, chosen in such a way that the following two conditions are satisfied

(1) the homotopy $\Gamma(t)$ forms a $(d+n+1)$-dimensional chain in $U_\rho$;

(2) the cycles $\Gamma(t)$ do not intersect

$$Z(\Sigma) + z = \{\zeta \in \mathbb{C}^d \colon \zeta - z \in Z(\Sigma)\}.$$

With these conditions satisfied the Stokes formula implies that

$$\int\limits_{\gamma} f(\zeta)\,\eta(\zeta - z) = \int\limits_{\Gamma(1)} f(\zeta)\,\eta(\zeta - z)$$

and the change of variables $\zeta \mapsto \zeta + z$ in the integral gives

$$\int\limits_{\mu^{-1}(\varepsilon)} f(\zeta + z)\,\eta(\zeta).$$

Since $\varepsilon \in K_\Sigma$, it follows from (4) that the integral equals $(2i)^r \pi^d \mathrm{Vol}(\Delta) f(z)$. Hence, it is left to point out such $\varepsilon$.

Notice the fact that $\varepsilon \in K_\Sigma$ automatically implies that the whole curve $R(t)$ lies in $K_\Sigma$. Then cycles from $\{\Gamma(t)\}$, $t \in [0,\, 1]$ constitute a continuous family, i.e. a $(d+n+1)$-dimensional chain. By the triangle inequality in the standard metric of $\mathbb{R}^{2d}$

$$\begin{aligned}\big(a_{i1}|\zeta_1 - tz_1|^2 + \cdots + a_{id}|\zeta_d - tz_d|^2\big)^{1/2} &\geq \\ \geq \big(a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2\big)^{1/2} &- t\,\big(a_{i1}|z_1|^2 + \cdots + a_{id}|z_d|^2\big)^{1/2};\end{aligned}$$

denoting the image $\mu(z)$ by $\mu$ we obtain

$$\big(a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2\big)^{1/2} \leq \sqrt{R_i(t)} + t\sqrt{\mu_i} \;\text{ for all } \zeta \in \Gamma(t).$$

Therefore, to satisfy the first condition for $\Gamma(t)$ it is enough to require

$$(\sqrt{\rho_i} - t\sqrt{\mu_i})^2 - R_i(t) \geq 0 \text{ for all } i = 1, \ldots, r.$$

These inequalities hold if $\varepsilon_i < ((\sqrt{\rho_i} - \sqrt{\mu_i})^2)$, $i = 1, \ldots, r$.

For all $\zeta \in \Gamma(t)$, we have

$$l_I(R(t)) = \sum_{i \in I} |\zeta_i - tz_i|^2 - \sum_{j \in J} c_j |\zeta_j - tz_j|^2.$$

Substituting any point of $Z + z$ into corresponding identity, we get

$$-\sum_{j \in J} c_j |\zeta_j - tz_j|^2 = l_I(R(t)) - (1-t)^2 \sum_{i \in I} |z_i|^2.$$

By definition $\sum\limits_{i \in I} |z_i|^2 < l_I(\rho)$ and get the right hand side being strictly positive. This proves that $\zeta \in Z(\Sigma) + z$ does not belong to the chain $\{\Gamma(t)\}$. □

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