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Analytic Bezout Identities*

CARLOS A. BERENSTEIN

Mathematics Department and Systems Research Center, University of Maryland, College Park, Maryland 20742

AND

ALAIN YGER

UER Mathematiques, Lab. Ass. CNRS 226, Université de Bordeaux I, 33405 Talence Cedex, France

1. INTRODUCTION

In a number of papers [2–4, 6, 7, 12] the problem of finding explicit solutions $h_1, \dots, h_m$ for the Bezout equation: $f_1 h_1 + \cdots + f_m h_m = 1$ has been considered. If $f_1, \dots, f_m$ are complex polynomials in n variables and they have no common zeros in $\mathbb{C}^n$, the existence of explicit analytic expressions for the corresponding polynomials $h_1, \dots, h_m$ has a number of applications to systems theory and commutative algebra.

For instance, the problem of finding a closed loop controller for certain distributed parameter systems reduces to the question of finding a matrix with polynomial entries which is a left inverse to a rectangular matrix of polynomials of maximal rank (see [10, 20, or 21] for details). It is easy to see that this problem reduces to solving the polynomial Bezout equation (see [3]). The case where the rank is not maximal has also considerable interest in systems theory, we refer to [4] for some open questions in this case.

In several contexts, for instance, in transcendental number theory, one is interested in finding the solutions $h_1, \dots, h_m$ of the algebraic Bezout equation with the smallest possible degrees. Up to recently the best estimate known for $\deg h_j$ was (of the order of magnitude of) $(\max \deg f_j)^{2^n}$. Using explicit analytic expressions for the h_j, Brownawell [11] has shown that one can find solutions h_j with $\deg h_j \leq n^2 (\max \deg f_j)^n$. This estimate is known

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to be close to optimal, as we explain in Section 3 below. Up to date no purely algebraic proof of this bound for the degrees of the h_j has been found. (The interested reader will find a short survey of this topic in [5].)

Similarly, in the deconvolution problem, one has functions $f_1, \dots, f_m$ which are the Fourier transforms of a strongly coprime family $\mu_1, \dots, \mu_m$ of distributions of compact support. One searches for a procedure to compute explicitly distributions of compact support $\nu_1, \dots, \nu_m$ such that $\mu_1 * \nu_1 + \cdots + \mu_m * \nu_m = \delta$. (Here $\hat{\nu}_j$ play the role of h_j in the equivalent formulation $\hat{\mu}_1\hat{\nu}_1 + \cdots + \hat{\mu}_m\hat{\nu}_m \equiv 1$). This question arises in problems of robust filtering, image processing, etc. [10]. In [7] we wrote down formulas for a solution $\nu_1, \dots, \nu_m$ of the deconvolution problem in terms of interpolation series. The problem we have faced until recently is that, while for the one-dimensional case these formulas can be easily implemented, in the higher dimensional case they are far too cumbersome. Some of them seem to be beyond the range of symbolic languages like MACSYMA upon which we had, perhaps too optimistically, relied. For that reason we present here a new version of our original deconvolution formulas which assumes extra conditions on the family $\mu_1, \dots, \mu_m$ but has as a payoff a very simple formula for the deconvolutors $\nu_1, \dots, \nu_m$. We give herein simple examples where these extra conditions are satisfied.

The problem of finding an efficient algorithm to compute the above-mentioned solutions to the algebraic Bezout equation being still open, we also analyze here the particular case in which those polynomials can be computed in terms of interpolation formulas. Finding an algorithm with a low complexity for this problem will have many important applications in the theory of distributed parameter systems and in robotics.

We have also found that a language barrier prevented our work [7] from being more easily available to some engineers, and we hope that the present paper will overcome those shortcomings.

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2. Analytic Case

We will consider only entire functions f of n complex variables satisfying inequalities of the form

$$|f(z)| \leq A(1 + |z|)^m d^{H(\operatorname{Im} z)}, \qquad z \in \mathbf{C}^n, \tag{1}$$

$\operatorname{Im} z = (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n) \in \mathbf{R}^n$, where H is a convex continuous function in $\mathbf{R}^n$, homogeneous of degree 1 (i.e., $H(\lambda x) = \lambda H(x)$ when $\lambda > 0$). We call such a function H a *supporting function*. By the Paley–Wiener theorem [18] there is a distribution μ of compact support in $\mathbf{R}^n$ such that $f = \hat{\mu}$, the Fourier transform of μ. Furthermore, the supporting function H_0 of $\operatorname{cv}\operatorname{supp}\mu$ will satisfy $H_0 \leq H$ (Here cv denotes the convex hull). Conversely, if $f = \hat{\mu}$ we can take $H = H_0$ in (1) and m is related to the order of μ in a simple manner. Hereafter we will just write $f \in \hat{\mathscr{E}}'$ $(= \hat{\mathscr{E}}'(\mathbf{R}^n))$ if f satisfies (1).

For simplicity denote $\rho(z) := \log(2 + |z|) + |\operatorname{Im} z|$. A family $f_1, \ldots, f_m$ of functions in $\hat{\mathscr{E}}'$ is said to be *strongly coprime* if there is a constant c such that

$$\sum_{j=1}^{m} |f_m(z)|^2 \geq e^{-c\rho(z)}, \qquad z \in \mathbf{C}^n. \tag{2}$$

It is well known [14] that (2) is a necessary and sufficient condition for the existence of functions $h_1, \ldots, h_m \in \hat{\mathscr{E}}'$ such that

$$\sum_{j=1}^{m} f_j h_j = 1. \tag{3}$$

In other words, a strongly coprime family is precisely a family for which the analytic Bezout equation (3) has a solution. If we consider (3) in terms of the distributions $\mu_1, \ldots, \mu_m$, $\nu_1, \ldots, \nu_m$ such that $\hat{\mu}_j = f_j$ and $\hat{\nu}_j = h_j$ then we have the identity

$$\mu_1 * \nu_1 + \cdots + \mu_m * \nu_m = \delta, \tag{4}$$

i.e., $\nu_1, \ldots, \nu_m$ solve the deconvolution problem stated in [7]. We will say sometimes that the family of distributions $\mu_1, \ldots, \mu_m$ is strongly coprime.

It might be useful to explain why (4) is called a deconvolution problem. If we have an unknown signal (function or even distribution or random process) φ then the usual data one measures would be $\psi_1, \ldots, \psi_m$ given by

$$\psi_1 := \mu_1 * \varphi, \ldots, \psi_m := \mu_m * \varphi. \tag{5}$$

The way to recover φ is by *deconvolution* (which is still given here by convolution with distributions or compact support).

$$\varphi = \nu_1 * \psi_1 + \cdots + \nu_m * \psi_m. \tag{6}$$

As we have mentioned in the Introduction our problem is to find easily computable functions h_j and corresponding distributions ν_j solving (3) and (4), respectively. We note that under the strongly coprime condition (2), or even under the weaker assumption that (2) is only satisfied for real values of z ($z \in \mathbf{R}^n$), there are readily available tempered distributions α_j solving the deconvolution problem, namely, let

$$\hat{\alpha}_j(z) = \frac{\bar{f}_j(z)}{\sum_{j=1}^{m} |f_j(z)|^2}, \qquad z \in \mathbf{R}^n. \tag{7}$$

The problem is that the α_j do not have compact support and furthermore, the α_j themselves are not so readily computable (except by inverting the Fourier transform). Nevertheless there are many situations where these α_j are still very useful, among other reasons because they minimize the noise amplification of the deconvolution process (6) (see [7] for an example of implementation in two dimensions). On the other hand, in many applications it is often not necessary to obtain an exact solution to (4) but one is allowed to replace the Dirac δ in (4) by a (sufficiently) smooth function u with small support, i.e., an approximation to δ. It is this approximate deconvolution problem that is more readily solvable, even with very good knowledge on the support of the distributions ν_j, which will turn out to be (reasonably) smooth functions.

Since the method we use relies on Koppelman-type formulas, like those developed in [1, 9], we need the following explicit relation whose proof is an immediate verification.

LEMMA 1. *Let μ be a distribution of compact support in $\mathbf{R}^n$, $1 \le k \le n$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$. The holomorphic function of $2n$ complex variables $g_k(z, \zeta)$ defined by*

$$g_k(z,\zeta) := \frac{\hat{\mu}(z_1, \dots, z_k, \zeta_{k+1}, \dots, \zeta_n) - \hat{\mu}(z_1, \dots, z_{k-1}, \zeta_k, \dots, \zeta_n)}{z_k - \zeta_k} \tag{8}$$

is the Fourier transform (for ζ fixed) of the distribution denoted $I = I(\mu, \zeta, k)$, which evaluated at $\varphi \in C_0^\infty(\mathbf{R}^n)$ has the value

$$\langle I, \varphi \rangle := -i \int \left[\int_0^{t_k} \varphi(t_1, \dots, t_{k-1}, u, 0, \dots, 0) e^{i\zeta_k(u - t_k)} du \right] \times e^{-i(t_{k+1}\zeta_{k+1} + \cdots + t_n\zeta_n)} d\mu(t). \tag{9}$$

(By abuse of language we have written $\int \psi(t)\, d\mu(t)$ to denote $\langle \mu, \psi \rangle$).

We note that for the distribution I we have $\operatorname{cv\,supp} I \subseteq \operatorname{cv\,supp} \mu$. Furthermore the collection of functions $g_1, \dots, g_n$ satisfies

$$g_1(z,\zeta)(z_1 - \zeta_1) + \cdots + g_n(z,\zeta)(z_n - \zeta_n) = \hat{\mu}(z) - \hat{\mu}(\zeta). \tag{10}$$

Associated to these holomorphic functions we have a (1, 0) differential form g in the variable ζ given by

$$g = g(z, \zeta, \mu) := \sum_{k=1}^{n} g_k(z, \zeta)\, d\zeta_k. \tag{11}$$

Given a family of m entire holomorphic functions $f_1, \dots, f_m$ its zero set $\mathscr{Z}$ is defined as

$$\mathscr{Z} := \{ z \in \mathbf{C}^n : f_1(z) = \cdots = f_m(z) = 0 \}. \tag{12}$$

In our applications we will only consider the case where the set $\mathscr{Z}$ is discrete. We say that $\mathscr{Z}$ is *almost real* if there is constant $A > 0$ such that

$$\mathscr{Z} \subseteq \{ z \in \mathbf{C}^n : |\mathrm{Im}\, z| \leq A \log(2 + |z|) \}.$$

It is well known that an almost real zero set $\mathscr{Z}$ is discrete [8, 15]. For a $\mathscr{Z}$ discrete set $\mathscr{Z}$, $r > 0$, we can define a *counting function* $n(\mathscr{Z}, r) := \#(\mathscr{Z} \cap \mathscr{Z} B_r\}$, $B_r = \{ z \in \mathbf{C}^n : |z| < r \}$ = Euclidean ball of center 0 and raidus r. The *distance function* is $d(z, \mathscr{Z}) := \min(1, \min\{ |z - \zeta| : \zeta \in \mathscr{Z} \})$.

Given a family of n distributions of compact support in $\mathbb{R}^n$, $\mu_1, \dots, \mu_n$ let us denote H_1 the supporting function of cv $\bigcup_1^n \mathrm{supp}\, \mu_j$, that is,

$$H_1(\theta) := \max_{1 \leq j \leq n} \max\{ x \cdot \theta : x \in \mathrm{supp}\, \mu_j \}, \qquad (\theta \in \mathbb{R}^n), \tag{13}$$

$x \cdot \theta = x_1\theta_1 + \cdots + x_n\theta_n$.

DEFINITION 1. A family of n distributions $\mu_1, \dots, \mu_n$ of compact support in $\mathbb{R}^n$ is *well behaved* if there exist positive constants A, B, N, κ, a supporting function H_0 such that $0 \leq H_0 \leq H_1$, such that the zero set $\mathscr{Z}$ of the functions $f_1 = \hat{\mu}_1, \dots, f_n = \hat{\mu}_n$, is almost real,

$$n(\mathscr{Z}, r) = O(r^A), \tag{14}$$

and, denoting

$$|f(z)| := \left[\sum_1^n |f_j(z)|^2 \right]^{1/2}, \tag{15}$$

the following inequality holds:

$$|f(z)| \geq \frac{B d(z, \mathscr{Z})^{\kappa} e^{H_0(\mathrm{Im}\, z)}}{(1 + |z|)^N}. \tag{16}$$

DEFINITION 2. A well-behaved family $\mu_1, \dots, \mu_n$ is *very well behaved* if there are constants $c_1, M, c_1 > 0$, such that for every $\zeta \in \mathscr{Z}$, we have

$$|J(\zeta)| := \left| \det \left[\frac{\partial f_j}{\partial z_i}(\zeta) \right]_{i,j} \right| \geq c_1 (1 + |\zeta|)^{-M}. \tag{17}$$

This condition implies that the common zeros of $f_1, \dots, f_n$ are simple, that we can take $\kappa = 1$ in (16), and that if $\zeta, \zeta' \in \mathscr{Z}$, $\zeta \neq \zeta'$ then $|\zeta - \zeta'| \geq c_2(1 + |\zeta|)^{-M'}$ for some positive constants c_2, M'.

We will say also that functions $f_1, \dots, f_n$ are (very) well behaved if the above properties hold.

Given a family $f_1, \dots, f_m$ in $\hat{\mathscr{E}}'(\mathbf{R}^n)$, $m > n$, with no common zeros, we introduce the following functions and differential forms. First, let $g^j = g^j(z, \zeta, \mu_j)$, $f_j = \hat{\mu}_j$, be the (1,0) differential forms in ζ given by (11), we write $g^j = \sum_{k=1}^n g_k^j d\zeta_k$. Recall the coefficients g_k^j are holomorphic in both z and ζ. Let F be the vector-valued holomorphic function $F := (f_1, \dots, f_m)$; we write

$$|F(\zeta)| := \left[\sum_{j=1}^m |f_j(\zeta)|^2 \right]^{1/2},$$

which is a nowhere vanishing C^∞ function of ζ. Let

$$\varphi = \varphi(z, \zeta) := \left[\sum_{j=1}^m \bar{f}_j(\zeta) f_j(z) \right] \Big/ |F(\zeta)|^2, \tag{18}$$

$$Q = Q(z, \zeta) := \left[\sum_{j=1}^m \bar{f}_j(\zeta) g^j(z, \zeta) \right] \Big/ |F(\zeta)|^2. \tag{19}$$

Therefore φ is a C^∞ function of (z, ζ), $\varphi(\zeta, \zeta) \equiv 1$ and, as a function of z, φ is a linear combination of the f_j. Q is a (1,0) differential form in ζ and its coefficients are C^∞ in (z, ζ) and holomorphic in z. Finally, the $n + 1$ functions Δ_j, C^∞ in (z, ζ) and holomorphic in z are defined by the identities

$$g^1 \wedge \cdots \wedge \hat{g^j} \wedge \cdots \wedge g^n \wedge Q = \Delta_j \, d\zeta_1 \wedge \cdots \wedge d\zeta_n, \qquad 1 \leq j \leq n \tag{20}$$

$$g^1 \wedge \cdots \wedge g^n = \Delta_{n+1} \, d\zeta_1 \wedge \cdots \wedge d\zeta_n. \tag{21}$$

It is clear that the Δ_j are simply $n \times n$ determinants whole entries are obtained from the coefficients of $g^1, \dots, g^n, Q$. Therefore, as functions of z, they are finite linear combinations of products of n among the functions g_k^j, $1 \leq k \leq n$. Note that these products are just Fourier transforms of convolutions of n distributions of the form $I(\mu_j, \zeta, k)$ (see (9)).

In order to obtain simple and easily computable deconvolution formulas we need to assume that a strongly coprime family of distributions $\mu_1, \ldots, \mu_m$ contains a (very) well-behaved subfamily $\mu_1, \ldots, \mu_n$. Furthermore, we need some control on the relation between the support of all the μ_j versus the supports of the first n. Let

$$H_2(\theta) = \max_{1 \le j \le n} \max\{x \cdot \theta \colon x \in \operatorname{supp} \mu_j\} \qquad (\theta \in \mathbf{R}^n). \tag{22}$$

One such relation between the supporting functions H_0, H_1, H_2 is given by

$$H_2 \le 2H_1, \tag{23}$$

and

$$2(n-1)H_1(\theta) + H_2(\theta) < 2nH_0(\theta) \quad \text{if} \quad \theta \neq 0. \tag{24}$$

The last condition is equivalent to

$$\exists r_0 > 0 \quad \text{such that} \quad r_0|\theta| \le 2nH_0(\theta) - 2(n-1)H_1(\theta) - H_2(\theta). \tag{25}$$

With all this notation in place we are now ready to state the first deconvolution formula.

THEOREM 1. *Let $\mu_1, \ldots, \mu_m$ be a strongly coprime family of distributions such that $\mu_1, \ldots, \mu_n$ is a very well-behaved subfamily. Assume further that* (23) *and* (25) *hold. For any $u \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} u \subseteq \{x \in \mathbf{R}^n \colon |x| \le r_0\}$ one can write*

$$\hat{u}(z) = \sum_{\zeta \in \mathscr{Z}} \hat{u}(\zeta) \frac{\Delta_{n+1}(z,\zeta)}{J(\zeta)} \varphi(z,\zeta) + \sum_{j=1}^{n} (-1)^{N+1-j} f_j(z) \sum_{\zeta \in \mathscr{Z}} \frac{\Delta_j(z,\zeta)}{J(\zeta)} \hat{u}(\zeta). \tag{26}$$

Formula (26) can be rewritten as

$$\hat{u}(z) = \sum_{j=1}^{m} h_j(z) f_j(z),$$

where the h_j are given by explicit interpolation formulas and they are Fourier transforms of a series of distributions which are computable in terms of the original $\mu_1, \ldots, \mu_m$. In the particular case where $m = n+1$

then formula (26) can be also rewritten as

$$\hat{u}(z) = \sum_{\zeta \in Z} \frac{\hat{u}(\zeta)}{J(\zeta) f_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z,\zeta) \cdots g_1^{n+1}(z,\zeta) \\ \vdots \qquad\qquad \vdots \\ g_n^1(z,\zeta) \cdots g_n^{n+1}(z,\zeta) \\ f_1(z) \cdots f_{n+1}(z) \end{vmatrix}. \tag{27}$$

Proof of Theorem 1. It follows the lines of Theorem 3 from our paper [7]. It uses the Koppelman-type generalization of the Cauchy integral representation formula, especially in the version due to Anderson and Berndtsson [1]. First one introduces a parameter $\varepsilon > 0$, a function φ_ε, and two $(1,0)$ differential forms in ζ as

$$\varphi_\varepsilon(z,\zeta) := \frac{\sum_{j=1}^{n} \bar{f}_j(\zeta) f_j(z) + \varepsilon}{|f(\zeta)|^2 + \varepsilon} \tag{28}$$

$$s(z,\zeta) := \sum_{j=1}^{n} \left(\bar{\zeta}_j - \bar{z}_j\right) d\zeta_j \tag{29}$$

$$Q_\varepsilon(z,\zeta) := \frac{\sum_{j=1}^{n} \bar{f}_j(\zeta) g_j(z,\zeta)}{|f(\zeta)|^2 + \varepsilon}, \tag{30}$$

where, as before, $f = (f_1, \dots, f_n)$ $|f(\zeta)|^2 = \sum_{j=1}^n |f_j(\zeta)|^2$. The procedure from [9, pp. 402, 409]) gives two kernels K_ε, P_ε (i.e., differential forms in the variable ζ of type $(n, n-1)$ and (n, n), respectively) such that if v is a holomorphic function in a neighborhood of $\bar{B}_R$, $z \in B_R$, then

$$v(z) = \frac{1}{(2\pi i)^n} \left\{ \int_{\partial B_R} v(\zeta) K_\varepsilon(z,\zeta) + \int_{B_R} v(\zeta) P_\varepsilon(z,\zeta) \right\}. \tag{31}$$

These two kernels are defined as follows. Let $G_1(t) = t^n$ and $G_2(t) = t$; we denote for any $\alpha \in \mathbb{N}$.

$$G_1^{(\alpha)} = G_1^{(\alpha)}(z,\zeta) := \frac{d^\alpha}{dt^\alpha} G_1 \Big|_{t=\varphi_\varepsilon(z,\zeta)} \tag{32}$$

$$G_2^{(\alpha)} = G_2^{(\alpha)}(z,\zeta) := \frac{d^\alpha}{dt^\alpha} G_2 \Big|_{t=\varphi_\varepsilon(z,\zeta)}, \tag{33}$$

where φ is given by (18) and φ_ε by (28). With Q_ε defined by (30) and Q by

(19), we define

$$K_\varepsilon(z,\zeta) := \sum_{\alpha_0+\alpha_1+\alpha_2=n-1} \frac{1}{\alpha_1!\alpha_2!} \frac{G_1^{(\alpha_1)}G_2^{(\alpha_2)}s \wedge (\bar\partial s)^{\alpha_0} \wedge (\bar\partial Q_\varepsilon)^{\alpha_1} \wedge (\bar\partial Q)^{\alpha_2}}{|z-\zeta|^{2(\alpha_0+\alpha_1)}}, \tag{34}$$

$$P_\varepsilon(z,\zeta) := \sum_{\alpha_1+\alpha_2=n} \frac{1}{\alpha_1!\alpha_2!} G_1^{(\alpha_1)}G_2^{(\alpha_2)}(\bar\partial Q_\varepsilon)^{\alpha_1} \wedge (\bar\partial Q)^{\alpha_2}, \tag{35}$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathbf{N}$. Everywhere the variable z is considered as a parameter and the $\bar\partial$ derivative is taken with respect to ζ. Due to our choice of function G_2, the index α_2 can only take the values 0 and 1. For this reason the expression for P_ε becomes particularly simple

$$P_\varepsilon = \varphi(\bar\partial Q_\varepsilon)^n + n\varphi_\varepsilon(\bar\partial Q_\varepsilon)^{n-1} \wedge \bar\partial Q. \tag{36}$$

The terms $(\bar\partial Q_\varepsilon)^{n-1}$ and $(\bar\partial Q_\varepsilon)^n$ must be computed, for instance,

$$\begin{aligned}(\bar\partial Q_\varepsilon)^n &= \bar\partial Q_\varepsilon \wedge \cdots \wedge \bar\partial Q_\varepsilon \ (n \text{ times})\\ &= (2i)^n n! \Delta_{n+1} \frac{\bar J(\zeta)\varepsilon}{\left(|f(\zeta)|^2+\varepsilon\right)^{n+1}} d\lambda,\end{aligned} \tag{37}$$

where $d\lambda = d\lambda(\zeta)$ = Lebesgue measure in $\mathbf{C}^n$. (We have eliminated the variables (z,ζ) where they were evident; we will use this convention freely in the rest of the paper.)

It is clear that φ_ε and Q_ε are singular when $\varepsilon = 0$ precisely at the points $\zeta \in \mathscr{Z}$. The expression (37) shows that the strength of this singularity is in one of the terms of P_ε. The strategy of the proof is to try to get very singular terms so that when $\varepsilon \to 0$ the volume integrals in (31) become sums, while the boundary integrals tend to zero when we set $v = \hat u$ and let $R \to \infty$ over a conveniently chosen sequence. The reason this idea works is the following lemma [7, Corollary 4.1.1].

LEMMA 2. *Let σ_0 be the measure which is the sum of direct masses at the points of $\mathscr{Z}$; i.e., for $\psi \in C_0^\infty(\mathbf{C}^n)$ we have $\int\psi\, d\sigma_0 = \sum_{\zeta\in\mathscr{Z}}\psi(\zeta)$. Then, the family of measures σ_ε given by*

$$d\sigma_\varepsilon(\zeta) = \frac{\varepsilon}{\left(|f(\zeta)|^2+\varepsilon\right)^{n+1}} d\lambda(\zeta) \tag{38}$$

converges when $\varepsilon \to 0$, to the measure

$$\frac{\pi^n}{n!} \frac{d\sigma_0}{|J(\zeta)|^2}, \tag{39}$$

where, as always, J denotes the determinant Jacobian of $f_1, \ldots, f_n$.

From (37) we see that the first term in (36) is amenable to Lemma 2. The second term is not singular enough, therefore it will be transformed using Stokes' formula in the corresponding integral of (31). Namely, due to type considerations, one obtains the first part of the identity

$$\begin{aligned} &d_\zeta\left\{v(\zeta)\varphi_\varepsilon(z,\zeta)\left((\bar\partial Q_\varepsilon(z,\zeta))\right)^{n-1} \wedge Q(z\zeta)\right\} \\ &= \bar\partial\left\{v\varphi_\varepsilon(\bar\partial Q_\varepsilon)^{n-1} \wedge Q\right\} \\ &= v\varphi_\varepsilon(\bar\partial Q_\varepsilon)^{n-1} \wedge \bar\partial Q + v\bar\partial\varphi_\varepsilon \wedge (\bar\partial Q_\varepsilon)^{n-1} \wedge Q. \end{aligned}$$

The last identity follows from the fact that v is a holomorphic function in ζ and the $(2n-2)$ form $(\bar\partial Q_\varepsilon)^{n-1}$ is $\bar\partial$ closed. Using this identity the representation formula (31) becomes

$$\begin{aligned} v(z) = &\frac{1}{(2\pi i)^n}\int_{\partial B_R} v(\zeta)\left\{K_\varepsilon + n\varphi_\varepsilon(\bar\partial Q_\varepsilon)^{n-1} \wedge Q\right\} \\ &+ \frac{1}{(2\pi i)^n}\int_{B_R} v(\zeta)\left\{\varphi(\bar\partial Q_\varepsilon)^n - n\bar\partial\varphi_\varepsilon \wedge (\bar\partial Q_\varepsilon)^{n-1} \wedge Q\right\}, \end{aligned} \tag{40}$$

where the integration is in the variable ζ and we have suppressed the dependency on (z,ζ) of the kernels.

LEMMA 3. *The following identity holds*

$$\begin{aligned} \bar\partial\varphi_\varepsilon \wedge (\bar\partial Q_\varepsilon)^{n-1} = (n-1)^{(n+1)n/2} &\frac{\varepsilon}{\left(|f(\zeta)|^2+\varepsilon\right)^{n+1}} \\ &\times\left[\sum_{j=1}^n (-1)^j (f_j(z) - f_j(\zeta)) \bigwedge_{k=1}^n g^k\right] \wedge \bigwedge_{k=1}^n \overline{\partial f_j}(\zeta), \end{aligned} \tag{41}$$

where the wedge products in (41) *are to be taken in their natural order, e.g.,* $\bigwedge_{k\neq 1} g^k = g^2 \wedge \cdots \wedge g^n$.

Proof of Lemma 3. We start by rewriting φ_ε,

$$\varphi_\varepsilon = \frac{\sum_{j=1}^n f_j(z)\bar{f}_j(\zeta) + \varepsilon}{|f(\zeta)|^2 + \varepsilon} = \frac{\sum_{j=1}^n \bar{f}_j(\zeta)\big(f_j(z) - f_j(\zeta)\big) + |f_j(\zeta)|^2 + \varepsilon}{|f(\zeta)|^2 + \varepsilon}$$

$$= 1 + \sum_{j=1}^n \big(f_j(z) - f_j(\zeta)\big)\frac{\bar{f}_j(\zeta)}{|f(\zeta)|^2 + \varepsilon}.$$

Denote $f_j = f_j(\zeta)$ and $\psi_j = \psi_j(\zeta) := (|f(\zeta)|^2 + \varepsilon)^{-1}\bar{f}_j$. Then we have

$$\varphi_\varepsilon = 1 + \sum_{j=1}^n \big(f_j(z) - f_j\big)\psi_j, \qquad Q_\varepsilon = \sum_{j=1}^n \psi_j g^j.$$

Therefore,

$$\bar\partial\varphi_\varepsilon \wedge \big(\bar\partial Q_\varepsilon\big)^{n-1} = \big[\textstyle\sum\big(f_j(z) - f_j\big)\bar\partial\varphi_j\big] \wedge \big[\textstyle\sum\bar\partial\varphi_j \wedge g^j\big]^{n-1}$$
$$= \sum_{j=1}^n \big(f_j(z) - f_j\big)\left[\bar\partial\psi_j \wedge \left[\sum_{k\neq j}\bar\partial\psi_k \wedge g^k\right]^{n-1}\right],$$

since $\bar\partial\psi_j \wedge \bar\partial\psi_j = 0$. Using that the 2-forms $\bar\partial\psi_k \wedge g^k$ commute and that the product of two of them with the same index vanishes, we have

$$\left[\sum_{k\neq j}\bar\partial\psi_k \wedge g^k\right]^{n-1} = (n-1)!\bigwedge_{k\neq j}\big(\bar\partial\psi_k \wedge g^k\big)$$
$$= (n-1)!(-1)^{(n-1)n/2}\Big(\bigwedge_{k\neq j} g^k\Big) \wedge \Big(\bigwedge_{k\neq j}\bar\partial\psi_k\Big).$$

Hence

$$\bar\partial\psi_j \wedge \left[\sum_{k\neq j}\bar\partial\psi_k \wedge g^k\right]^{n-1} = (n-1)!(-1)^{n(n+1)/2}\Big(\bigwedge_{k\neq j} g^k\Big) \wedge \Big(\bigwedge_{k=1}\bar\partial\psi_k\Big).$$

Now, we have $\bar\partial\psi_k = (|f|^2 + \varepsilon)^{-1}\overline{\partial f_k} - (|f|^2 + \varepsilon)^{-2}\bar{f}_k\,\bar\partial|f|^2$. Therefore we can use that $\bar\partial|f|^2 \wedge \bar\partial|f|^2 = 0$ and obtain

$$\bigwedge_{k=1}^n \bar\partial\psi_k = \frac{\bigwedge_{k=1}^n \overline{\partial f_k}}{\big(|f|^2 + \varepsilon\big)^n} - \frac{\sum_{j=1}^n\left[f_j \bigwedge_{1\le n<j}^n \overline{\partial f_k} \wedge \bar\partial|f|^2 \wedge \bigwedge_{j<k\le n}^n \overline{\partial f_k}\right]}{\big(|f|^2 + \varepsilon\big)^{n+1}}$$

If we now expand $\bar\partial|f|^2 = \sum_k f_k \overline{\partial f_k}$, we see that only the term $f_j \overline{\partial f_j}$ remains in the triple product above. Hence

$$\bigwedge_{k=1}^{n} \bar\partial\psi_k = \frac{\bigwedge_{k=1}^{n} \overline{\partial f_k}}{\left(|f|^2+\varepsilon\right)^n} - \frac{\left[\sum_{j=1}^{n}|f_j|^2\right]\bigwedge_{k=1}^{n}\overline{\partial f_k}}{\left(|f|^2+\varepsilon\right)^{n+1}} = \frac{\varepsilon}{\left(|f|^2+\varepsilon\right)^{n+1}}\bigwedge_{k=1}^{n}\overline{\partial f_k}.$$

This concludes the proof of Lemma 3. □

Lemma 3 tells us that $\bar\partial\varphi_\varepsilon \wedge (\bar\partial Q_\varepsilon)^{n-1} \wedge Q$ is the product of a measure with a smooth density, independent of ε, and the function $\varepsilon(|f(\zeta)|^2+\varepsilon)^{n-1}$. Lemma 2 can now be invoked to see that the volume integral in (40) reduces to a sum when $\varepsilon \to 0$. In fact, let us choose R so that $|f(\zeta)|^2 = |f_1(\zeta)|^2 + \cdots + |f_n(\zeta)|^2 \neq 0$ when $|\zeta| = R$. This choice is always possible since $\mathscr{Z}$ is a discrete set by assumption. In this case none of φ_ε, K_ϵ, Q_ε have singularities when $\varepsilon = 0$ and $\zeta \in \partial B_R$. We set φ_0, K_0, Q_0 to be the correspondent quantities. Therefore

$$v(z) = \frac{1}{(2\pi i)^n}\int_{\partial B_R} v(\zeta)\left\{K_0 + n\varphi_0\left((\bar\partial Q_0)^{n-1}\wedge Q\right)\right\}$$
$$+\frac{1}{(2\pi i)^n}\lim_{\varepsilon\to 0^+}\int_{B_R} v(\zeta)\left[\varphi(\bar\partial Q_\varepsilon)^n - n\bar\partial\varphi_\varepsilon \wedge (\bar\partial Q_\varepsilon)^{n-1}\wedge Q\right].$$

Recall that $f_j(\zeta) = 0$ if $\zeta \in \mathscr{Z}$ and $1 \le j \le n$. Using Lemma 2 and the definition (20) and (21) of the Δ_j we can compute explicitly the limit and obtain

$$v(z) = \frac{1}{(2\pi i)^n}\int_{\partial B_R} v(\zeta)\left(K_0 + n\varphi_0\left((\bar\partial Q_0)^{n-1}\wedge Q\right)\right)$$
$$+\sum_{\zeta\in\mathscr{Z}\cap B_R} v(\zeta)\frac{\Delta_{n+1}(z,\zeta)\varphi(z,\zeta)}{J(\zeta)} + \sum_{j=1}^{n}(-1)^{n+1-j}f_j(z)$$
$$\times \sum_{\zeta\in\mathscr{Z}\cap B_R} v(\zeta)\frac{\Delta_j(z,\zeta)}{J(\zeta)}. \quad (42)$$

Up to this moment we have only used that $\mathscr{Z}\cap\partial B_R = \varnothing$ and that v is holomorphic in a neighborhood of $\bar B_R$. To let $R\to\infty$ we have to choose a sequence $R\to\infty$ judiciously. Recall that $n(\mathscr{Z}, r) = \#(\mathscr{Z}\cap B_r) \le Cr^A$ for some positive constants A, D, and all $r \ge 1$. Let M be the smallest integer $\geq C(R+1)^A + 1$, divide the shell $\overline{B}_{R+1} \setminus B_R$ into M concentric subshells but choosing the boundaries to be $\partial B_{(R+j/M)}$, $0 \leq j \leq M$. There is at least one such subshell that is free from points of $\mathscr{Z}$; choose R' to be the mid-radius of this subshell, then $d(\zeta, \mathscr{Z}) \geq (2M)^{-1}$ if $|\zeta| = R'$. Starting from the sequence $R = q = 1, 2, \dots$ we construct a sequence R_q, $q < R_q < q+1$, such that for some positive constants A_1, N_1,

$$|f(\zeta)| \geq A_1 q^{-N_1} e^{H_0(\operatorname{Im}\zeta)} \quad \text{if} \quad |\zeta| = R_q. \tag{43}$$

This follows from (16) and the choice of R_q.

We are now ready to estimate the terms in the boundary integral of (42) for $R = R_f$. We will assume $|z| \leq C_0 < \infty$ and consider those q such that $R_q \geq C_0 + 1$.

First, let us observe that the functions g_k^j, $1 \leq j \leq m$, $1 \leq k \leq n$, satisfy an estimate of the form

$$|g_k^j(z,\zeta)| \leq C_1(1+|z|)^{M_1}(1+|\zeta|)^{M_1} e^{H_2(\operatorname{Im} z) + H_2(\operatorname{Im}\zeta)}, \tag{44}$$

for some constants $M_1, C_1 > 0$. If $1 \leq j \leq n$, we can replace H_2 by H_1. We can now estimate the coefficients of differential form Q. Denote $\|Q(z,\zeta)\|$ the largest absolute value of the coefficients of $d\zeta_k$ at the point (z,ζ). We proceed as follows. First,

$$|F(\zeta)| \geq |f(\zeta)| \geq A_1 q^{-M_1} e^{H_0(\operatorname{Im}\zeta)} \quad \text{if } |\zeta| = R_q. \tag{45}$$

Therefore,

$$\|Q(z,\zeta)\| \leq \frac{1}{|F(\zeta)|}\left[\sum_{j=1}^{n} \|g^j(z,\zeta)\|^2\right]^{1/2}$$
$$\leq \frac{C_1(1+|z|)^{M_1} e^{H_2(\operatorname{Im} z)}(1+R_q)^{M_1} e^{H_2(\operatorname{Im} z)}}{A_1 q^{-N_1} e^{H_0(\operatorname{Im}\zeta)}}$$

which leads to

$$\|Q(z,\zeta)\| \leq C_3 q^{N_2} e^{H_2(\operatorname{Im}\zeta) - H_0(\operatorname{Im}\zeta)}. \tag{46}$$

The constant C_3 depends in fact on z, but $|z| \leq C_0$ and $N_2 = M_1 + N_1$. (In fact C_3 can be estimated in terms of $e^{2H_2(\operatorname{Im} z)}$ and polynomials in $|z|$.) Similarly, with possibly different values for the constants C_3, N_2 appearing below, we have

$$\|\bar\partial Q(z,\zeta)\| \le C_3 q^{N_2} e^{H_2(\operatorname{Im} z) - 2H_0(\operatorname{Im} z)} \tag{47}$$

$$\|\bar\partial Q_0(z,\zeta)\| \le C_3 q^{N_2} e^{2H_1(\operatorname{Im} z) - 2H_0(\operatorname{Im} z)} \tag{48}$$

$$|\varphi_0(z,\zeta)| \le C_3 q^{N_1} e^{-H_0(\operatorname{Im} z)} \tag{49}$$

$$|\varphi(z,\zeta)| \le \frac{|F(z)|}{|F(\zeta)|} \le C_3 q^{N_1} e^{-H_0(\operatorname{Im} z)}. \tag{50}$$

To estimate K_0 we recall that α_2 can only take the values 0 and 1 in (34). In case $\alpha_2 = 0$ we have to estimate terms of the form $\varphi_0^{n-\alpha_1}\varphi(\bar\partial Q_0)^{\alpha_1}$, with $0 \le \alpha_1 \le n-1$. There are powers of q that we will disregard, the estimate is then the function of $\operatorname{Im}\zeta$,

$$\left\|\varphi_0^{n-\alpha}\varphi\left(\bar\partial Q_0\right)^{\alpha_1}\right\| \ll e^{-(n+\alpha_1+1)H_0 + 2\alpha_1 H_1}.$$

Since $H_1 \ge H_0$, the worst case estimate occurs when $\alpha_1 = n-1$. Hence the terms corresponding to $\alpha_2 = 0$, $\alpha_0 + \alpha_1 = n-1$, in the definition of K_0 can all be estimated by

$$C_3 q^{N_3} e^{-2nH_0(\operatorname{Im}\zeta) + 2(n-1)H_1(\operatorname{Im} z)}, \qquad |\zeta| = R_q. \tag{51}$$

The terms with $\alpha_1 = 1$, $\alpha_0 + \alpha_1 = n-2$ corresponds to the estimate of $\|\varphi_0^{n-\alpha_1}(\bar\partial Q_0)^{\alpha_1} \wedge \bar\partial Q\|$. The worst case occurs this time when $\alpha_1 = n-2$ and we obtain an estimate of the form

$$C_3 q^{N_3} e^{-2nH_0(\operatorname{Im}\zeta) + 2(n-2)H_1(\operatorname{Im}\zeta) + 2H_2(\operatorname{Im}\zeta)}, \qquad |\zeta| = R_q. \tag{52}$$

In (42) we have one more term to estimate for $|\zeta| = R_q$,

$$\left\|\varphi_0\left(\bar\partial Q_0\right)^{n-1} \wedge Q\right\| \le C_3 q^{N_3} e^{-2nH_0(\operatorname{Im}\zeta) + 2(n-1)H_1(\operatorname{Im}\zeta) + H_2(\operatorname{Im}\zeta)}. \tag{53}$$

The conditions (23) and (25) imply that the largest exponential factor in (51), (52), and (53) is the one in (53) and it satisfies

$$-2nH_0(\operatorname{Im}\zeta) + 2(n-1)H_1(\operatorname{Im}\zeta) + H_2(\operatorname{Im}\zeta) \le -r_0|\operatorname{Im}\zeta|. \tag{54}$$

Since we have assumed that $u \in C_0^\infty(\bar B_{r_0})$, we have

$$|\hat u(\zeta)| \le C_4(1+|\zeta|)^{-N_3-2n} e^{r_0|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbf{C}^n, \tag{55}$$

which allows us to conclude that, with $v = \hat u$,

$$\lim_{q\to\infty} \int_{\partial B_{R_q}} \hat u(\zeta)\left[K_0(z,\zeta) + n\varphi_0(z,\zeta)\left(\bar\partial Q_0\right)^{n-1} \wedge Q(z,\zeta)\right] = 0. \tag{56}$$

To conclude the proof of Theorem 1 we need to show that the series appearing in the representation formula (26) converge absolutely and uniformly in compact subsets of $\mathbf{C}^n$ to functions in $\hat{\mathscr{E}}'$. Since we have assumed that $\mathscr{Z}$ is almost real, the estimates of all the terms Δ_j, φ, J^{-1} are in terms of powers of $|\zeta|$. Using that $n(\mathscr{Z}, r) = O(r^A)$ and $|\hat{u}(\zeta)|$ decreases as fast as $|\zeta|^{-N_4}$ for $\zeta \in \mathscr{Z}$, we have the desired convergence once N_4 is chosen sufficiently large. The support of all the distributions thus obtained is contained in the convex set K whose supporting function is $(n+1)H_2$. □

Remarks 1. One can see that the condition $2H_1 \geq H_2$ cannot be relaxed if the other conditions of the theorem remain the same, otherwise the exponent in (52) would become positive and we would not be able to prove (56).

2. A way to weaken the conditions in Theorem 1 is to impose some better lower bound on $|F|$ than (45) that only depends on the first n functions. We will do so in Theorem 2 below.

3. It is clear that one only needs $u \in C_0^N(\overline{B}_{r_0})$ for N sufficiently large to obtain (26).

The following example shows how Theorem 1 enormously simplifies the computation of the deconvolution formula proposed in [7].

Let μ_1, μ_2, μ_3 be the characteristic functions of the squares centered at 0, of sides parallel to the axes and of lengths $2\sqrt{3}, 2\sqrt{2}, 2$, respectively. One can easily show [7, 17] that μ_1, μ_2 is a very well-behaved family with

$$H_0(\operatorname{Im}\zeta) = \sqrt{2}\left(|\operatorname{Im}\zeta_1| + |\operatorname{Im}\zeta_2|\right)$$

Here $H_1(\operatorname{Im}\zeta) = H_2(\operatorname{Im}\zeta) = \sqrt{3}(|\operatorname{Im}\zeta_1| + |\operatorname{Im}\zeta_2|)$. In this case the main hypotheses (24) reduces to verify that $4\sqrt{2} - 3\sqrt{3} > 0$!. Since $|x_1| + |x_2| = \sqrt{x_1^2 + x_2^2}$ for $x \in \mathbb{R}^2$, we have $r_0 = 4\sqrt{2} - 3\sqrt{3} \geq 0.2$. The variety $\mathscr{Z}$ in this case is given by

$$\mathscr{Z} = \left\{\left[\frac{j\pi}{\sqrt{3}}, \frac{k\pi}{\sqrt{2}}\right] : j, k \in \mathbf{Z}^*\right\} \cup \left\{\left[\frac{j\pi}{\sqrt{2}}, \frac{k\pi}{\sqrt{3}}\right] : j, k \in \mathbf{Z}^*\right\}.$$

(There were about forty different types of terms in [7] to compute.)

Before we proceed to state Theorem 2 we need to point out that the representation formula (31) does not depend on the particular choice of the differential forms g^j we have chosen, rather on the fact that (10) is satisfied. That is,

$$\sum_{k=1}^{n} g_k^j(z,\zeta)(z_k - \zeta_k) = f_j(z) - f_j(\zeta).$$

Now, let $f = \hat{\mu}$ and $h(z) = \sin Bz_1$ for some $B > 0$, and denote by g the differential forms associated to f by (8) and (11). Let us define a differential form γ by

$$\gamma(z, \zeta) := f(\zeta)\frac{\sin Bz_1 - \sin B\zeta_1}{z_1 - \zeta_1}d\zeta_1 + h(z)g(z, \zeta), \tag{57}$$

writing $\gamma = \Sigma\gamma_k\, d\zeta_k$ we have

$$\gamma_1(z, \zeta)(z_1 - \zeta_1) + \cdots + \gamma_n(z, \zeta)(z_n - \zeta_n) = f(z)h(z) - f(\zeta)h(\zeta).$$

Therefore we can associate γ to the product $f \cdot h$. It is also clear that as a function of z, the γ_k are Fourier transforms of distribution of compact support, easily computable terms of μ and B. Obviously we can replace $\sin Bz_1$ by $\sin Bz_j$ without any problems; hence, given a family $f_1, \ldots, f_m$ we can construct an *augmented family* $f_1, \ldots, f_m, f_{m+1} := f_1 \sin Bz_1, \ldots, f_{2m} := f_m \cdot \sin Bz_1, \ldots, f_{(n+1)m} := f_m \cdot \sin Bz_n$. The corresponding g^j for $j \geq m + 1$ are computed following the procedure (57). It is clear that if $f_1, \ldots, f_m$ was strongly coprime, the augmented family remains strongly coprime. If $f_1, \ldots, f_n$ form a very well-behaved family we will keep the notation H_0, H_1, H_2 to indicate the support functions corresponding to the m original members of the augmented family $f_1, \ldots, f_{(n+1)m}$.

THEOREM 2. *Let $f_1, \ldots, f_m$ be a strongly coprime family such that the subfamily $f_1, \ldots, f_n$ is very well behaved. There are constants $B_0 \geq 0$, $r_0 > 0$, such that for any $B \geq B_0$, and any $u \in C_0^\infty(\overline{B}_{r_0})$, the representation formula (26) is valid for the augmented family $f_1, \ldots, f_m, \ldots, f_{(n+1)m}$ defined above if either of the following two conditions holds*:

$$H_2 \leq 2H_1 \quad \text{and} \quad 2(n-1)H_1 < (2n-1)H_0 \tag{58}$$

$$2H_1 < H_2 \quad \text{and} \quad 2(n-2)H_1 + H_2 < (2n-1)H_0. \tag{59}$$

Proof. The proof is exactly the same as that of Theorem 1 except for improvements on the estimates (46), (47), and (50) for the new Q and φ. Recall that it is there where all the functions $f_1, \ldots, f_{(n+1)m}$ appear. Let $F_1 := (f_1, \ldots, f_{(n+1)m})$ and keep the notation $F = (f_1, \ldots, f_m)$ as before. We have

$$|F_1(\zeta)| = |F(\zeta)|\left(1 + |\sin B\zeta_1|^2 + \cdots + |\sin B\zeta_n|^2\right)^{1/2} = |F(\zeta)| \cdot \vartheta(\zeta). \tag{60}$$

It is clear that for some positive constants c_n, c'_n we have

$$\begin{aligned}\vartheta(\zeta) &= \left(1 + |\sin B\zeta_1|^2 + \cdots + |\sin B\zeta_n|^2\right)^{1/2}\\ &\geq c'_n\left[e^{2B|\operatorname{Im}\zeta_1|} + \cdots + e^{2B|\operatorname{Im}\zeta_n|}\right]^{1/2}\\ &\geq c_n e^{2(B/n)|\operatorname{Im}\zeta|_1} \geq c_n^{2(B/n)|\operatorname{Im}\zeta|},\end{aligned}$$

where $|\operatorname{Im}\zeta|_1 = \sum_{j=1}^n |\operatorname{Im}\zeta_j|$, $|\operatorname{Im}\zeta| = [\sum_{j=1}^n |\operatorname{Im}\zeta_j|^2]^{1/2}$. We estimate first g^j, $j = m + 1$, for $|z - \zeta| \geq 1$, since that is the only case that appears in the proof of (56). As it follows from (57) we have, for some i, k $(1 \leq i \leq n$, $1 \leq k \leq m)$, the estimate

$$\begin{aligned}\|g^j(z,\zeta)\| &\leq ce^{2B|\operatorname{Im} z_i|}\left[|f_k(\zeta)|\left(1 + |\sin B\zeta_i|^2\right)^{1/2} + \|g^k(z,\zeta)\|\right]\\ &\leq C(1 + |z|)^{N_1}(1 + |\zeta|)^{N_1} e^{B|\operatorname{Im} z_i|}\\ &\quad\times\left[|f(\zeta)|\vartheta(\zeta) + e^{H_2(\operatorname{Im} z)}e^{H_2(\operatorname{Im}\zeta)}\right].\end{aligned}$$

It follows that

$$\begin{aligned}\|Q(z,\zeta)\| &\leq \frac{1}{|F_1(\zeta)|}\left[\sum_{j=1}^{(n+1)m}\|g^j(z,\zeta)\|^2\right]^{1/2}\\ &\leq C_1(1 + |z|)^{N_1}(1 + |\zeta|)^{N_1} e^{B|\operatorname{Im} z| + H_2(\operatorname{Im} z)}\\ &\quad\times\left\{\frac{e^{H_2(\operatorname{Im}\zeta)}}{|F_1(\zeta)|} + \frac{|F(\zeta)|\vartheta(\zeta)}{|F_1(\zeta)|}\right\}.\end{aligned}$$

Therefore, for a positive constant C_2 depending on z, we obtain

$$\|Q(z,\zeta)\| \leq C_2(1 + |\zeta|)^{N_1}\left[1 + e^{H_2(\operatorname{Im}\zeta) - H_0(\operatorname{Im}\zeta) - (B/n)|\operatorname{Im}\zeta|_1}\right]. \tag{61}$$

Similarly,

$$|\varphi(z,\zeta)| \leq \frac{|F_1(z)|}{|F_1(\zeta)|} \leq C_2(1 + |\zeta|)^{N_1} e^{H_0(\operatorname{Im}\zeta) - (B/n)|\operatorname{Im}\zeta|_1}.$$

Finally,

$$\|\bar\partial Q(z,\zeta)\| \leq C \max_{1 \leq j,\, k \leq (n+1)m} \frac{\|\partial f_j(\zeta)\|\,\|g^k(z,\zeta)\|}{|F_1(\zeta)|^2}.$$

Now, for $j \geq m + 1$ we have for some $1 \leq i \leq n$, $1 \leq l \leq m$,

$$\| \partial f_j(\zeta) \| = \| \sin B\zeta_1 \cdot \partial f_l(\zeta) + f_l(\zeta) B \cdot \cos B\zeta_i \, d\zeta_i \|$$
$$\leq C_1(1 + |\zeta|)^{N_2} \big(\vartheta(\zeta) e^{H_2(\operatorname{Im}\zeta)} + |F(\zeta)| \vartheta(\zeta) \big).$$

It follows that

$$\| \bar\partial Q(z,\zeta) \| \leq C_2 \frac{(1 + |\zeta|)^{N_2}}{|F_1(\zeta)|^2} \Big[|F(\zeta)|^2 \vartheta(\zeta)^2 + 2|F(\zeta)| \vartheta(\zeta)^2 e^{H_2(\operatorname{Im}\zeta)} + \vartheta(\zeta) e^{2H_2(\operatorname{Im}\zeta)} \Big]$$
$$\leq C_2(1 + |\zeta|)^{N_2} \big[1 + 2e^{H_2(\operatorname{Im}\zeta) - H_0(\operatorname{Im}\zeta)} + e^{2H_2(\operatorname{Im}\zeta) - 2H_0(\operatorname{Im}\zeta) - (B/n)|\operatorname{Im}\zeta|_1} \big].$$

Choose $B_0 \geq 0$ such that

$$2H_2(\operatorname{Im}\zeta) - 2H_0(\operatorname{Im}\zeta) + (B_0/n)|\operatorname{Im}\zeta|_1 \leq 0. \tag{63}$$

When $B \geq B_0$ we will have $\|Q(z,\zeta)\| \leq C_2(1 + |\zeta|)^{N_1}$ and

$$\| \bar\partial Q(z,\zeta) \| \leq C_2(1 + |\zeta|)^{N_1} e^{H_2(\operatorname{Im}\zeta) - H_0(\operatorname{Im}\zeta)}, \tag{64}$$

where C_2 still denotes a constant depending on z of the form

$$C_2 = \operatorname{const}(1 + |z|)^{N_1} e^{H_2(\operatorname{Im} z) + B|\operatorname{Im} z|_1}.$$

We can now return to the proof of Theorem 1 at the point where we obtained the estimates (51), (52), (53). Ignoring powers of q, the exponential factors are

$$\exp(-2nH_0(\operatorname{Im}\zeta) + 2(n-1)H_1(\operatorname{Im}\zeta) - (B/n)|\operatorname{Im}\zeta|_1) \tag{51$'$}$$
$$\exp(-(2n-1)H_0(\operatorname{Im}\zeta) + 2(n-2)H_1(\operatorname{Im}\zeta) + H_2(\operatorname{Im}\zeta)) \tag{52$'$}$$
$$\exp(-(2n-1)H_0(\operatorname{Im}\zeta) + 2(n-2)H_1(\operatorname{Im}\zeta)). \tag{53$'$}$$

Under hypothesis (58) the largests of these three is (53′) and its exponent satisfies

$$-(2n-1)H_0(\operatorname{Im}\zeta) + 2(n-1)H_1(\operatorname{Im}\zeta) \leq -r_0|\operatorname{Im}\zeta|, \tag{53$''$}$$

for some $r_0 > 0$. If the hypothesis (59) holds, then the largest exponent is

(52′) and we define $r_0 > 0$ by

$$-(2n-1)H_0(\operatorname{Im}\zeta) + 2(n-2)H_1(\operatorname{Im}\zeta) + H_2(\operatorname{Im}\zeta) \le -r_0|\operatorname{Im}\zeta|. \tag{52″}$$

In either case the rest of the proof is the same as that of Theorem 1.

EXAMPLE. As [7] shows the family μ_1, μ_2, μ_3 obtained by taking $\mu_1 =$ characteristic function of the unit square $= \chi_{[-1,1]\times[-1,1]}$, μ_2 a rotation of μ_1 by 36° and μ_3 a rotation of μ_1 by 45° satisfies the first conditions of Theorem 1 and Theorem 2 with $H_0(\theta) = |\theta|$, since the squares contain the unit disk. One can easily convince oneself that the hypothesis (24) does not hold (e.g., take $\theta = (t, 0)$, $t > 0$.) On the other hand, one can take $H_1(\theta) = H_2(\theta) = \sqrt{2}\,|\theta|$, since all the squares are contained in the disk of radius $\sqrt{2}$. We are in the situation of hypothesis (58) and its verification reduces to the fact that

$$r_0 = 3 - 2\sqrt{2} > 0.$$

Furthermore, $B_0 = 4(\sqrt{2} - 1)$ works in this case.

3. POLYNOMIAL CASE

The conditions on Theorems 1 and 2 imply that the convex set defined by H_0 contains a ball. If we want to prove an algebraic version of (26), the fact that this condition is not satisfied plays a role. Such a representation was stated in [3, 4] without proof. We analyze here the conditions under which it is valid.

THEOREM 3. *Let $p_1, \dots, p_m$ be a family of polynomials in $\mathbf{C}^n$ without common zeros, suppose further that*:

(*a*) $D := \max_{1\le j\le m} \deg p_j = \deg p_i$ *for* $1 \le i \le n$.

(*b*) $\mathscr{Z} = \{z \in \mathbf{C}^n\colon p_i(z) = 0;\ 1 \le i \le n)$ *is discrete.*

(*c*) $J(z) :=$ *Jacobian determinant of* $p_1, \dots, p_n$ *at* z *is* $\neq 0$ *for all* $z \in \mathscr{Z}$.

(*d*) $p_1, \dots, p_n$ *have no common zeros at infinity, i.e.,* $\#\mathscr{Z} = D^n$.

Then

$$1 = \sum_{\zeta\in\mathscr{Z}} \frac{\Delta_{n+1}(z,\zeta)}{J(\zeta)}\varphi(z,\zeta) + \sum_{j=1}^{n} (-1)^{n+1-j} p_j(z) \sum_{\zeta\in\mathscr{Z}} \frac{\Delta_j(z,\zeta)}{J(\zeta)}, \tag{65}$$

where φ, Δ_j *are defined as in* (18)–(21) *with respect to the polynomials* $p_1, \dots, p_m$.

Remarks 1. The functions g_k^j defined by (8) are obviously polynomials of degree $D - 1$. It follows that (66) has the form

$$p_1(z)A_1(z) + \cdots + p_m(z)A_m(z) = 1 \tag{66}$$

for some polynomials $A_j \in \mathbf{C}[z_1, \ldots, z_n]$ of degree of most $n(D-1)$. This follows from the fact they are given as $n \times n$ determinants involving the g_k^j.

2. As before, the case $m = n + 1$ leads to a particularly pleasing form of (66),

$$1 = \sum \frac{1}{J(\zeta)} \frac{1}{p_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z,\zeta) \cdots g_1^{n+1}(z,\zeta) \\ \vdots \qquad\qquad \vdots \\ g_n^1(z,\zeta) \cdots g_n^{n+1}(z,\zeta) \\ p_1(z) \cdots p_{n+1}(z) \end{vmatrix}. \tag{67}$$

3. The two statements in condition (d) above are really a form of Bezout's theorem [19]. The meaning of the expression "$p_1, \ldots, p_n$ have no common zeros at infinity" is that if we introduce homogeneous polynomials $H_j(z_0, \ldots, z_n) := z_0^D p_j[z_1/z_0, \ldots, z_n/z_0]$ then the subset of $\mathbf{C}^{n+1}$ defined by $\{z_0 = 0, H_1 = \cdots = H_n = 0\}$ is $\{0\}$. This is equivalent to the statement

$$|p(z)| = \left[\sum_{j=1}^{n} |p_j(z)|^2\right]^{1/2} \geq C|z|^D \quad \text{if } |z| \geq R_0 \geq 1. \tag{68}$$

If we call $p_1^0(z)$ the leading homogeneous polynomial of p_j, then the estimate (68) is also equivalent to

$$\{z \in \mathbf{C}^n : p_1^0(z) = \cdots = p_n^0(z) = 0\} = \{0\}. \tag{69}$$

We also note that (69) implies (b) above. That is, condition (d) above implies condition (b).

Proof of Theorem 3. The proof is the same as that of Theorem 1. This time we take $v = 1$ and R arbitrary $\geq R_0$ (cf. (68)). One can estimate $\|Q(z, \zeta)\| \leq C/|\zeta|$, $\|\bar\partial Q(z, \zeta)\| \leq C/|\zeta|^2$, $|\varphi_0(z, \zeta)| \leq C/|\zeta|^D$, $\|\bar\partial Q_0(z, \zeta)\| \leq C/|\zeta|^2$ if $|\zeta| = R$ and $|z| \leq K \leq R - 1$, with $C = C(k) > 0$. These estimates imply that the boundary integral in (40) tends to zero when $R \to \infty$. □

If $p_1, \ldots, p_n$ are such that their leading terms p_j^0 satisfy (69) but their degrees $\deg p_j = D_j$ are not all equal or $\max\{D_j : 1 \leq j \leq n\}$ is smaller than $D = \max\{D_j : 1 \leq j \leq m\}$, then we can still prove a version of Theorem 3. That corresponds to the analytic counterpart of Theorem 1, that is, to

Theorem 2. For the moment we continue to assume that $J(z) \neq 0\ \forall z \in \mathscr{Z} = \{p_1 = \cdots = p_n = 0\}$.

Let $L_1(z) = u_1 z_1 + \cdots + u_n z_n$ be a linear homogeneous polynomial with generic coefficients. The condition that $\{z \in \mathbf{C}^n\colon L_1 = p_2^0 = \cdots = p_2^0 = 0\} \neq \{0\}$ is an algebraic condition on the coefficients of L_1. Therefore we can choose L_1 such that for any integer $d_1 \geq 0$ we have $\{[p_1^0 L_1^{d_1} = p_2^0 = \cdots = p_2^0 = 0\} = \{0\}$. Continuing in this fashion we can choose $L_1, \ldots, L_n$, $d_1, \ldots, d_n$, such that for any choice of constants $\varepsilon_1, \ldots, \varepsilon_n$, if we define $\tilde{p}_j = p_j(L_j + \varepsilon_j)^{d_j}$, then $\{\tilde{p}_1^0 = \cdots \tilde{p}_n^0 = 0\} = \{0\}$ and $\deg \tilde{p}_j = D$ for $1 \leq j \leq n$. It is clear now that for most choices of ε_j we still have that all common zeros of $\tilde{p}_1^0 = \cdots \tilde{p}_n^0$ are simple and $\tilde{p}_1^0 = \cdots \tilde{p}_n^0, p_{n+1}, \ldots, p_m$ have no common zeros. Theorem 3 can now be applied to this new family; one obtains polynomials A_j,

$$\sum_{j=1}^{m} A_j p_j = 1, \qquad \deg A_j \leq n(D-1), \tag{70}$$

and such that they have a representation of the type (65).

We remark that a representation such as (65) cannot be valid if the $p_1, \ldots, p_n$ have common zeros at infinity. For instance, in the example of Masser and Philippon in [11],

$$p_1 = z_1^D,\ p_2 = z_1 - z_2^D, \ldots, p_{n-1} = z_{n-2} - z_{n-1}^D,\ p_n = 1 - z_{n-1} z_n^{D-1}$$

one knows that $\delta = D^n - D^{n-1}$ is the best estimate possible for the degrees of A_j solving the polynomial Bezout equation. The polynomials

$$A_1 = z_n^\delta,\ A_2 = -z_n^\delta \frac{\left[z_1^D - z_2^{D^2}\right]}{z_1 - z_2^D}, \qquad A_3 = -z_n^\delta \frac{\left[z_2^D - z_3^{D^3}\right]}{z_2 - z_3^D}, \ldots,$$

$$A_{n-1} = -z_n^\delta \frac{\left[z_{n-2}^{D^{n-2}} - z_{n-1}^{D^{n-1}}\right]}{z_{n-2} - z_{n-1}^D}, \qquad A_n = \frac{1 - z_{n-1}^{D^{n-1}} z_n^\delta}{1 - z_{n-1} z_n^{D-1}},$$

have exactly this degree. On the other hand, if we had a representation like (65) we could conclude that there are solutions A_j of the polynomial Bezout equation with $\deg A_j \leq n(D-1)$ as in (70).

We would like now to show that the condition (c) for the simplicity of the zeros in Theorem 3 is not necessary. Regretfully, we only know how to do this in the case where $m = n + 1$.

THEOREM 4. *Let $p_1, \ldots, p_{n+1}$ be a fimily of polynomials in $\mathbf{C}^n$ without any common zeros, $D = \deg p_1 = \cdots = \deg p_n \geq \deg p_{n+1}$ and $\{p_1^0 = \cdots = p_n^0 = 0\} = \{0\}$. Then we can find polynomials A_j of degree $\leq$*

$n(D-1)$ satisfying the identity $\sum_1^{n+1} A_j p_j = 1$. The coefficients of the A_j can be written in terms of the values of p_{n+1} and values of the derivatives of p_{n+1}, and the coefficients of $g_k^{n+1}(z,\zeta)$ (when considered as polynomials in z), all of these evaluated at the points of $\mathscr{Z} = \{\zeta : p_1(\zeta) = \cdots = p_n(\zeta) = 0\}$.

Proof. For $0 < \varepsilon_j \ll 1$ the function $1/p_{n+1}$ is holomorphic in a neighborhood of $\Pi = \{|p_1| \le \varepsilon_1, \ldots, |p_n| \le \varepsilon_n\}$. This set is a compact polynomially convex set. (The compactness follows from condition (68)). By Sard's theorem one can choose the ε_j so that the sets $\{|p_j| = \varepsilon_j\}$ are real analytic submanifolds of $\mathbf{C}^n$. (In fact, we only need it in a neighborhood of Π.) For an $v \in \mathscr{H}(\overline{\Pi})$ we have that integral

$$\operatorname{Res}_{\mathscr{Z}}(v\,d\zeta_1 \wedge \cdots \wedge d\zeta_n) := \frac{1}{(2\pi i)^n} \int_{\substack{|p_1| = \sigma_1 \\ \vdots \\ |p_n| = \sigma_n}} v(\zeta) \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_n}{p_1(\zeta) \cdots p_n(\zeta)} \tag{71}$$

is independent of choice of $\sigma_1, \ldots, \sigma_n$ as long as $0 < \sigma_j \le \varepsilon_j$ and the $\{|p_j| = \sigma_j\}$ are smooth. Furthermore, if v is in the ideal generated by $p_1, \ldots, p_n$ in $\mathscr{H}(\Pi)$ then this residue is zero. Therefore, it depends only on the values of v at $\mathscr{Z}$ and a certain number of derivatives of v at $\mathscr{Z}$ (as it follows from the Nullstellensatz as presented, e.g., in [13, 16]). In other words, the integral (71) can be considered as an operator defined by a certain linear combination of the Dirac masses δ_ζ and their derivatives $(\partial^{|\alpha|}/\partial\zeta^\alpha)\,\delta_\zeta$, $\zeta \in \mathscr{Z}$, applied to the holomorphic function v. This operator is very hard to compute explicitly except in very simple cases but it is perfectly defined as the common value of all the integrals (71). It is called the *residue current* of $\mathscr{Z}$.

Let us consider now the polynomial $B(z)$ defined by

$$B(z) = \operatorname{Res}_{\mathscr{Z}} \frac{1}{p_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z,\zeta) & \cdots & g_1^{n+1}(z,\zeta) \\ \vdots & & \vdots \\ g_n^1(z,\zeta) & \cdots & g_n^{n+1}(z,\zeta) \\ p_1(z) & \cdots & p_{n+1}(z) \end{vmatrix} d\zeta_1 \wedge \cdots \wedge d\zeta_n. \tag{72}$$

This polynomial is in fact of the form $\sum_{j=1}^{n+1} A_j(z) p_j(z)$, with A_j polynomials of degree $\le n(D-1)$ whose coeffcients are computed in terms of the values of derivatives of p_{n+1} and the coefficients of g_k^j (as polynomials in z) evaluated over $\mathscr{Z}$. The only problem is to show $B(z) \equiv 1$. We fix values σ_j, $0 < \sigma_j < \varepsilon_j$. Consider complex numbers $\alpha_1, \ldots, \alpha_n$ sufficiently small and so chosen that:

(i) All the common zeros of $p_1 - \alpha_1, \ldots, p_n - \alpha_n$ are simple and lie in $\{|p_1| < \frac{1}{2}\sigma_1, \ldots, |p_n| < \frac{1}{2}\sigma_n\}$.

(ii) $p_1 - \alpha_1, \ldots, p_n - \alpha_n, p_{n+1}$ have no common zeros.

Note that $p_1 - \alpha_1, \dots, p_n - \alpha_n$ still do not have any common zeros at ∞. Let us denote $\mathscr{Z}_\alpha = \{z \in \mathbf{C}^n\colon p_1 - \alpha_1 = \cdots = p_n - \alpha_n = 0\}$. Then

$$\begin{aligned}
&\operatorname{Res}_{\mathscr{Z}_\alpha}(v\, d\zeta_1 \wedge \cdots \wedge d\zeta_n) \\
&= \frac{1}{(2\pi i)^n} \int_{\substack{|p_1| = \sigma_1 \\ \vdots \\ |p_n| = \sigma_n}} v(\zeta) \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_n}{(p_1(\zeta) - \alpha_1) \cdots (p_n(\zeta) - \alpha_n)} \\
&= \sum_{\zeta \in \mathscr{Z}_\alpha} v(\zeta)/J(\zeta),
\end{aligned} \tag{73}$$

where

$$J(\zeta) = \frac{\partial(p_1 - \alpha_1, \dots, p_n - \alpha_n)}{\partial(\zeta_1, \dots, \zeta_n)} = \frac{\partial(p_1, \dots, p_n)}{\partial(\zeta_1, \dots, \zeta_n)}$$

and the last identity follows from Stokes' theroem. (Replace the contour by little spheres about the distinct points of $\mathscr{Z}_\alpha$). This last identity is the effective computation of the residue current of $\mathscr{Z}_\alpha$, namely,

$$\operatorname{Res}_{\mathscr{Z}_\alpha} = \sum_{\zeta \in \mathscr{Z}_\alpha} J(\zeta)^{-1} \delta_\zeta.$$

The first identity in (73) shows that $\operatorname{Res}_{\mathscr{Z}_\alpha} \to \operatorname{Res}_{\mathscr{Z}}$ as $\sigma \to 0$, i.e., the residue currents at $\alpha = 0$ are continuous when acting on holomorphic $(n, 0)$ forms. On the other hand, by Theorem 3 we have

$$\operatorname{Res}_{\mathscr{Z}_\alpha}\left(\frac{1}{p_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z,\zeta) \cdots g_1^{n+1}(z,\zeta) \\ g_n^1(z,\zeta) \cdots g_n^{n+1}(z,\zeta) \\ p_1(z) - \alpha_1 \cdots p_{n+1}(z) \end{vmatrix} d\zeta_1 \wedge \cdots \wedge d\zeta_n \right) \equiv 1.$$

(Note that the g^j corresponding to p_j and to $p_j - \alpha_j$ coincide.) By continuity we obtain $B \equiv 1$. This concludes the proof of Theorem 4. □

Remarks 1. We can obviously obtain the same result without assuming the degrees of $p_1, \dots, p_n$ coincide or that they are larger or equal than that of p_{n+1}.

2. The reasoning of Theorem 4 extends to a strongly coprime family of $n + 1$ elements whose first n members form a well-behaved family. Under the other conditions of Theorem 1 or Theorem 2, we obtain a series representation of the solutions of Bezout equation which we computed in terms of the residue current associated to $\mathscr{Z}$. This time the series converges after grouping of terms.

3. The interest of the theorems in this section lies in the search for explicit algorithms to obtain solutions A_j for the algebraic Bezout equation $\sum_1^m A_j p_j = 1$ which satisfy Brownawell's estimate, $\deg A_j \le 3\mu n D^\mu$, $\mu = \min(n, m)$.

4. Conclusion

We have shown how explicit solutions to the analytic and algebraic Bezout equations can be obtained under natural restrictions on the original functions $f_1, \ldots, f_m$. This work has applications to the implementation of deconvolution for multidetector systems.

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