Trace, residue currents and multidimensional residues, duality and division, Graduate course, Spring 2007

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Contents

Chapter 1

Lesson 1 : Macaulay *versus* Kronecker

1.1 Why does complex analysis interfere with algebraic problems ?

A large part of this course will be devoted to commutative polynomial algebra in $\mathbb{K}[X_1, ..., X_n]$, where $\mathbb{K}$ is field. We will essentially be interested into effective geometric problems :

- describe in terms of its generators the zero set in $\overline{\mathbb{K}}^n$ (where $\overline{\mathbb{K}}$ denotes an algebraic closure of $\mathbb{K}$) of an ideal $I \subset \mathbb{K}[X_1, ..., X_n]$;
- estimate some geometric invariants related to this set from the geometric point of view) ;
- exhibit an effective solution for algebraic division or interpolation problems (degree estimates in Hilbert's nullstellensatz, membership to an ideal , to its integral closure, or to the Chow ideal of an algebraic cycle[1]).

When $\mathbb{K}$ has characteristic zero, we can in fact reduce ourselves to the case when $\overline{\mathbb{K}} = \mathbb{C}$ and profit from the tools involved in analysis in n complex variables analysis.

When $\mathbb{K}$ has positive characteristic, some of the ideas introduced dealing with the $\overline{\mathbb{K}} = \mathbb{C}$ situation can be (at least partially) imitated and suggest hints to deal with the questions mentioned above.

When $\mathbb{K}$ is a number field (or simply $\mathbb{K} = \mathbb{Q}$), one can also study effectivity not only from the geometric or algebraic points of view (the natural "indicator" being the degree), but also from the arithmetic point of view (the indicators being both the degree and the notion of "height" for the polynomials or ideals involved). We will show here also, because of the well known "product formula" [2] in arithmetics, that the pairing between arithmetic objects and analytic ones appears as a necessity ; note for example that the Mahler measure of a polynomial in $\mathbb{Z}[X_1, ..., X_n]$, namely

$$\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log |P(e^{i\theta_1}, ..., e^{i\theta_n})| \, d\theta_1 \cdots d\theta_n$$

[1] All these notions will of course be defined all along this course.

[2] When a is a non zero rational number, the product of <u>all</u> absolute values of a equals 1, but we have to take into account both archimedian (linked with arithmetics) and non archimedian (linked with analysis) valuations ; this holds also when a belongs to a number field.

(deeply connected with Jensen's formula or Nevanlinna theory) is indeed an analytic object ! Such a measure will play an important role in the definition of the logarithmic height of polynomials with coefficients in a number field.

In $\mathbb{C}^n$, we have at our disposal n complex variables $z_1, ..., z_n$, which appear as the natural "specializations" of the generators $X_1, ..., X_n$ of $\mathbb{C}[X_1, ..., X_n]$. But it is important not to forget, besides this, that $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, and therefore that we may use in fact $2n$ real variables $x_1, y_1, x_2, y_2, ..., x_n, y_n$, linked with the z_j through the relations

$$z_j = x_j + iy_j \ , \ j = 1, ..., n \, .$$

This allows much more freedom ! For example, it is possible to realize locally finite partitions of unity

$$1 = \sum_\iota \varphi_\iota$$

with smooth (that is C^∞) functions φ_ι localized *ad-hoc*, which is totally impossible with such "rigid" objects as polynomials in $z_1, ..., z_n$, or more generally holomorphic functions in the n complex variables $z_1, ..., z_n$.

Usually $\mathbb{C}^n$ is oriented so that the $2n$ differential form

$$dx_1 \wedge dy_1 \wedge \cdots dx_n \wedge dy_n$$

is positive (this is just a convention) and, instead of using $x_1, y_1, ..., x_n, y_n$ as "variables" in $\mathbb{C}^n$, it is more clever to use $z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n$, where

$$\overline{z}_j := x_j - iy_j \, .$$

Related to that choice, the linear differential operators (with complex coefficients) that we will use will be

$$\begin{aligned} \frac{\partial}{\partial z_j} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_j}\Big) \ , \ j = 1, ..., n \\ \frac{\partial}{\partial \overline{z}_j} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_j}\Big) \ , \ j = 1, ..., n \, . \end{aligned}$$

One way to solve algebraic problems involving the polynomial algebra $\mathbb{C}[X_1, ..., X_n]$ *with analytic techniques is to try to solve them using <u>all</u> the variables*

$$z_1, z_2, ..., z_n, \overline{z}_1, ..., \overline{z}_n$$

and then (in a second step), show that in the solution which has been proposed, the "antiholomorphic" variables $\overline{z}_1, ..., \overline{z}_n$ *play some <u>neutral</u> role (namely, are treated just as constants playing the role of irrelevant parameters). This "philosophy" will be the central philosophy of this course.*

A key reason why the antiholomorphic variables are so useful is that they are essential in order to materialize the crucial notion of <u>positivity</u> in $\mathbb{C}^n$. Positivity is indeed an essential property when one has in mind to establish estimates (which is the final goal of effectivity in polynomial algebra, algebraic or arithmetic geometry). The trivial inequality

$$\sum_{j=1}^n z_j \overline{z}_j \geq 0 \ , \ \forall z \in \mathbb{C}^n$$

is a capital one, together with the positivity of the differential form

$$(-1)^{n(n-1)/2}(2i)^{-n}d\overline{z} \wedge dz\,,$$

where

$$\begin{aligned} dz &:= \bigwedge_{j=1}^{n} dz_j \\ d\overline{z} &= \bigwedge_{j=1}^{n} d\overline{z_j} \end{aligned}$$

(taking into account the convention about the orientation of $\mathbb{C}^n$).
The notion of distribution (introduced by physicists like Paul Dirac around 1920) and formalized by L. Schwartz [Sch], will be for us an essential tool in order to profit of the "smoothness" of analysis ($2n$ degrees of freedom instead of n) compared to the "rigidity" of polynomial algebra or even analytic geometry (involving holomorphic objects, that is objects depending <u>in a true sense</u> only of the holomorphic variables $z_1, ..., z_n$).
When U is an open subset in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, the vectorial space

$$\mathcal{D}(U) := \{\varphi \ : U \longrightarrow \mathbb{C}\,;\, \mathrm{Supp}\,\varphi \subset\subset U\,,\ \varphi \text{ is } C^\infty\}$$

(called space of test functions in U) can be equipped with an inductive limit topology. We will not describe precisely this topology here (refer to the text book [Y] for more details) ; it will be enough for our purpose to say what mean the convergence of a sequence $(\varphi_n)_n$ toward a function φ when n tends to infinity. It means :

- that, for n large enough, all sets $\mathrm{Supp}\,\varphi_n$ are included in some compact subset K of U (so for the support of the limit function φ) ;
- The convergence of $(\varphi_n)_n$ toward φ on K is the uniform convergence (on the compact K) of any sequence of derivatives (at any order) $(D^l\,\varphi_n)_n$ toward the test function $D^l\,\varphi$, for any $l \in \mathbb{N}^{2n}$.

It will be enough for us to formulate the definition of a distribution in U as follows :

Definition 1.1 *A distribution $T \in \mathcal{D}'(U)$ is a continuous linear form on $\mathcal{D}(U)$ when $\mathcal{D}(U)$ is equipped with the above topology.*

An important "prototype" of distribution is the Dirac mass as the origin, defined by the duality bracket

$$\langle \delta_0\,,\,\varphi\rangle := \varphi(0)\,.$$

Radon measures

$$\varphi \longmapsto \langle T\,,\,\varphi\rangle := \int \varphi\,d\mu$$

are examples of distributions (one says such particular distributions have order 0 since their action on a test function does not imply any true derivative of the test function).

Distributions in some open subset U of $\mathbb{C}^n$ may be differentiated (at any order) through repetitions of the duality formulas

$$\begin{aligned}\Big\langle \frac{\partial T}{\partial x_j}\,,\,\varphi\Big\rangle &:= -\Big\langle T\,,\,\frac{\partial \varphi}{\partial x_j}\Big\rangle,\ j=1,...,n\\ \Big\langle \frac{\partial T}{\partial y_j}\,,\,\varphi\Big\rangle &:= -\Big\langle T\,,\,\frac{\partial \varphi}{\partial y_j}\Big\rangle,\ j=1,...,n\end{aligned}$$

(the sign being here to mimic the process of integration by parts).

The concept of *current*[3] (since it allows to take more into account the notion of positivity) will be more important for us than the concept of distribution. It has been essentially introduced by P. Lelong [Lel] who realized the major fact that represents the notion of positivity in $\mathbb{C}^n$. It will be enough for us to know that a (p,q) current is an open subset U of $\mathbb{C}$ is just a (p,q) *differential from with distribution coefficients*, that is some object denoted as

$$T = \sum_{1\le k_1<k_2<\cdots<k_p\le n}\ \sum_{1\le l_1<l_2<\cdots<l_q\le n} T_{k\,,\,l} \bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\,,$$

where the coefficients $T_{k\,,\,l}$ are distributions in U. This just means that T is a linear continuous form on the space of $(n-p,n-q)$ test forms with test functions coefficients as follows : if

$$\begin{aligned}\{1,...,n\}\setminus\{k_1,...,k_p\} &= \{\kappa_1,...,\kappa_{n-p}\}\\ \{1,...,n\}\setminus\{l_1,...,l_q\} &= \{\lambda_1,...,\lambda_{n-q}\}\end{aligned}$$

(in increasing order), the action of such T (as above) on the smooth form

$$\varphi \bigwedge_{j=1}^{n-p} dz_{\kappa_j}\wedge \bigwedge_{j=1}^{n-q} d\overline{z}_{\lambda_j}\,,\ \ \varphi\in\mathcal{D}(U)\,,$$

is just

$$\pm(2i)^n\langle T_{k\,,\,l}\,,\,\varphi\rangle$$

the sign corresponding to the signature of the permutation that transform

$$\bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\wedge \bigwedge_{j=1}^{n-p} dz_{\kappa_j}\wedge \bigwedge_{j=1}^{n-q} d\overline{z}_{\lambda_j}$$

into

$$\bigwedge_{j=1}^{n}(d\overline{z}_j\wedge dz_j) = (2i)^n dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n\,.$$

Currents may be differentiated through the differential operators ∂d and $\overline{\partial}$, with the convention

$$\begin{aligned}\partial\Big[T_{k\,,\,l}\bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\Big] &= \Big(\sum_{j=1}^{n}\frac{\partial T_{k\,,\,l}}{\partial z_j}dz_j\Big)\wedge \bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\\ \overline{\partial}\Big[T_{k\,,\,l}\bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\Big] &= \Big(\sum_{j=1}^{n}\frac{\partial T_{k\,,\,l}}{\partial \overline{z_j}}d\overline{z}_j\Big)\wedge \bigwedge_{j=1}^{p} dz_{k_j}\wedge \bigwedge_{j=1}^{q} d\overline{z}_{l_j}\,.\end{aligned}$$

[3]There will be a lecture introducing the notions of current, positive current, Lelong number of a positive current, integration current on an analytic set, in the “Atelier” for students, starting with the presentation in [Lel].

1.2 Some examples of classical division problems in $\mathbb{K}[X_1, ..., X_n]$

1.2.1 Bézout identity

One of the most important questions (for robotics, control theory, recovering of blurred data) related to division problems in the polynomial algebra $\mathbb{K}$ is the search (when it is possible) for a Bézout type identity involving polynomial entries $P_1, ..., P_m$ in $\mathbb{K}[X_1, ..., X_n]$. Let us state first the fundamental result :

Theorem 1.1 [G. Hermann, 1929] *Let $P_1, ..., P_m$ m polynomials in the polynomial algebra $\mathbb{K}[X_1, ..., X_n]$, where $\mathbb{K}$ is an arbitrary commutative field. Assume $P_1, ..., P_m$ have no common zero in $\overline{\mathbb{K}}^n$, where $\overline{\mathbb{K}}$ is some integral closure of K. Then, there are polynomials $A_1, ..., A_m$ in $\mathbb{K}[X_1, ..., X_n]$ such that*

$$1 = A_1 P_1 + \cdots + A_m P_m \tag{1.1}$$

and

$$\max(\deg A_j) \leq 2(2D)^{2^{n-1}} . \tag{1.2}$$

Sketch of the proof. The original reference for G. Hermann's algorithmic proof is [Her] ; there is also a modern version in [MW]. Since we will see later in this course that the degree estimates (1.2) have been drastically sharpened [4] around 1985-1995, we will not insist here on the precise proof. We just present here the general line of ideas (inspired by elimination theory ideas) which sustain the construction of G. Hermann's algorithm. Our reference will be [VdW], volume II, chapter 11, satz 77-78-79 (see also [L], chapter IX)[5].

We may assume $\mathbb{K}$ is infinite (just add a transcendental parameter t in order to work in $\mathbb{K}(t)$ and get rid of he parameter at the end). One can perform a linear (invertible) change of coordinates $X = AY$ (with coefficients in $\mathbb{K}$) so that in the new set of coordinates $Y_1, ..., Y_n$, one has (if $d_j = \deg P_j$, $j = 1, ..., m$)

$$P_j(Y) = Y_1^{d_j} + \sum_{k=1}^{d_j} Y_1^{d_j - k} p_{j,k}(Y_2, ..., Y_n)$$

with $\deg p_{j,k} \leq k$. In order to do that, it is enough to choose the first column vector $(\xi_1, ..., \xi_n)$ of A such that the product of the homogeneous parts of higher degree of $P_1, ..., P_n$ does not vanish at the point ξ ; then, complete the matrix using just the incomplete basis theorem.

We now take P_1 as the polynomial of higher degree in the list and introduce additional transcendental parameters $\lambda_2, ..., \lambda_n$; then, we form the Sylvester resultant of P_1 and $P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$ (see section 1.3 for a brief recall and references about such notion), considered as polynomials in the single variable Y_1. This is a $2d_1 \times 2d_1$ determinant, which appears as a polynomial in the variables $\lambda_2, ..., \lambda_n, Y_2, ..., Y_n$,

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[4] Originally, precisely under the impulsion of ideas inspired by analysis.

[5] A detailed presentation of Hermann's proof (actualized in [MW]) and [VdW] approach toward elimination theory through resultant systems will be the theme of a lecture by the students in the "Atelier".

with coefficients in $\mathbb{Z}$[coefficients of $P_1, ..., P_m$]. This Sylvester determinant can be written as

$$R_1(\lambda, Y_2, ..., Y_n) = \sum_{\alpha \in \mathbb{N}^{m-1}} \lambda_2^{\alpha_2} \cdots \lambda_m^{\alpha_m} R_{1,\alpha}(Y_2, ..., Y_n) .$$

Each of the $R_{1,\alpha}$ (there are only a finite number of them which are non zero since R_1 is polynomial in λ) can be written as

$$R_{1,\alpha} = \sum_{j=1}^{m} U_{\alpha,j} P_j , \tag{1.3}$$

where the $U_{\alpha,j}$ are polynomials (with integer coefficients) in the coefficients of $P_1, ..., P_m$. Moreover, if the polynomials $R_{1,\alpha}$ have a common root $(\xi_2, ..., \xi_n)$ in $\overline{\mathbb{K}}^{n-1}$, then, for generic values of λ, the polynomials $P_1(Y_1, \xi)$ and

$$\lambda_2 P_2(Y_1, \xi) + \cdots + \lambda_m P_m(Y_1, \xi)$$

have a common root. If we specialize $\lambda_2, ..., \lambda_m$ to particular values in $\mathbb{K}$ (remember $\mathbb{K}$ is assumed to be infinite), we can see (because of the "box principle"[6]) that the polynomials $P_1(Y_1, \xi), ..., P_m(Y_1, \xi)$ should have a common zero. Since this is impossible, the system $(R_{1,,\alpha})_\alpha$ is a collection of polynomials with coefficients in $\mathbb{K}$,in $n-1$ variables $Y_2, ..., Y_n$, with no common zero in $\overline{\mathbb{K}}^n$. One can repeat the procedure with this new system instead of the original one $(P_1, ..., P_m)$ and continue that way ; if we get an identity

$$1 = \sum_{\alpha} A_{1,\alpha} R_{1,\alpha} ,$$

then an identity $1 = A_1 P_1 + \cdots + A_m P_m$ will follow because of the relations (1.3). This ends the synopsis of the proof. Hermann's bounds follow when one analyzes precisely the control of degree estimates all along the procedure (as done in details in [MW] for example)[7]. $\diamondsuit$

Methods presented here are based on elimination theory, which was extensively developed at the end of the XIX-th century and culminated at the beginning of the XX-th century with the work of F. Macaulay (see for example [Mac]). Sylvester resultant appears as a particular case of the notion of resultant $\mathcal{R}(F_0, ..., F_n)$ of $n+1$ homogeneous forms (with respective degrees $d_0, d_1, ..., d_n$) in the $n+1$ variables $X_0, ..., X_n$. Such a resultant is an irreducible multi homogeneous polynomial with integer coefficients, homogeneous with degree $\prod_{l \neq j} d_l$ in the coefficients of the polynomial F_j. The non vanishing of $\mathcal{R}(F_0^{(\xi)}, ..., F_{n+1}^{(\xi)})$ for specialized coefficients ξ in the homogeneous forms F_j means precisely that the corresponding projective algebraic sets

$$\{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n(\overline{\mathbb{K}})\,;\, F_j^{(\xi)}(z_0, ..., z_n) = 0\} ,\ j = 0, ..., n$$

[6]This principle, which was popularized by C. L. Siegel (following Dirichlet), is the following : if we have to organize the repartition of M matches in N boxes (with $M > N$), any configuration will be such that there are at least two matches in at least one among the boxes !

[7]Note that, as soon as the degree of the A_j can be predicted, the search for $A_1, ..., A_m$ can be done solving a system of linear equations which is known to be compatible.

do not intersect in $\mathbb{P}^n(\overline{\mathbb{K}})$. Moreover, there is a formula

$$\mathcal{R}(F_0,...,F_{n+1}) = \sum_{j=0}^{n} R_j(X_0,...,X_n)F_j(X_0,...,X_n)\,,$$

such that the coefficients of the R_j are polynomials (with integer coefficients) in the coefficients of the homogeneous forms $F_0,...,F_n$. For elimination theory following Macaulay, see for example the introductive presentation in [L], chapter IX.

1.2.2 Hilbert's nullstellensatz

Let $\mathbb{K}$ a commutative field and I is an ideal in $\mathbb{K}[X_1,...,X_n]$. The radical of I ($\sqrt{I}$) is defined as the set of polynomials Q in $\mathbb{K}[X_1,...,X_n]$ such that Q vanishes on the zero set of I

$$V(I) := \{z \in \overline{\mathbb{K}}^n\,;\, P(z)=0\,,\ \forall P \in I\} \subset \overline{\mathbb{K}}^n\,,$$

which is an affine algebraic subspace in $\overline{\mathbb{K}}^n$. The radical $\sqrt{I}$ contains I, but the inclusion

$$I \subset \sqrt{I}$$

is in general strict. A *radical* ideal is an ideal which equals its radical, a *prime* ideal is an ideal I such that

$$PQ \in I \Longrightarrow (P \in I) \vee (Q \in I)\,;$$

a *primary* ideal is a ideal whose radical is prime.

A geometric approach (as we will see in this course) toward an ideal I will only provide information on the radical of I ; combined with analytic techniques involving Lelong numbers for positive currents [8], we will see in this course that such geometric approach carries information about "multiplicities" attached to irreducible components of the algebraic set $V(I)$ (we will explain later "multiplicities" in which sense). Neither geometric nor analytic approach can allow a complete vision of the algebraic setting. In fact, any ideal in $\mathbb{K}[X_1,...,X_n]$ admits a primary decomposition

$$I = \bigcap_k \mathfrak{Q}_k$$

where $\mathfrak{Q}_k$ is a primary ideal (see [Mat] for example, this is a consequence of finitely generated modules theory) ; though such a decomposition is not unique, the list $\mathrm{Ass}\,(\mathbb{K}[X_1,...,X_n]/I)$ of prime ideals involved in such decomposition is uniquely determined by I. The associated primes $\mathfrak{P}$ of an ideal can be organized respect to the order

$$\mathfrak{P}_1 \leq \mathfrak{P}_2 \Longrightarrow \mathfrak{P}_1 \subset \mathfrak{P}_2\,.$$

The minimal primes in the family $\mathrm{Ass}\,(I)$ are called the *isolated* associated primes of I ; the other ones are called the *embedded* ones. For example

$$(X_1, X_1X_2) = (X_1) \cap (X_1, X_2)\,,$$

so that here the principal ideal (X_1) is an isolated associated prime, the ideal (X_1, X_2) being a non-isolated associated one. The geometric approach leads to

[8] All these notions will be defined later on.

the identification of the zero sets of the isolated primes (called the isolated components) in the decomposition of $V(I)$; it does not provide any information about the non-isolated ones. One of the purposes of this course is to show that the analytic approach, though it is unable to carry all the information related to the embedded components, carries some information on ideals $\overline{I}$ and $I^{\rm Chow}$ which are intermediate between I and $\sqrt{I}$ ($I \subset \overline{I} \subset \sqrt{I}$, $I \subset I^{\rm Chow} \subset \sqrt{I}$). Such information will deeply help for the effectiveness of the Hilbert's nullstellensatz we formulate below :

Theorem 1.2 [D. Hilbert's nullstellensatz] *Let $I = (P_1, ..., P_m)$ be a finitely generated ideal in $\mathbb{K}[X_1, ..., X_n]$ ($\mathbb{K}$ being a commutative field). There exists an integer q (depending on I) such that $(\sqrt{I})^q \subset I$.*

Proof. The proof is based on Bézout identity combined with the ingenious Rabinovitch's trick. Let us take Q in $\sqrt{(P_1, ..., P_m)}$ and let X_0 be an additional variable. Consider in $\mathbb{K}(X_0, X_1, ..., X_n]$ the $m+1$ polynomials

$$P_1(X_1, ..., X_n), ..., P_m(X_1, ..., X_n), 1 - X_0 Q(X_1, ..., X_n)\,.$$

Since $Q \in \sqrt{I}$, these polynomials have no common zero in $\overline{\mathbb{K}}^{n+1}$. So, one can apply theorem 1.1 and get polynomials $A_0(X_0, ..., X_n)$, ..., $A_m(X_0, X_1, ..., X_n)$ such that

$$1 = (1 - X_0 Q(X_1, ..., X_n)) A_0(X_0, X_1, ..., X_n) + \sum_{j=1}^{m} A_j(X_0, X_1, ..., X_n) P_j(X_1, ..., X_n)\,.$$

From such a polynomial identity, we get, if we specialize $X_0 = 1/Q$, the following rational identity :

$$1 = \sum_{j=1}^{m} A_j\Big(\frac{1}{Q(X_1, ..., X_n)}, X_1, ..., X_n\Big) P_j(X_1, ..., X_n)\,;$$

raising denominators, we show there exists some exponent q such that

$$Q^q \in \sqrt{I}\,.$$

The result is proved ; how efficient q can be depends of course on the degree estimates for Bézout identity[9]. $\diamondsuit$

1.2.3 The intrinsic hardness of the "membership" problem

Another important example of division problem in polynomial algebra is the membership problem : given an ideal $I = (P_1, ..., P_m)$ in $\mathbb{K}[X_1, ..., X_n]$ ($\mathbb{K}$ being a commutative field) and a polynomial Q which is known to lie in I, how intrinsically difficult is it to express it explicitly as

$$Q = Q_1 P_1 + \cdots + Q_m P_m\,.$$

[9]The reason why Hilbert's nullstellensatz happens to be so important (not only in mathematics, but also in computer science and logic), is that this problem is known to be an NP complete one (for the basic notions about the classes of complexity problems P or NP and adequate references, we refer for example to the survey presentation in [Y7]). The problem to decide whether the Hilbert's nullstellensatz over $\mathbb{C}$ can be solved in polynomial time is still not known (if it was not, it would answer the conjecture $P \neq NP$ over $\mathbb{C}$). The classical $P \neq NP$ conjecture can be reformulated saying that the algebraic nullstellensatz over $\mathbb{F}_2$ cannot be solved in polynomial time ; this is another challenge, which of course remains also today still unknown.

A disappointing result about this question came in 1988, when computer scientists E. Mayr and A. Meyer constructed in [MM], for each integer $D \geq 5$, for each integer $k > 1$, a collection $F_1, ..., F_{10k+1}$ binomials in n variables (with integer coefficients) such that $X_1 \in (F_1, ..., F_{10k+1})$ and the minimum of the degrees of the Q_j, $j = 1, ..., 10k+1$, in any polynomial identity

$$X_1 = Q_1 F_1 + \cdots + Q_{10k+1} F_{10k+1}$$

is greater than $(D-2)^{2^{k-1}}$; it means there is no hope to solve the membership problem under double exponential time ! This does not mean of course that special cases of the membership problem (such as Bézout identity or the closely related Hilbert's nullstellensatz) could not be solved with much better bounds (in fact, we will see they may be).

1.3 Several ways to solve the Bézout identity in dimension 1

Let us point out at the beginning of this course that there are (at least) four methods (which are intrinsically different) in order to solve the famous Bézout identity (with two polynomials) in the polynomial algebra $\mathbb{C}[X]$; namely, given two polynomials P_1 and P_2 with no common zero in $\mathbb{C}$, construct two polynomials A_1 and A_2 (with respective degrees $\deg P_2 - 1$ and $\deg P_1 - 1$) such that

$$A_1 P_1 + A_2 P_2 \equiv 1\,.$$

The first method lies on the use of Euclidean division algorithm, following back the computations that lead to the GCD (here 1) of P_1 and P_2. What makes the advantage of this method (among all others we will present here) is that it is the only one which is <u>algorithmic</u>. It works when P_1 and P_2 have coefficients in any field $\mathbb{K}$.

Besides this method, one can use basic elimination theory (as in the G. Hermann's algorithm, see [Her, VdW]) : if P_1 and P_2 are two polynomials in $\mathbb{K}[X_1]$, such that P_1 and P_2 have no common zero in some algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, such that $\deg P_1 = p$, $\deg P_2 = q$ and

$$\begin{aligned} P_1(X) &= a_0 X^p + a_1 X^{p-1} + \cdots + a_p \\ P_2(X) &= b_0 X^q + b_1 X^{q-1} + \cdots + b_q\,, \end{aligned}$$

one can form the Sylvester resultant, that is the $(p+q, p+q)$ determinant.

$$\mathcal{R}(P_1,P_2) := \begin{vmatrix} a_0 & a_1 & \dots & a_p & 0 & \dots & 0 & 0 \\ 0 & a_0 & \dots & a_{p-1} & a_p & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \dots & a_p & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & a_{p-1} & a_p \\ b_0 & b_1 & \dots & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & b_{q-1} & b_q \end{vmatrix}\,.$$

Linear algebra manipulations show such resultant can be expressed in the ideal generated by (P_1, P_2) in $\mathbb{K}[X]$ as

$$\mathcal{R}(P_1, P_2) = U_1(X)P_1(X) + U_2(X)P_2(X)$$

(note that U_1 and U_2 have coefficients which can be expressed as polynomial expressions with integer coefficients of the coefficients of P_1, P_2). Moreover, $\mathcal{R}(P_1, P_2) \neq 0$ is equivalent to the fact that P_1 and P_2 are co prime in $\mathbb{K}[X]$ (that is, have no common zero in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$) ; for an elementary approach of such basic results in elimination theory, one can for example refer to the textbook of J.M. Arnaudiès and J. Lelong-Ferrand [ArnLF] or to the recent textbook by F. Apéry and J.P. Jouanolou [ApJ]. This second method is intrinsically different from the first one since it is based on a formula instead of an algorithm.

The third method is based on also on a formula, but it lies on a more analytic idea, the use of Lagrange interpolation formula. When P_1 and P_2 have simple zeroes, one can take as

$$\begin{aligned} A_1(X) &:= \sum_{P_2(\beta)=0} \frac{1}{P_1(\beta)P_2'(\beta)} \frac{P_2(X)}{X-\beta} \\ A_2(X) &:= \sum_{P_1(\alpha)=0} \frac{1}{P_1'(\alpha)P_2(\alpha)} \frac{P_1(X)}{X-\alpha} \, ; \end{aligned}$$

when the zeroes of P_1 or P_2 are not simple simple anymore, Lagrange interpolators are more involved and one needs to take

$$\begin{aligned} A_1(X) &:= \operatorname{Res} \left[\begin{matrix} \frac{1}{P_1(\zeta)} \frac{P_2(X)-P_2(\zeta)}{X-\zeta}\, d\zeta \\ P_2(\zeta) \end{matrix} \right] \\ A_2(X) &:= \operatorname{Res} \left[\begin{matrix} \frac{1}{P_2(\zeta)} \frac{P_1(X)-P_1(\zeta)}{X-\zeta}\, d\zeta \\ P_1(\zeta) \end{matrix} \right] , \end{aligned}$$

where the notation

$$\operatorname{Res} \left[\begin{matrix} R(\zeta)\, d\zeta \\ Q(\zeta) \end{matrix} \right]$$

denotes (when R is a rational function and Q a polynomial such that R is regular on $\{P = 0\}$), the total sum of residues (at all zeroes of Q) of the $(1,0)$-rational form

$$\omega(\zeta) = \frac{R(\zeta)}{Q(\zeta)}\, d\zeta \, .$$

Of course, formula $1 \equiv A_1P_1 + A_2P_2$ is easy to check since $1 - (A_1P_1 + A_2P_2)$ appears as a polynomial with degree $\deg P_1 + \deg P_2 - 1$ which vanishes at exactly $\deg P_1 + \deg P_2$ points (namely the zeroes of P_1 and P_2 counted with multiplicities) in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$.

The last method we propose here (when $\overline{\mathbb{K}} = \mathbb{C}$, that is in fact, for any $\mathbb{K}$ with characteristic zero) is deeply connected with the central formula in one variable complex analysis that is Cauchy formula :

$$1 = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{d\zeta}{\zeta - z} \, , \ \forall z \, , \ |z| < R \, . \tag{1.4}$$

Though such a formula looks an analytic formula at first glance, note that the integral symbol has just a formal meaning, since one knows that it can be replaced by any symbol

$$\int_{\gamma_z}$$

where γ_z is a closed loop with support in $\mathbb{C} \setminus \{z\}$ such that $\mathrm{Ind}(\gamma, z) = 1$; nevertheless, we will use here the analytic model since it will appear as very convenient. We take for the moment R strictly bigger than the modulus of all zeroes of P_1 (we assume $\deg P_1 \geq 1$, otherwise Bézout identity $1 = A_1 P_1 + A_2 P_2$ can be trivially realized), so that the open disk $D(0, R)$ contains all zeroes of P_1. Let us write

$$P_1(\zeta) - P_1(z) = g_1(\zeta, z)(\zeta - z)\,,$$

where g_1 is a polynomial in the two variables ζ and z (this can be done using the well known identities $\zeta^k - z^k = (\zeta - z)(\zeta^{k-1} + \cdots + z^{k-1})$, note that such trivial identities will be a key ingredient several times in this course). Then, one can rewrite (1.4) as

$$1 = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{g_1(\zeta, z)\, d\zeta}{P_1(\zeta) - P_1(z)}\,.$$

If z is sufficiently closed from one particular zero of P_1 (let us say for example α), one can assume

$$|P_1(z)| < \min_{|\zeta|=R} |P_1(\zeta)|$$

and therefore develop

$$\frac{1}{P_1(\zeta) - P_1(z)}$$

as

$$\sum_{k=0}^{\infty} \frac{(P_1(z))^k}{(P_1(\zeta))^{k+1}} \tag{1.5}$$

for any ζ with $|\zeta| = R$, the convergence of the series in (1.5) being uniform on the circle $\{|\zeta| = R\}$. It follows that for such z close to α, one can write

$$1 = \sum_{k=0}^{\infty} \Big(\frac{1}{2i\pi} \int_{|\zeta|=R} \frac{g_1(\zeta, z)\, d\zeta}{(P_1(\zeta))^{k+1}} \Big) (P_1(z))^k\,. \tag{1.6}$$

Thanks to residue formula (again an analytic tool !), formula (1.6) can be rewritten as

$$1 = \sum_{k=0}^{\infty} \mathrm{Res} \left[\begin{matrix} g_1(\zeta, X)\, d\zeta \\ P_1^{k+1}(\zeta) \end{matrix} \right] P_1^k(z)\,. \tag{1.7}$$

Here comes another crucial remark, due also to Lagrange : when N and D are two polynomials in $\mathbb{C}[X]$ such that $\deg N \leq \deg D - 2$, then the total sum of residues of the rational form

$$\frac{N(\zeta) d\zeta}{D(\zeta)}$$

at all its poles is zero ; this can be checked either estimating (for $r >> 1$) the integral

$$\Big| \int_{|\zeta|=r} \frac{N(\zeta)\,d\zeta}{D(\zeta)} \Big|$$

and letting r tend to infinity, either using the decomposition of the rational function N/D in simple elements in $\mathbb{C}[X]$; note that if $D(X) = \delta_0 X^d + \cdots + \delta_d$, and

$$[N : D] = u_0 X^{d-1} + \cdots + u_{d-1}$$

(the remainder in Euclidean division of N by D), then

$$\mathrm{Res}\left[\begin{matrix} N(\zeta)\,d\zeta \\ D(\zeta) \end{matrix}\right] = u_0/d_0$$

(check that as an exercise, which show again that if $\deg N \leq \deg D - 2$, then $u_0 = 0$, so that the total sum of residues of $N\,d\zeta/D$ equals zero). This remark ensures us that formula (1.7) (still for z close to α) can be reduced just to to

$$1 = \mathrm{Res}\left[\begin{matrix} g_1(\zeta, X)\,d\zeta \\ P_1(\zeta) \end{matrix}\right]. \tag{1.8}$$

Let us introduce now the second polynomial P_2, together with $g_2(z, \zeta)$ such that

$$P_2(\zeta) - P_2(z) = g_2(\zeta, X)\,(\zeta - z)\,.$$

One can write the polynomial identity

$$\begin{vmatrix} g_1(\zeta, X) & g_2(\zeta, X) \\ 0 & P_2(\zeta) \end{vmatrix} = \begin{vmatrix} g_1(\zeta, X) & g_2(\zeta, X) \\ P_1(X) - P_1(\zeta) & P_2(X) \end{vmatrix}\,.$$

If we inject such an identity in (1.8), one gets (still for z close to α),

$$1 = \mathrm{Res}\left[\begin{matrix} \frac{1}{P_2(\zeta)} \begin{vmatrix} g_1(\zeta, z) & g_2(\zeta, z) \\ P_1(z) & P_2(z) \end{vmatrix}\,d\zeta \\ P_1(\zeta) \end{matrix}\right],$$

which is a Bézout identity $1 = A_1(z)P_1(z) + A_2(z)P_2(z)$ valid at least for z close to α ; since it is an algebraic identity, it is valid everywhere and we are done ! The Bézout identity we obtained this way is

$$1 = \mathrm{Res}\left[\begin{matrix} \frac{1}{P_2(\zeta)} \begin{vmatrix} g_1(\zeta, X) & g_2(\zeta, X) \\ P_1(X) & P_2(X) \end{vmatrix}\,d\zeta \\ P_1(\zeta) \end{matrix}\right]. \tag{1.9}$$

If $P_1, p_2, ..., p_m$ are m polynomials in $\mathbb{K}[X]$ with no common zero and if $\mathbb{K}$ is assumed to be infinite (which one may assume, introducing artificially an auxiliary transcendental parameter t, so that we work in $\mathbb{K}(t)$ instead of $\mathbb{K}$), one can find a linear combination (with coefficients in $\mathbb{K}$)

$$P_2 := \lambda_1 P_1 + \lambda_2 p_2 + \cdots + \lambda_m p_m$$

such that P_1 and P_2 have no common zero. This remark is based on the use of the "box principle" (as in the proof of theorem 1.1) [10]. The four methods which have been proposed below (dealing with two polynomials) can be used to recover a Bézout identity

$$a_1 P_1 + a_2 p_2 + \cdots + a_m p_m = 1 \,.$$

The last one is deeply inspired by L. Kronecker's and C. Jacobi's approach (following in fact Bézout, see [Alf]) toward such questions (in the XIX-th century). What makes the last approach (as well as the approach based on the use of Lagrange's interpolation formula) intrinsically different from the approach based on the euclidean algorithm is that, in order to express A_1 and A_2, one uses the action of a linear form

$$\mathrm{Res}^P \; : R\, d\zeta \longmapsto \begin{bmatrix} R\, d\zeta \\ P_1 \end{bmatrix}$$

(acting here on the $\mathbb{C}$-vectorial space of rational forms $Rd\zeta$ with no poles on the algebraic set $\{P_1 = 0\}$).

Formula (1.8) will be interpreted as a *trace* formula (the right-hand side will be interpreted later as the trace of a linear operator from a finite dimensional $\mathbb{C}$-vectorial space into itself).

[10] Suggestion of exercise : *complete for example here the exercise consisting in writing precisely the argument that leads from the box principle to the construction of P_2 ; explain also how one can get rid of the auxiliary parameter t when the field $\mathbb{K}$ is not infinity.*

Chapter 2

Lesson 2 : the notion of multidimensional residue

2.1 The case $n = 1$

When f and g are germs of holomorphic functions at the origin in one complex variable z ($f, g \in {}_0\mathcal{O}_1$), such that

$$f(z) = \sum_{k=m}^{\infty} a_k z^k$$

with $m \geq 0$ and $a_m \neq 0$, one defines the *local residue at the origin* of the germ of meromorphic differential $\dfrac{g(\zeta)}{f(\zeta)}\, d\zeta$ as

$$\mathrm{Res}_0\Big(\frac{g(\zeta)}{f(\zeta)}\, d\zeta\Big) = \begin{bmatrix} g(\zeta)\, d\zeta \\ f(\zeta) \end{bmatrix}_0 := \frac{1}{2i\pi} \int_{|\zeta|=\epsilon} \frac{g(\zeta)}{f(\zeta)}\, d\zeta\,,$$

where one takes representatives for the germs f and g in $D(0, r)$, ϵ being small enough so that 0 is the only (eventual) zero of f in $\overline{D(0, \epsilon)}$.

We need to make several remarks :

1. The reason why one considers the residue of a $(1, 0)$-germ of meromorphic form (instead of the residue of a germ of meromorphic function) is inherent to the geometric signification of the local residue at the origin ; the local residue

   $$\begin{bmatrix} g(\zeta)\, d\zeta \\ f(\zeta) \end{bmatrix}_0$$

   materializes the obstruction for the $(1, 0)$ holomorphic form $\omega = f(\zeta)/g(\zeta)\, d\zeta$ in $D(0, \epsilon) \setminus \{0\}$ to be exact in $D(0, \epsilon) \setminus \{0\}$, since the residue map induces an isomorphism

   $$\dot{\omega} \in \frac{Z^1_{\text{abel}}(D(0, \epsilon) \setminus \{0\})}{B^1_{\text{abel}}(D(0, \epsilon) \setminus \{0\})} \longmapsto \quad \mathrm{Res}_0\, \omega \in \mathbb{C}$$

   (here $Z^1_{\text{abel}}(U)$ denotes the $\mathbb{C}$-vectorial space of holomorphic – or *abelian* – $(1, 0)$ forms in the open set U, $B^1_{\text{abel}}(U)$ the $\mathbb{C}$-vectorial subspace of d-exact holomorphic – or *abelian* – ones).

2. The local residue $\mathrm{Res}_0(g(\zeta)\,d\zeta/f(\zeta))$ is the Laurent coefficient a_{-1} in the Laurent development
$$\frac{g(\zeta)}{f(\zeta)} = \sum_{-m}^{\infty} a_k \zeta^{-k}$$
of the meromorphic function g/f about the origin ; when $m = 1$ (the pole is simple), one has
$$\left[\frac{g(\zeta)\,d\zeta}{f(\zeta)}\right]_0 = \frac{g(0)}{f'(0)}\,,$$
while
$$\left[\frac{g(\zeta)\,d\zeta}{f(\zeta)}\right]_0 = \frac{1}{(m-1)!}\Big(\frac{d}{d\zeta}\Big)^{m-1}\Big[\zeta^m \frac{g(\zeta)}{f(\zeta)}\Big]_{\zeta=0}$$
in the general case[1] ; in fact, the notion at a point of residue of a $(1,0)$-germ of meromorphic form makes when the germ of form is a germ of (1.0) form at a point z_0 on some Riemann surface S[2].

3. We have the formula
$$\left[\frac{df(\zeta)}{f(\zeta)}\right]_0 = \left[\frac{f'(\zeta)\,d\zeta}{f(\zeta)}\right]_0 = m\,,$$
m being the order of vanishing of f at 0, that is $m = \mathrm{mult}_0(f)$; such a formula will be quite important not only in algebra, but also dealing with arithmetics (we will see its role in Arakelov theory [L]).

2.2 The notion of regular sequence in $\mathcal{O}_n$

In this section, we will deal with sequences $(f_1, ..., f_n)$ of elements in the local ring $\mathcal{O}_n$ of germs of holomorphic functions about the origin in $\mathbb{C}^n$. Recall that the maximal ring of $\mathcal{O}_n$ (which is a local ring) is the ring $(\zeta_1, ..., \zeta_n)$.

Definition 2.1 *A sequence $(f_1, ..., f_k)$, $k \leq n$, of elements in $\mathcal{O}_n$ is called weakly regular if and only if $f_1 \not\equiv 0$ and, for any $j = 1, ..., k-1$, f_{j+1} is not zero-divisor in the quotient ring $\mathcal{O}_n/(f_1, ..., f_j)$, which means*
$$f_{j+1}\, h \in (f_1, ..., f_j) \Longrightarrow h \in (f_1, ..., f_j)\,.$$
It is called regular if, in addition
$$(f_1, ..., f_k)\mathcal{O}_n$$
is a proper ideal in $\mathcal{O}_n$.

[1] In fact, it is better to forget such a formula since it could give the false idea that residue calculus would not make sense in the positive characteristic context ; the division by $(m-1)!$ is here a fictive division.

[2] When the form is expressed in a local coordinate w on S centered at z_0, the coefficient a_{-1} in the Laurent development (about the origin) of the form expressed in w appears as the only Laurent coefficient (among all the a_k's) which has an intrinsic geometric definition, that is does not depend on the choice of the local coordinate w.

For references about the notion of regular sequence (called also M-sequence) in a commutative ring R, we refer for example to ([Eis], [Ha1], [Mat], [North]).

When R is a n-dimensional local ring (such as $\mathcal{O}_n$) with maximal ideal $\mathfrak{M}$, the notion of regularity for a sequence $(f_1, ..., f_k)$, $k \leq n$, does not depend on the order[3]. This may be false when R is not local : for example, in $\mathbb{C}[X_1, X_2, X_3]$, the sequence

$$(X_1(1-X_3), X_2(1-X_3), X_3)$$

is not regular (when taken in this order) since

$$X_2(1-X_3) \times X_1 \in \Big(X_1(1-X_3)\Big)$$

and X_1 is not in the ideal $(X_1(1-X_3))$! Nevertheless, when taken in the order $(X_1(1-X_3), X_3, X_2(1-X_3))$, it happens to be regular in $\mathbb{C}[X_1, X_2, X_3]$.

When R is a local ring, a necessary and sufficient condition for a sequence $(f_1, ..., f_k)$ of elements in the maximal ideal $\mathfrak{M}$ to be regular is that the *Koszul complex* built from the sequence $(f_1, ..., f_k)$ is exact at any degree.

Koszul complex (a short presentation) : Let R be a commutative ring and M be a R-module ; let $a_1, ..., a_k$ be k elements in R ; for any $j \in \{0, ..., k\}$, consider the module

$$\Big(\bigwedge_j R^k\Big) \otimes M = \bigoplus_{1 \leq i_1 < \cdots < i_j \leq k} (e_{i_1} R \wedge \cdots \wedge e_{i_j} R) \otimes M\,,$$

$e_1, ..., e_k$ being the canonical basis of R^k, with the convention

$$\Big(\bigwedge_0 R^k\Big) \otimes M = M\,.$$

The Koszul complex $K(a_1, ..., a_k \,|\, M)$ attached to $(a_1, ..., a_k)$ in reference to M is the complex

$$0 \xrightarrow{d_{k+1}} M \cdots \longrightarrow \Big(\bigwedge_j R^k\Big) \otimes M \xrightarrow{d_j} \Big(\bigwedge_{j-1} R^k\Big) \otimes M \longrightarrow \cdots M \xrightarrow{d_0} 0$$

where

$$d_j \;: \bigwedge_{l=1}^{p} e_{i_l} \longmapsto \sum_{l=1}^{p} (-1)^{l-1} a_{i_l} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_p}\,,$$

the symbol $\widehat{\cdot}$ meaning the element is deleted. When $M = R$ is a nœtherian ring, and $a_1, ..., a_k$ lie in the Jacobson radical of R (namely the intersection of all maximal ideals), the regularity of the sequence $a_1, ..., a_k$ is equivalent to the fact that all obstructions

$$H^j(K^*(a_1, ..., a_k \,|\, R)\,,\; j > 0$$

equal zero[4](for a proof of this result, see for example [North], section 8.5).

The notion of *quasi-regularity* will be also interesting for us (outside the setting of local rings) since it does not depend on the order.

Definition 2.2 *Let $(a_1, ..., a_k)$ a sequence of elements in a commutative ring R ; the sequence $(a_1, ..., a_k)$ is called quasi-regular if and only if, for any $p \in \mathbb{N}$, any relation*

$$\sum_{\{\alpha \in \mathbb{N}^k\,;\, \alpha_1 + \cdots + \alpha_k = p\}} r_{\underline{\alpha}}\, a_1^{\alpha_1} \cdots a_k^{\alpha_k} \in (f_1, ..., f_k)^{p+1}$$

implies that the coefficients $r_{\underline{\alpha}}$ all lie in $(a_1, ..., a_k)$.

[3]Note that necessarily, the f_j all lie in $\mathfrak{M}$ if $(f_1, ..., f_k)$ is regular.

[4]It is enough to set ensure it for $j = 1$.

The notion of quasi-regularity clearly does not depend on the ordering of the list $(a_1, ..., a_k)$ (which is not the case for the notion of regularity).

The regularity of a sequence $(f_1, ..., f_k)$ $(k \leq n)$ in the local ring $\mathcal{O}_n$ is equivalent to the fact that all f_j lie in the maximal ideal $\mathfrak{M}$ and that the dimension of any irreducible component of the germ of analytic set

$$\{\zeta\,;\, f_1(\zeta) = \cdots = f_j(\zeta) = 0\}$$

equals $n - j$ (in fact, it is enough to ensure this for the extreme case $j = k$).

The circularity of a sequence $(P_1, ..., P_k)$, $k \leq n$ in $\mathbb{K}[X_1, ..., X_n]$, where $\mathbb{K}$ is any commutative field, is equivalent to the fact that all irreducible components of the algebraic set

$$\{z \in \overline{\mathbb{K}}^n\,;\, P_1(z) = \cdots = P_k(z) = 0\} \subset \overline{\mathbb{K}}^n$$

have pure dimension equal to $n-k$ (or, which is equivalent, that all isolated[5] primes in $\mathrm{Ass}(\mathbb{K}[X_1, ..., X_n]/(P_1, ..., P_k))$ have Krull dimension equal to $n-k$). For example, the sequence $(X_1(1 - X_3), X_2(1 - X_3), X_3)$ is quasi-regular in $\mathbb{K}[X_1, X_2, X_3]$). We say in such a case that the k-uplet $(P_1, ..., P_k)$ defines a *complete intersection* in $\overline{\mathbb{K}}^n$.

The notion of complete intersection can be extended to a larger (geometric) context : given $k \leq n$ holomorphic functions $f_1, ..., f_k$ in some open subset $U \subset \mathfrak{X}$, where $\mathfrak{X}$ denotes a n-dimensional complex manifold, the k-uplet $(f_1, ..., f_k)$ defines a *complete intersection* in U if and only if the closed analytic subset

$$V(f) := \{x \in U\,;\, f_1(x) = \cdots = f_k(x) = 0\}$$

has dimension at most $n - k$; this is equivalent to say that at any point x_0 in $V(f)$, the local complex dimension of the germ of analytic set defined by the germs $f_{1,x_0}, ..., f_{k,x_0}$ in $\mathcal{O}_{x_0}(\mathfrak{X})$ equals exactly $n - k$. When U is a Stein manifold [6], this is equivalent to say (this is an *algebraic* formulation instead of a *geometric* one) that $(f_1, ..., f_k)$ defines a quasi-regular sequence in the ring $\mathcal{O}(U)$[7].

When $(P_1, ..., P_k)$ defines a quasi-regular sequence in the nœtherian polynomial algebra $\mathbb{K}[X_1, ..., X_n]$, one can say more about the primary decomposition of $(P_1, ..., P_k)$. It is a classical fact from commutative algebra (see for example [Mat]) that any proper ideal $\mathfrak{I}$ in $\mathbb{K}[X_1, ..., X_n]$ can be decomposed as

$$\mathfrak{I} = \bigcap_{j=1}^{l} \mathfrak{Q}_j\,, \tag{2.1}$$

where $\mathfrak{Q}_1, ..., \mathfrak{Q}_l$ are primary ideals, with radicals the prime distinct ideals $\mathfrak{P}_1, ..., \mathfrak{P}_l$ (that is, for any $j = 1, ..., l$, there is some integer $\nu_j \in \mathbb{N}^*$ such that $\mathfrak{P}_j^{\nu_j} \subset \mathfrak{Q}_j$), in

[5] All this terminology will be clarified below.

[6] For the definition of a Stein manifold, we refer for example to [GR] ; what is enough for us here is to know in this case that a function $f \in \mathcal{O}(U)$ lies in the ideal generated by $(f_1, ..., f_k)$ in $\mathcal{O}(U)$ if and only if for any $x_0 \in U$, the germ f_{z_0} of f in the local ring $\mathcal{O}_{x_0}(U)$ lies in the ideal generated in $\mathcal{O}_{x_0}(U)$ by the germs $f_{1,x_0}, ..., f_{k,x_0}$, this follows from H. Cartan's theorem A (see [GR]) ; a n-dimensional Stein manifold can be embedded *via* a proper biholomorphic mapping in $\mathbb{C}^{2n+1}$.

[7] It is significant to note here that the "*algebraic*" characterization (the quasi regularity of the sequence) looks much more precise (and rich of information) than its "*geometric*" pendant (namely that $(f_1, ..., f_k)$ defines a complete intersection in U) ; nevertheless, these two properties of the system $(f_1, ..., f_k)$ in $\mathcal{O}(U)$ are equivalent when U is a Stein manifold.

such a way that for any $j = 1, ..., l$,

$$\bigcap_{\substack{j=1\\ l\neq j}}^{l} \mathfrak{Q}_l \not\subset \mathfrak{Q}_j\,,\ j=1,...,l\,.$$

Such a result could be the pendant in the nœtherian ring $\mathbb{K}[X_1, ..., X_n]$ of the fundamental theorem of arithmetic : any integer such that $|n| \geq 2$ can be decomposed in a unique way as

$$n = \pm \prod_{j=1}^{l} p_j^{\nu_j}\,,$$

where $p_1, ..., p_l$ are distinct prime numbers. The major stumbling block here (which makes a crucial difference between the arithmetic situation in $\mathbb{Z}$ and the situation in $\mathbb{K}[X_1, ..., X_n]$) is that such a decomposition (2.1) is not unique ! For example, one has in $\mathbb{K}[X, Y]$:

$$(X^2, XY) = (X) \cap (X, Y)^2 = (X) \cap (X^2, Y)\,; \tag{2.2}$$

the two inclusions $(X^2, XY) \subset (X) \cap (X, Y)^2$ and $(X^2, XY) \subset (X) \cap (X^2, Y)$ are trivial ; if $P = aX = \alpha X^2 + \beta XY + \gamma Y^2$ is in $(X) \cap (X, Y)^2$, one can see immediately that X divides γ by Gauss lemma, so that $P \in (X^2, XY)$; on the other hand, if $P = aX = \alpha X^2 + \beta Y$, X divides β for the same reason, so that $P \in (X^2, XY)$ and the equalities (2.2) are proved.

What is unique in decomposition (2.1) is the list of distinct prime ideals $\mathfrak{P}_1, ..., \mathfrak{P}_l$ involved in the decomposition (2.1) as radicals of the primary factors. The prime ideals $\mathfrak{P}_1, ..., \mathfrak{P}_l$ which are uniquely associated to $\mathfrak{I}$ through such a primary decomposition (of the form (2.1)) form the list of prime ideals associated with $\mathbb{K}[X_1, ..., X_n]/\mathfrak{I}$:

$$\mathrm{Ass}\,(\mathbb{K}[X_1, ..., X_n]/\mathfrak{I}) = \{\mathfrak{P}_1\,, ...,\ \mathfrak{P}_l\}\,.$$

The elements of $\mathrm{Ass}\,(\mathbb{K}[X_1, ..., X_n]/\mathfrak{I})$ can be compared respect to the inclusion order ; the minimal elements of the list are called the *isolated* primes associated with $\mathbb{K}[X_1, ..., X_n]/\mathfrak{I}$; the other ones are called the *embedded* primes associated with $\mathbb{K}[X_1, ..., X_n]/\mathfrak{I}$. Note that the affine algebraic variety $V(\mathfrak{I}) \subset \overline{\mathbb{K}}^n$ defined as

$$V(\mathfrak{I}) := \{x \in \overline{\mathbb{K}}^n\,;\, P(x) = 0\,,\ \forall\, P \in \mathfrak{I}\}$$

can be described as

$$V(\mathfrak{I}) = \bigcup_{\{\mathfrak{P}\ \text{isolated}\,(\mathbb{K}[X_1,...,X_n]/\mathfrak{I})\}} V(\mathfrak{P})\,.$$

The *geometric* point of view does not provide any information on the embedded primes (and corresponding *embedded components*, zero loci of such primes) ; this is a basic fact, namely that we loose any information about embedded objects when keeping to the *geometric* approach (and forgetting about the *algebraic* one). We will see in this course that the analytic point of view could help to get at least some partial information on the embedded world ; it will play the role of some kind of “compromise”. The optimum exponents ν_j attached to primary ideal $\mathfrak{Q}_j$ (in decomposition 2.1) related to isolated primes through

$$\nu_j = \inf\{\nu \in \mathbb{N}^*\,;\, \mathfrak{P}_j^{\nu} \subset \mathfrak{Q}_j\}$$

will of course be reachable through analytic techniques (they will appear as Lelong numbers and will play the role of *multiplicities*) but analysis will also provide some insight about such ν_j related to embedded primes (though the decomposition of $\mathfrak{I}$ is not unique !).

If $(P_1, ..., P_k)$ $(k \leq n)$ is a quasi-regular sequence in $\mathbb{C}[X_1, ..., X_n]$, then <u>all</u> associated primes with $\mathbb{C}[X_1, ..., X_n]/(P_1, ..., P_k)$ are isolated. This follows from the local result : in the local nœtherian ring $\mathcal{O}_n$ (where the primary decomposition and the classification of associated primes in isolated and embedded can be carried exactly in the same way), there is <u>no embedded prime</u> attached to $\mathcal{O}_n/(f_1, ..., f_k)$ whenever $f_1, ..., f_k$ is a regular sequence in $\mathcal{O}_n$ (this follows from the so-called *Macaulay unmixed theorem*, see theorem 17.3 page 134 in [Mat]). The same is true if $\mathbb{C}$ is replaced by any commutative field $\mathbb{K}$ (whatever the characteristic is)[8].

2.3 The *absolute* case $k = n$ in $\mathcal{O}_n$

2.3.1 The local Grothendieck residue : an *analytic* approach

In this section, we deal with n germs $f_1, ..., f_n$ defining a regular sequence in the local ring $\mathcal{O}_n$. We will denote $\mathfrak{M}$ the maximal ideal $(\zeta_1, ..., \zeta_n)$.

Since $(f_1, ..., f_n)$ is a regular sequence, all f_j lie in $\mathfrak{M}$. Let $J(f_1, ..., f_n)$ be the Jacobian determinant of $(f_1, ..., f_n)$ that is

$$J(f_1, ..., f_n) = J(f) := \det \left[\frac{\partial f_k}{\partial \zeta_l}\right]_{1\leq k,l\leq n}.$$

Moreover 0 is the only common zero of $f_1, ..., f_n$ in a small neighborhood of the origin where we have selected representative for the germs $\dot{f}_j$, $j = 1, ..., n$.

When $J(f)(0) \neq 0$, it follows from the local inversion theorem that there exists a local biholomorphism $\zeta \longmapsto w = \varphi(\zeta)$ between two neighborhoods of 0 such that

$$f \circ \varphi^{-1} \; : (w_1, ..., w_n) \longmapsto (w_1, ..., w_n)\,.$$

The ideal $(f_1, ..., f_n)$ coincides in this case with $\mathfrak{M}$ and, up to a change of coordinates[9], we may assume that $f_1, ..., f_n$ are just the coordinate functions $w_1, ..., w_n$. The situation therefore is well known and we will not be interested in this case for the moment.

Much more interesting is the case when $J(f)(0) = 0$. Our objective in this section is to *divide by* $J(f)$, exactly as we where trying to do it sideways when writing (in the one variable setting)

$$\mathrm{Res}_0\Big(\frac{g(\zeta)}{f(\zeta)}\,d\zeta\Big) = \left[\begin{matrix} g(\zeta)\,d\zeta \\ f(\zeta)\end{matrix}\right]_0 := \frac{1}{2i\pi}\int_{|\zeta|=\epsilon} \frac{g(\zeta)}{f(\zeta)}\,d\zeta$$

[8] This follows from the fact that a sequence $(P_1, ..., P_k)$ in $\mathbb{K}[X_1, ..., X_n]$ is quasi-regular if and only if its image $\phi(P)$ in any localization $\mathbb{K}[X_1, ..., X_n]_{\mathfrak{P}}$ (where $\mathfrak{P}$ is a maximal (*resp.* prime) ideal containing $(P_1, ..., P_k)$) is regular ; this is in fact a general fact, valid in any commutative ring R instead of $\mathbb{K}[X_1, ..., X_n]$. The *unmixed Macaulay theorem, theorem 17.3, page 134 in [Mat]* for regular sequences in a nœtherian ring (here any of the localizations $\mathbb{K}[X_1, ..., X_n]_{\mathfrak{P}}$) can be applied here to confirm the absence of embedded associated primes with $\mathbb{K}[X_1, ..., X_n]_{\mathfrak{P}}/\phi(P)$.

[9] A *base change* if one follows the algebraic terminology.

when 0 is a multiple zero of f, so that the usual definition of the residue for simple poles, that is $g(0)/f'(0) = g(0)/J(f)(0)$ does not make sense anymore. We will do it analytically first, then algebraically next (and we will see the algebraic procedure seems more interesting respect to its consequences).

We will consider the real setting and introduce the germ of C^∞ map from $\mathbb{R}^{2n}$ itself (at the origin)

$$F\ : (x_1, y_1, ..., x_n, y_n) \mapsto \Big(\mathrm{Re}\, f_1(x+iy), \mathrm{Im}\, f(x+iy), ..., \mathrm{Re}\, f(x+iy), \mathrm{Im}\, f(x+iy)\Big).$$

The determinant of the jacobian matrix of this map equals (thanks to the Cauchy-Riemann equations) $|J(f)(x+iy)|^2$. *Critical points* of a C^∞ map from an open set $U \subset \mathbb{R}^{m_1}$ with values in $\mathbb{R}^{m_2}$ are by definition the points where the rank of the jacobian matrix is strictly less than m_2 ; the so called *critical values* are by definition the images of critical points. In our situation, critical points are the points $(\zeta_1, ..., \zeta_n) = \zeta$ such that $J(f)(\zeta) = 0$.

A major result in differential geometry, A. Sard's lemma[10], asserts that the set of *critical values* (not of course of critical points !) of a C^∞ map

$$F\ : U \subset \mathbb{R}^{m_1} \longrightarrow \mathbb{R}^{m_2}$$

has Lebesgue measure equal to 0 in $\mathbb{R}^{m_2}$ (the result is even more precise, in terms of Hausdorff measure). As a consequence, we can ensure here that, despite $J(f)(0) = 0$, <u>almost</u> all $u_1, ..., u_n$ close to zero are non critical values of F, which means that for all pre-images ξ such that $f_1(\xi) = u_1, ..., f_n(\xi) = u_n$, the jacobian $J(f)(\xi)$ does not vanish.

We will show next two important facts :

- the cardinal of $f^{-1}(u)$ remains finite and constant (equal to some $\mu > 1$ which is precisely the topological degree of F as we will see) when u remains close to the origin[11] ;
- the function
$$u \longmapsto \prod_{f(\xi)=u} J(f)(\xi) = \delta(u)$$
is a non zero analytic function of u [12] ; such a function will be called the *discriminant* of f.

If g denotes some element in $\mathcal{O}_n$, It makes sense to define, for almost all u close to zero, the *trace function*[13]

$$T[g\, d\zeta; f]\ : (u_1, ..., u_n) \longmapsto \sum_{f(\xi)=u} \frac{g(\xi)}{J(f)(\xi)}$$

[10] See [Sa] for the original statement or also [HenY] (exercise 1.5 and its correction) for a detailed proof in the case $m_1 = m_2$ which interests us ; in fact, A. Sard's lemma is the analytic substitute for Bertini's result (see [Jo]) in algebraic geometry.

[11] Think for example about Rouché's theorem in the one dimensional situation.

[12] Note that this confirms indeed Sard's lemma, since an analytic hypersurface in $U \subset \mathbb{C}^n$ has Lebesgue measure zero

[13] The terminology used here will be justified later on by algebraic considerations.

This occurs to be in fact a meromorphic function with δ as denominator. In order to compute its "value" at the origin (note the origin is a pole, so we cannot do that abruptly !), we notice that, if one chooses conveniently $(\eta_1, ..., \eta_n)$ so that all values $(\eta_1 e^{i\theta_1}, ..., \eta_n e^{i\theta_n})$, $\theta \in \mathbb{R}^n$ are non critical for the map

$$\widetilde{F} \ : (x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n} \longmapsto (|f_1(x+iy)|^2, ..., |f_n(x+iy)|^2) \in \mathbb{R}^n$$

(such a choice is possible and implies that any $(\eta_1 e^{i\theta_1}, ..., \eta_n e^{i\theta_n})$ is in fact non critical for F), then it follows from Fubini's and Lebesgue's dominated convergence theorems that

$$\frac{1}{(2i\pi)^n} \int\limits_{\substack{|u_1| = \eta_1 \\ \vdots \\ |u_n| = \eta_n}} T[g\, d\zeta; f](u) \bigwedge_{j=1}^{n} \frac{du_j}{u_j} = \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1| = \eta_1 \\ \vdots \\ |f_n| = \eta_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)}\,, \tag{2.3}$$

where

$$d\zeta := \bigwedge_{j=1}^{n} d\zeta_j\,.$$

Here the orientation of the n-dimensional cycle

$$\{|u_1| = \eta_1, ..., |u_n| = \eta_n\}$$

is the standard one in order that Cauchy formula holds ; the cycle in the left-hand side integral in (2.3) is parametrized by

$$u_j = \eta_j e^{i\theta_j}\ ,\ \ \theta_j \in [0, 2\pi]\,, j = 1, ..., n\,,$$

so that

$$\begin{aligned} &\frac{1}{(2i\pi)^n} \int\limits_{\substack{|u_1| = \eta_1 \\ \vdots \\ |u_n| = \eta_n}} T[g\, d\zeta; f](u) \bigwedge_{j=1}^{n} \frac{du_j}{u_j} \\ &\qquad = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} T[g\, d\zeta; f](\eta_1 e^{i\theta_1}, ..., \eta_n e^{i\theta_n})\, d\theta_1 \dots d\theta_n\,. \end{aligned}$$

By Stokes's theorem, this defines (almost everywhere) a function of η which happens to be locally constant ; in fact, it will even appear as globally constant[14] (taking into account that is only defined for almost all η), we will prove it below, and its value will be our definition of the local Grothendieck residue

$$\mathrm{Res} \begin{bmatrix} g(\zeta)\, d\zeta \\ f_1, ..., f_n \end{bmatrix}_0\,.$$

[14]This confirms here the fact that the integral symbol in the right-hand side integral in (2.3) has more some formal meaning than really some analytic one ; nevertheless, it will be sometimes useful to profit of such an integral symbol just as if it had some true analytic meaning (we will see examples later in the course).

This will be our "trick" to turn around the division by the jacobian and define sideways nevertheless $T[gd\zeta; f](0)$ though 0 is known to be a critical value of F !

Let us proof first (as a consequence of Stokes's theorem) that, for any (generic) choices of $(\eta_1, ..., \eta_n)$ and $(\tilde{\eta}_1, ..., \tilde{\eta}_n)$, one has

$$\frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1| = \eta_1 \\ \vdots \\ |f_n| = \eta_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)} = \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1| = \tilde{\eta}_1 \\ \vdots \\ |f_n| = \tilde{\eta}_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)} \,. \tag{2.4}$$

It is enough to prove this result when $\eta_2 = \tilde{\eta}_2, ..., \eta_n = \tilde{\eta}_n$, and $\eta_1 < \tilde{\eta}_1$, since after this step, one can prove it for distinct η_2 and $\tilde{\eta}_2$, and so on ... The $2n - (n-1)$-dimensional "annulus ", considered as a $n+1$-dimensional cycle γ

$$\{\eta_1 \leq |f_1| \leq \tilde{\eta}_1 \,,\, |f_2| = \eta_2 \,, \cdots \,, |f_n| = \eta_n\}$$

has for boundary $\partial\gamma$ the n-cycle

$$\{|f_1| = \tilde{\eta}_1 \,,\, |f_2| = \eta_2, \ldots, |f_n| = \eta_n\} - \{|f_1| = \eta_1 \,,\, |f_2| = \eta_2, \ldots, |f_n| = \eta_n\} \,.$$

Since the $(n, 0)$-differential form

$$\omega(\zeta) := \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)}$$

is d-closed in a neighborhood of the cycle γ,

$$\begin{aligned} \int_{\partial\gamma} \omega &= \int\limits_{\substack{|f_1| = \eta_1 \\ \vdots \\ |f_n| = \eta_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)} - \int\limits_{\substack{|f_1| = \tilde{\eta}_1 \\ \vdots \\ |f_n| = \eta_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)} \\ &= \int_\gamma d\omega = 0 \,. \end{aligned}$$

This proves (2.4) when $\tilde{\eta}_2 = \eta_2$,...,$\tilde{\eta}_n = \eta_n$ and therefore completes the definition of the local residue.

Definition 2.3 *Let $f_1, ..., f_n$ be n elements defining a regular sequence in $\mathcal{O}_n$. Let $g \in \mathcal{O}_n$. The mapping*

$$(\epsilon_1, ..., \epsilon_n) \longmapsto \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1| = \epsilon_1 \\ \vdots \\ |f_n| = \epsilon_n}} \frac{g(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)}$$

is almost everywhere defined (and constant) for $(\epsilon_1, ..., \epsilon_n)$ sufficiently close to zero (depending of the representative which have been chosen for the classes $\dot{f}_1, ..., \dot{f}_n, \dot{g}$).

Its value is the Grothendieck local residue[15]*, which will be denotes as*

$$\mathrm{Res}_0\Big(\frac{g(\zeta)d\zeta}{f_1(\zeta)\cdots f_n(\zeta)}\Big) \qquad or \qquad \mathrm{Res}\begin{bmatrix} g(\zeta)\, d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,,\ldots,\; f_n(\zeta) \end{bmatrix}_0 .$$

The first notation is more familiar to analytic geometers, the second one to algebraists.

Moreover, Let μ denote the topological degree of the map F, that is precisely the degree of the map

$$\underline{F}\ : x\in\mathbb{S}^{2n-1}\longmapsto \frac{F(\rho x)}{\|F(\rho x)\|}\in\mathbb{S}^{2n-1}$$

when $\rho>0$ is sufficiently small. This map is a smooth morphism from a compact manifold (the unit sphere in $\mathbb{C}^{2n}$) into itself ; choosing an orientation on $\mathbb{S}^{2n-1}$ (which is equivalent to choose a $2n-1$ volume form Ω^{2n-1}), we recall that the degree of $\underline{F}$ is defined as the positive[16] integer μ such that the pull-back $\underline{F}^*[\Omega^{2n-1}]$ equals $\mu\Omega^{2n-1}$ (for a brief presentation of degree theory from the point of view of differential geometry, see for example [HenY], section 3.6.2). It follows from the degree theorem ([HenY], theorem 3.5 for example) that μ is also the cardinal of the set $f^{-1}(u)$ for $(u_1,\ldots,u_n)$ generic and close to $(0,\ldots,0)$. Therefore, we have the following immediate result :

Proposition 2.1 *For any $g\in\mathcal{O}_n$, one has*

$$\mathrm{Res}\begin{bmatrix} g(\zeta)\, df_1\wedge\ldots\wedge df_n \\ f_1(\zeta)\,,\ldots,\; f_n(\zeta) \end{bmatrix}_0 = \mu g(0)\,.$$

Proof. This follows from the fact that for $(u_1,\ldots,u_n)$ generic

$$\mathrm{Tr}\,(g(\zeta)J(f)(\zeta)\,d\zeta;u) = \sum_{\xi\in f^{-1}(u)}\frac{g(\xi)J(f)(\xi)}{J(f)(\xi)} = \sum_{\xi\in f^{-1}(u)} g(\xi)\,, \tag{2.5}$$

which takes the value $\mu g(0)$ when $u=0$. $\diamondsuit$

Remark. Formula (2.5) will be interpreted later as some factorization formula for the integration current on an analytic set (multiplicities being taken into account) as the product of the "*residue current*" and the jacobian differential form $df_1\wedge\cdots\wedge df_n$.

Another key immediate property of the Grothendieck residue is the following :

Proposition 2.2 *If h lies in the ideal generated by $(f_1,\ldots,f_n)$ in $\mathcal{O}_n$, then*

$$\forall g\in\mathcal{O}_n\,,\ \mathrm{Res}\begin{bmatrix} g(\zeta)h(\zeta)\, d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,,\ldots,\; f_n(\zeta) \end{bmatrix}_0 = 0\,. \tag{2.6}$$

[15]The concept was introduced by Grothendieck from the formal point of view (it is hard to find here a precise reference, it seems to more an oral presentation) toward the realization of duality theory, but it was essentially presented in such a way by P. Griffiths ([Gr2] and [GH], chapter 6) following the analytic geometric point of view and by R. Hartshorne [Ha2], E. Kunz [Ku0] and later by J. Lipman [Li] following an algebraic point of view.

[16]It is automatically strictly positive since $f_1,\ldots,f_n$ are holomorphic functions and the real jacobian of a n-valued holomorphic map is always positive.

Proof. It is enough to prove this result when $h = af_1$, with $a \in \mathcal{O}_n$. Let $(\epsilon_1, ..., \epsilon_n)$ generic in $(R^{+*})^n$, such that

$$\mathrm{Res}\begin{bmatrix} g(\zeta)a(\zeta)f_1(\zeta)g(\zeta)(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,, ...,\, f_n(\zeta)\end{bmatrix}_0 = \frac{1}{(2i\pi)^n}\int_{\substack{|f_1|=\epsilon_1\\ \vdots \\ |f_n|=\epsilon_n}} \frac{a(\zeta)g(\zeta)\,d\zeta}{f_2(\zeta)\cdots f_n(\zeta)}\,.$$

Consider the $2n-(n-1)=n+1$ cycle γ

$$\{|f_1|\le\epsilon_1,\ |f_2|=\epsilon_2, ..., |f_n|=\epsilon_n\}\,.$$

One has, by Stokes's theorem again,

$$\int_{\partial\gamma}\frac{a(\zeta)g(\zeta)d\zeta}{f_2(\zeta)...f_n(\zeta)} = \int_\gamma d\Big[\frac{a(\zeta)g(\zeta)d\zeta}{f_2(\zeta)...f_n(\zeta)}\Big] = 0$$

since the integrand form is closed on the interior of the support of γ. $\diamondsuit$

The main theorem we will establish later (that is the *duality theorem*) can be stated as follows :

Theorem 2.1 [local duality theorem] *Let $f_1, ..., f_n$ be a regular sequence in $\mathcal{O}_n$. An element $h \in \mathcal{O}_n$ lies in the ideal $(f_1, ..., f_n)$ if and only if, for any $g \in \mathcal{O}_n$, one has*

$$\mathrm{Res}\begin{bmatrix} g(\zeta)h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,, ...,\, f_n(\zeta)\end{bmatrix}_0\,.$$

Remark. When $n = 1$ and $f(\zeta) = \zeta^m$, the result is clear : if

$$h(\zeta) = \sum_{\alpha=0}^{\infty} a_\alpha \zeta^\alpha\,,$$

then for any $k \in \mathbb{N}$,

$$\mathrm{Res}\begin{bmatrix} \zeta^k h(\zeta)\,d\zeta \\ \zeta^m \end{bmatrix} = a_{m-1-k}\,;$$

if all such numbers are 0, h is in the ideal (ζ^m). Of course, the single condition

$$\mathrm{Res}\begin{bmatrix} h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,, ...,\, f_n(\zeta)\end{bmatrix}_0$$

does not imply anything relatively to the membership of h to the ideal $(f_1, ..., f_n)$!

Before ending this section, we would like to mention that the topological degree μ of the morphism $\underline{F}$ is always smaller than the product $m(f_1)\times\cdots\times m(f_n)$ of multiplicities at the origin of $f_1, ..., f_n$. We recall here that $m(f_j)$, $j = 1, ..., n$ is defined as the degree of the homogeneous part f_j^{init} of lowest degree in the Taylor expansion

$$f_j(\zeta) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(f)\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n}$$

of f_j about the origin. Define the tangent cone $C(f_j)$ to the hypersurface $\{f_j = 0\}$ at the origin as the projective hypersurface

$$C(f_j) := \{[\zeta_1 : ... : \zeta_n] \in \mathbb{P}^{n-1}(\mathbb{C})\,;\ f_j^{\mathrm{init}}(\zeta) = 0\}\,,\ j = 1, ..., n$$

(in the particular case $n = 2$, C_j is a finite set of points in $\mathbb{P}^1(\mathbb{C})$). Assume that

$$C(f_1) \cap C(f_2) \cap \cdots \cap C(f_n) = \emptyset \,. \tag{2.7}$$

This is generically the case since n homogeneous forms $(\varphi_1, ..., \varphi_n)$ of respective degrees

$$m(f_1), ..., m(f_n)$$

in n variables define a non-empty set in $\mathbb{P}^{n-1}(\mathbb{C})$ if and only if the resultant

$$\mathcal{R}_{m(f_1),...,m(f_n)}(\varphi_1, ..., \varphi_n)$$

specialized at their coefficients, is zero (see for example [L2], elimination theory). One can show in such case that the topological degree μ of the morphism $\underline{F}$ equals exactly

$$\mu = m(f_1) \times \cdots \times m(f_n) \,.$$

(see for example proposition 5.10 in [HenY] for an "analytic" proof of this result which is in the spirit of this course). This is a local version of Bézout theorem since μ is also called the *intersection multiplicity* at the origin of the germs of hypersurfaces (considered taking into account multiplicities) $\{f_1 = 0\}, ..., \{f_n = 0\}$. In general, adapting the deformation argument used in the proof of proposition 5.10 in [HenY], one can show that in general (if hypothesis (2.7) is not fulfilled) , one has always the inequality

$$\mu \le m(f_1) \times \cdots \times m(f_n) \,,$$

the inequality being strict if and only if the tangent cones $C(f_j)$, $j = 1, ..., n$, intersect in $\mathbb{P}^{n-1}(\mathbb{C})$.

Of course, in order to profit completely from the fact that in $\mathbb{C}^n = \mathbb{R}^{2n}$, one has $2n$-real degrees of freedom, it will be quite important in the future to extend the action of the residue to germs of C^∞ functions

$$\zeta \longmapsto \varphi(\zeta, \overline{\zeta})$$

and be able to define

$$\mathrm{Res}\begin{bmatrix} \varphi(\zeta,\overline{\zeta})\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1(\zeta)\,, ..., \; f_n(\zeta) \end{bmatrix}_0 .$$

Of course, Stokes's theorem cannot be applied anymore in order to show that the function

$$(\epsilon_1, ..., \epsilon_n) \longmapsto \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1| = \epsilon_1 \\ \vdots \\ |f_n| = \epsilon_n}} \frac{\varphi(\zeta,\overline{\zeta})\, d\zeta}{f_1(\zeta)\cdots f_n(\zeta)}$$

is locally constant ! Nevertheless, if one is very careful and take the "limit" when $(\epsilon_1, ..., \epsilon_n)$ tends to 0 in some particular way, we will see (our algebraic presentation in the next section will help us for this) that the limit exists, is independent of the path used to approach the origin (once such path is "admissible") and deals with the φ just as if φ was considered as the holomorphic function

$$\zeta \longmapsto \varphi(\zeta, 0)$$

(that is, anti-holomorphic coordinates $\overline{\zeta}_j$, $j = 1, ..., n$, play only some "neutral" role and are treated as constants). This is known to be the case already in the one variable case, where one knows that if

$$\varphi(\zeta, \overline{\zeta}) = \sum_{\alpha} \sum_{\beta} a_{\alpha,\beta} \zeta^{\alpha} \overline{\zeta}^{\beta},$$

then

$$\lim_{\epsilon \to 0} \Big(\frac{1}{2i\pi} \int_{|\zeta|=\epsilon} \frac{\varphi(\zeta, \overline{\zeta})\, d\zeta}{\zeta^m} \Big) = \mathrm{Res} \left[\frac{\Big(\sum\limits_{\alpha} a_{\alpha,0} \zeta^{\alpha} \Big)\, d\zeta}{\zeta^m} \right] = a_{m-1,0}$$

(check this just using the parametrization of $|\zeta| = \epsilon$ by $\zeta = \epsilon e^{i\theta}$, $\theta \in [0, 2\pi]$).

Once this will be achieved, the action of the residue on germs of $(n, 0)$ test-forms $\varphi d\zeta$ will make sense and define the action of a $(0, n)$ current that will be denoted (for natural reasons that we will explain later)

$$\bigwedge_{j=1}^{n} \overline{\partial}(1/f_j)\,.$$

The result established in proposition 4.2 will become a particular case of a fundamental formula, *Lelong-Poincaré formula*, which says that the integration current $[f = 0]$ on the analytic set $\{f = 0\}$ (considered with multiplicities taken into account)

$$\varphi \in C_0^{\infty}(\mathbb{C}^n) \longmapsto \mu \int_{\{f=0\}} \varphi(\zeta) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

can be factorized as

$$[f = 0] = \Big(\bigwedge_{j=1}^{n} \overline{\partial}(1/f_j) \Big) \wedge df_1 \wedge \ldots \wedge df_n\,. \tag{2.8}$$

Here $C_0^{\infty}(\mathbb{C}^n)$ denotes the germs at the origin of C^{∞} functions from $\mathbb{C}^n$ to $\mathbb{C}$.

2.3.2 The local Grothendieck residue : an *algebraic* approach

2.3.4

The definition of the Grothendieck residue symbol (which has been given in 2.3) is a local one, but, since it involves integration on the n-dimensional cycle

$$\{|f_1| = \epsilon_1, ..., |f_n| = \epsilon_n\}$$

(which support lies in a neighborhood of the origin), it also carries some semi-local aspects. Playing with both contexts (local and semi-local) will be essential to settle the main properties of the Grothendieck residue symbol.

We will present here the approach developed by J. Lipman in [Li], chapter 3, which leads to the simultaneous definition of all residue symbols

$$\mathrm{Res} \begin{bmatrix} h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1^{k_1+1}(\zeta), ..., f_n^{k_n+1}(\zeta) \end{bmatrix}_0$$

for $h \in \mathcal{O}_n$, $(f_1, ..., f_n)$ being a regular sequence in $\mathcal{O}_n$.

The general frame will be the frame of a commutative A-algebra R over a commutative ring A. In our context R will be the local ring $\mathcal{O}_n$ and $A \simeq \mathbb{C}$ its quotient field $\mathcal{O}_n/\mathfrak{M}$.

Let $f_1, ..., f_n$ be a quasi-regular sequence in R (in our context, a quasi-regular sequence in $\mathcal{O}_n$ [17]). So, we will retain that the sequence $f_1, ..., f_n$ satisfies the following : for any $m \in \mathbb{N}^*$, any time we have (in R) a relation

$$\sum_{|\underline{k}|=m} a_{\underline{k}} f_1^{k_1} \cdots f_n^{k_n} = 0\,,\ a_{\underline{k}} \in A\,,$$

then all $a_{\underline{k}}$ lie in $Rf_1 + \cdots + Rf_n$. Recall that in our example $R = \mathcal{O}_n$ and $A = \mathbb{C}$.

The other hypothesis we will need is that the quotient A-module

$$P = \frac{R}{f_1 R + \cdots + f_n R}$$

is projective[18] and finitely generated. What will be in fact important for us is the possibility to define a trace map $\mathrm{Tr}\ : E = \mathrm{Hom}_A(P, P) \longrightarrow A$ (this is of course possible when P is projective and finitely generated).

In our context, such condition is fulfilled since $\mathcal{O}_n/(f_1, ..., f_n)$ is a finitely dimensional $\mathbb{C}$-vectorial space : since the ideal $(f_1, ..., f_n)$ is $\mathfrak{M}$-primary, there is some exponent $\nu \in \mathbb{N}^*$ such that

$$\mathfrak{M}^\nu \subset (f_1, ..., f_n)\,; \tag{2.9}$$

the smallest $\nu \in \mathbb{N}^*$ such that 2.9 holds is called the *Nœther exponent* of $(f_1, ..., f_n)$. We will see in the next section that the Nœther exponent is always smaller[19] than the intersection multiplicity μ that has been introduced in the previous section. Since $\zeta_j^\nu \in (f_1, ..., f_n)$, $j = 1, ..., n$, one has necessarily

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_n)} < \infty\,.$$

In fact, one can say more : there is a polynomial $H_f \in \mathbb{Q}[X]$ of the form

$$H_f(X) = \frac{\mu}{n!} X^n + \text{lower degree terms}\,,$$

such that, for any $k \in \mathbb{N}$ sufficiently large,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_n)^k} = H_f(k)\,.$$

The polynomial H_f is called the *Hilbert-Samuel polynomial* (for references, see for example [Ha1] (chapter 1) or [ZS]. We will see later that when $(f_1, ..., f_n)$ is a regular sequence in $\mathcal{O}_n$, then

$$\dim_C \frac{\mathcal{O}_n}{(f_1, ..., f_n)} = \mu\ ;$$

[17] Though quasi-regularity is equivalent to regularity in a local ring such $\mathcal{O}_n$, it is very important to stick here to this notion of *quasi-regularity* (see definition 2.2) since we will have to play simultaneously in the local and semi-local context (as the analytic approach suggests it).

[18] This means there is a A-module Q such that $P \oplus Q$ is a free A-module.

[19] It may be strictly smaller.

it is easy to guess that since we know that if we "perturb" $f_1, ..., f_n$ in $f_1 - \epsilon_1, ..., f_n - \epsilon_n$ (as we have done in section 2.3.1), we get exactly $\mu = \mu(f)$ distinct simple common zeroes[20].

Let us now start with Lipman's construction. The first basic idea is to introduce the $(f_1, ..., f_n)$-*adic completion* of R (here of $\mathcal{O}_n$). That is, we introduce on R the "pseudo-valuation" :

$$v_f \ : a \longmapsto \sup \Big\{ p \in \mathbb{N} \, ; \, a \in (f_1, ..., f_n)^p \Big\} \in [0, ..., +\infty]$$

and

$$(h_1, h_2) \longmapsto d_f(a_1, a_2) := \exp(-v_f(a_1 - a_2)) \, .$$

The completion of R respect to the pseudo "metric" d_f will be denoted as $\widehat{R}$; it is the $(f_1, ..., f_n)$-*adic completion* of R (note is is also a A-commutative algebra)[21].

We now choose a A-linear map

$$\sigma \ : P = \frac{R}{(f_1, ..., f_n)} \longrightarrow R$$

such that $\pi \circ \sigma = \mathrm{Id}_P$ when $\pi \ : R \longrightarrow P$ is the quotient map ; this can be done easily since P is assumed to be projective and finitely generated ; for example in the situation $R = \mathcal{O}_n$ and $f = (f_1, ..., f_n)$ is a regular sequence, we just take representatives for elements $\dot{a}_1, ..., \dot{a}_\mu$ which form a basis for the finite dimensional $\mathbb{C}$-vectorial space $\mathcal{O}_n/(f_1, ..., f_n)$ and then extend σ by linearity.

We start with the fundamental lemma :

Lemma 2.1 *For any $h \in R$, one can find a unique list of elements $\dot{h}_{\underline{k}}$ in P (depending of course of the choice of σ) such that h has the following development (when considered as an element in $\widehat{R}$) :*

$$h = \sum_{\underline{k} \in \mathbb{N}^n} \sigma(\dot{h}_{\underline{k}}) \, f_1^{k_1} \cdots f_n^{k_n} \, . \tag{2.10}$$

Proof. We use here the full strength of the quasi-regularity condition. Clearly $\dot{h} = \pi(\sigma(h_{\underline{0}}))$, so that $h_{\underline{0}} = \dot{h} = \pi(h)$. Now, if we have

$$h - \sigma(\dot{h}_{\underline{0}}) = \sum_{\underline{k} \in \mathbb{N}^n} \sigma(\dot{h}_{\underline{k}}) \, f^{\underline{k}} = \sum_{\underline{k} \in \mathbb{N}^n} \sigma(\tilde{h}_{\underline{k}}) \, f^{\underline{k}} \, ,$$

we have

$$\sum_{j=1}^n \sigma\Big(\dot{h}_{(0,..,\overset{j}{1},..,0)} - \tilde{h}_{(0,..,\overset{j}{1},..,0)} \Big) \, f_j \in (f_1, ..., f_n)^2 \, .$$

[20] When $(f_1, ..., f_m)$ is still $\mathfrak{M}$-primary, the dimension of the quotient does not coincide with the topological degree of the map $x \longmapsto \tilde{f}(\rho x)/\|\tilde{f}(\rho x)\|$ (where $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n)$ consists in n generic linear combinations of $f_1, .., f_m$) for $\rho > 0$ small : for example if $(f_1, f_2, f_3) = (\zeta_1^2, \zeta_2^2, \zeta_1\zeta_2) \in \mathcal{O}_2$, one has $\dim \mathcal{O}_2/(f_1, f_2, f_3) = 3$ while the topological degree of the map $x \longmapsto \tilde{f}(\rho x)/\|\tilde{f}(\rho x)\|$ when $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ equals 4 !

[21] As examples, think about the construction of $\widehat{\mathcal{O}_n} = \mathbb{C}\{\{\zeta\}\}$ when $(f_1, ..., f_n) = \mathfrak{M}$ or $\widehat{\mathbb{K}[X]} = \mathbb{K}[[X]]$ when $\mathbb{K}$ is a commutative field and $f = (X_1, ..., X_n)$.

Applying the quasi-regularity condition with $m = 1$, one can see that

$$\sigma\Big(\dot{h}_{(0,..,\overset{j}{1},..,0)} - \tilde{h}_{(0,..,\overset{j}{1},..,0)}\Big) \in (f_1, ..., f_n)$$

for any $j = 1, ..., n$, so that

$$\pi\Big(\sigma\Big(\dot{h}_{(0,..,\overset{j}{1},..,0)} - \tilde{h}_{(0,..,\overset{j}{1},..,0)}\Big)\Big) = \dot{0}\,,\ j = 1, ..., n\,,$$

therefore

$$\dot{h}_{(0,..,\overset{j}{1},..,0)} = \tilde{h}_{(0,..,\overset{j}{1},..,0)}\,,\ j = 1, ..., n\,,$$

so that the $\dot{h}_{\underline{k}}$ for $|\underline{k}| = 1$ are uniquely determined. It is clear how to continue this procedure step by step (using next the quasi-regularity condition for $m = 2$ and so on...). $\diamondsuit$

Now, we are going to associate to any $h \in R$ a list of operators $h^\sharp_{\underline{k}} \in \mathrm{Hom}_A(P, P)$ (in our situation, these are linear operators from the μ-dimensional $\mathbb{C}$-vectorial space $P = \mathcal{O}_n/(f_1, ..., f_n)$ into itself). Namely :

$$h^\sharp_{\underline{k}}\ : \dot{r} \in P \longmapsto \dot{h}_{\dot{r},\underline{k}} \in P\,,$$

where

$$h \cdot \sigma(\dot{r}) = \sum_{\underline{k}\in\mathbb{N}^n} \sigma(\dot{h}_{\dot{r},\underline{k}})\, f_1^{k_1} \cdots f_n^{k_n}$$

(when developed in $\widehat{R}$). Because of the previous lemma, the $\dot{h}_{\dot{r},\underline{k}}$, $\underline{k} \in \mathbb{N}^n$, are uniquely determined (depending of course of the choice of the section σ). We now attach to the list $(h^\sharp_{\underline{k}})_{\underline{k}\in\mathbb{N}^n}$ the formal power series

$$\sum_{\underline{k}\in\mathbb{N}^n} h^\sharp_{\underline{k}}\ f_1^{k_1} \cdots f_n^{k_n} \in \mathrm{Hom}_A(P, P)\,[[f_1, ..., f_n]]\,.$$

We are now in good shape to introduce the procedure which will "mimic" the construction of

$$(\zeta_1, ..., \zeta_n) \longmapsto \frac{1}{J\,(f_1, ..., f_n)}$$

(when $J(f_1, ..., f_n)(0) = 0$) we had to by-pass in order to construct the residue symbols analytically in section 2.3.1.

One can associate to any coordinate function $\zeta_1, ..., \zeta_n$ the element

$$\zeta_j^\sharp = \sum_{\underline{k}\in\mathbb{N}^n} \zeta^\sharp_{j,\underline{k}}\ f_1^{k_1} \cdots f_n^{k_n} \in \mathcal{O}_n/(f_1, ..., f_n)[[f_1, ..., f_n]]\,,$$

together with the n elements

$$\frac{d}{df_i}\,\zeta_j^\sharp := \sum_{\underline{k}\in\mathbb{N}^n} k_i \zeta^\sharp_{j,\underline{k}}\ f_1^{k_1} \cdots f_i^{k_i-1} \cdots f_n^{k_n}\,,\ i = 1, ..., n\,.$$

There is a natural (non commutative) product between elements in

$$\frac{\mathcal{O}_n}{(f_1, ..., f_n)}\,[[f_1, ..., f_n]]\ ;$$

it is defined on monomials in f by

$$(T\ f^{\underline{k}}) \circ (S\, f^{\underline{l}}) := (T \circ S)\ f^{\underline{k}+\underline{l}}$$

and then extended by linearity to formal power series if f ; using such product, one may define (taking into account the non-commutativity) :

$$h^\sharp \circ \begin{vmatrix} \frac{d}{df_1}\zeta_1^\sharp & \cdots & \frac{d}{df_n}\zeta_1^\sharp \\ \vdots & \vdots & \vdots \\ \frac{d}{df_1}\zeta_n^\sharp & \cdots & \frac{d}{df_n}\zeta_n^\sharp \end{vmatrix} = \sum_{\underline{k}\in\mathbb{N}^n} \delta^\sharp_{h,\underline{k}}\ f^{\underline{k}}\,.$$

The key final point here is that the traces of the operators $\delta^\sharp_{h,\underline{k}}$, $\underline{k} \in \mathbb{N}^n$, **do not depend** of the choice of the section σ. This is slightly more delicate to see and we will just sketch the idea. The section σ may be extended (by extension of scalars) to a $A[[X]]$-linear map

$$\sigma^*\ : P \otimes_A A[[X]] \longrightarrow \widehat{R}\,.$$

Such map is bijective[22], which shows us that $\widehat{R}$ is a projective and finitely generated $A[[X]]$-module (just as P is a projective and finitely generated A-module)[23]. There is also a natural trace map

$$\mathbf{Tr}\ : \mathrm{Hom}_A(P,P)\,[[f]] = \mathrm{Hom}_{A[[X]]}(\widehat{R},\widehat{R}) \longrightarrow A[[X]]$$

and one has

$$\mathbf{Tr}\Big(\sum_{\underline{k}\in\mathbb{N}^n} \delta^\sharp_{h,\underline{k}}\ f^{\underline{k}}\Big) = \sum_{\underline{k}\in\mathbb{N}^n} \mathrm{Tr}[\delta^\sharp_{h,\underline{k}}]\ f^{\underline{k}} \tag{2.11}$$

for any $h \in R$. Since changing the section amounts to make a change of basis in P, therefore in $P \otimes_A A[[X]] \simeq \widehat{R}$, the traces or the $\delta^\sharp_{h,\underline{k}}$ are all independent of the choice of σ (because of 2.11 and the fact that the trace of an linear operator does not depend on the basis in which it is represented).

We now define (as the same time,which is rather instructive, compared to what we have done in section 2.3.1 with analytic methods) the whole list of residue symbols

$$\mathrm{Res}\begin{bmatrix} h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1^{k_1+1}(\zeta)\,, ...,\ f_n^{k_n+1}(\zeta) \end{bmatrix}_0 := \mathrm{Tr}\,(\delta^\sharp_{h,\underline{k}})\,,\ \underline{k} \in \mathbb{N}^n\,.$$

We will see later (it is not yet clear at this point) that the two definitions of the residue symbol (for $\underline{k} = (0, ..., 0)$), namely the analytic one and the algebraic one, coincide.

In the general setting (R, A), we may define symbols of the form

$$\mathrm{Res}\begin{bmatrix} h\, dr_1 \wedge \ldots \wedge dr_n \\ f_1^{k_1+1}(\zeta)\,, ...,\ f_n^{k_n+1}(\zeta) \end{bmatrix}_0\,,\ \underline{k} \in \mathbb{N}^n\,,$$

as soon as $r_1, ..., r_n$ are n elements (arbitrary) in the A-algebra R and $(f_1, ..., f_n)$ a quasi-regular sequence of elements in R such that the quotient A-module

$$\frac{R}{f_1R + \cdots f_nR}$$

is projective and finitely generated.

[22] Since P is finitely generated.
[23] In our case $A = \mathbb{C}$.

2.3.3 A key property : the *transformation law*

The following law will be a crucial ingredient for multivariate residue calculus. In this section, we will stick to the analytic approach of the residue symbol introduced in section 2.3.1.

Theorem 2.2 (Transformation Law for local residues) *Let $(f_1, ..., f_n)$ and $(g_1, ..., g_n)$ two regular sequences in $\mathcal{O}_n$ such that*

$$g_j = a_{j,1} f_1 + \cdots + a_{j,n} f_n \,,\ j = 1, ..., n \,,$$

where the $a_{j,k}$, $j, k = 1, ..., n$ are elements in $\mathcal{O}_n$. Then, for any $h \in \mathcal{O}_n$,

$$\operatorname{Res}\begin{bmatrix} h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1(\zeta)\,, ...,\ f_n(\zeta) \end{bmatrix}_0 = \operatorname{Res}\begin{bmatrix} h(\zeta)\ \det\,[a_{j,k}]_{\substack{1\le j\le n\\ 1\le k\le n}} d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ g_1(\zeta)\,, ...,\ g_n(\zeta) \end{bmatrix}_0 . \tag{2.12}$$

Application. Let us introduce the Nœther exponent $\nu = \nu(f)$ and therefore write

$$\zeta_j^\nu = \sum_{k=1}^n a_{j,k}\, f_k \,,\ j = 1, ..., n \,.$$

Because of the transformation law, we have, for any $h \in \mathcal{O}_n$,

$$\operatorname{Res}\begin{bmatrix} h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1(\zeta)\,, ...,\ f_n(\zeta) \end{bmatrix}_0 = \operatorname{Res}\begin{bmatrix} h(\zeta)\ \det\,[a_{j,k}]_{\substack{1\le j\le n\\ 1\le k\le n}} d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ \zeta_1^\nu\,, ...,\ \zeta_n^\nu \end{bmatrix}_0 . \tag{2.13}$$

The right hand side of 2.13 can be computed immediately since the variables are separated. Computations here amount to iterated computations of residues in the one variable setting (which can be performed straightforward).

Proof. The transformation law and its interest (in the analytic context) were pointed out by P. Griffiths (see [GR2], [GH]). In order to prove the result, let us recall that, for $\eta > 0$,

$$\frac{(n-1)! \sum\limits_{k=1}^n (-1)^{k-1} \epsilon_k \bigwedge\limits_{\substack{l=1\\ l\neq k}}^n d\epsilon_l}{\eta^n}$$

is the normalized volume form on the $(n-1)$-simplex

$$\Big\{(\epsilon_1, ..., \epsilon_n) \in [0, \infty[^n\,;\ \epsilon_1 + \cdots + \epsilon_n = \eta\Big\} \subset \mathbb{R}^n$$

equipped with the orientation induced by the canonical orientation of $\mathbb{R}^n$ (the oriented normal being the vector $(1, ..., 1)$).

Let $h \in \mathcal{O}_n$ (we choose a representative defined about the origin). Since

$$(\epsilon_1, ..., \epsilon_n) \longmapsto \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1|^2 = \epsilon_1\\ \vdots \\ |f_n|^2 = \epsilon_n}} \frac{h(\zeta)\, d\zeta}{f_1(\zeta) \cdots f_n(\zeta)}$$

is almost everywhere defined and almost everywhere equal to a constant, namely the residue symbol

$$\mathrm{Res}\begin{bmatrix} h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,,\ldots,\,f_n(\zeta)\end{bmatrix}_0 ,$$

on Σ_η when $\eta>0$ is small enough, we have

$$\begin{aligned}&\mathrm{Res}\begin{bmatrix} h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1(\zeta)\,,\ldots,\,f_n(\zeta)\end{bmatrix}_0 \\ &= \frac{(n-1)!}{\eta^n}\int_{\epsilon\in\Sigma_\eta}\Bigg(\frac{1}{(2i\pi)^n}\int\limits_{\substack{|f_1|^2=\epsilon_1\\ \vdots \\ |f_n|^2=\epsilon_n}}\frac{h(\zeta)\,d\zeta}{f_1(\zeta)\cdots f_n(\zeta)}\Bigg)\sum_{k=1}^n(-1)^{k-1}\epsilon_k\bigwedge_{\substack{l=1\\ l\neq k}}^n d\epsilon_l\,.\end{aligned} \tag{2.14}$$

We now notice that, in the integral above :

$$d\epsilon_j = d[|f_j|^2] = \overline{f}_j\,df_j + f_j\,d\overline{f}_j\,,\ j=1,\ldots,n\,.$$

After we proceed to the substitutions (and re-ordering of differential forms) in the double integral

$$\int_{\epsilon\in\Sigma_\epsilon}\int\limits_{\substack{|f_1|^2=\epsilon_1\\ \vdots \\ |f_n|^2=\epsilon_n}}\frac{h(\zeta)\,d\zeta}{f_1(\zeta)\cdots f_n(\zeta)}\,d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge\Big(\sum_{k=1}^n(-1)^{k-1}\epsilon_k\bigwedge_{\substack{l=1\\ l\neq k}}^n d\epsilon_l\Big)\,,$$

we finally obtain from 2.14

$$\begin{aligned}\mathrm{Res}\begin{bmatrix} h\,d\zeta \\ f_1\,,\ldots,\,f_n\end{bmatrix}_0 &= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{\eta^n(2i\pi)^n}\int_{\|f\|^2=\eta} h\Big(\sum_{k=1}^n(-1)^{k-1}\overline{f}_k d\overline{f}_{[k]}\Big)\wedge d\zeta \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|f\|^2=\eta}\frac{h\Big(\sum_{k=1}^n(-1)^{k-1}\overline{f}_k d\overline{f}_{[k]}\Big)\wedge d\zeta}{\|f\|^{2n}} \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|f\|^2=\eta} h\Big(\sum_{k=1}^n(-1)^{k-1}s_k ds_{[k]}\Big)\wedge d\zeta\end{aligned} \tag{2.15}$$

where we have used the abridged expressions :

$$\|f\|^2 := |f_1|^2+\cdots+|f_n|^2 \qquad d\overline{f}_k := \bigwedge_{\substack{l=1\\ l\neq k}}^n d\overline{f}_l\,,\ k=1,\ldots,n \qquad d\zeta := d\zeta_1\wedge\ldots\wedge d\zeta_n$$

$$s_k = \frac{\overline{f}_k}{\|f\|^2} \qquad ds_{[k]} = \bigwedge_{\substack{l=1\\ l\neq k}}^n ds_l \qquad (k=1,\ldots,n)\,.$$

One can see that, outside the origin (that is when $\eta \geq \|f\|^2 > 0$ pour η small enough),

$$d\Big[\Big(\sum_{k=1}^{n}(-1)^{k-1}s_k ds_{[k]}\Big)\wedge d\zeta\Big] = ds_1\wedge ds_2\wedge\cdots\wedge ds_n\wedge d\zeta = 0$$

since

$$s_1f_1 + s_2f_2 + \cdots + s_nf_n = \langle s\,,\, f\rangle \equiv 1$$

outside the origin. So, we conclude from Stokes's theorem that

$$\operatorname{Res}\begin{bmatrix} h\,d\zeta \\ f_1\,,\dots,\, f_n\end{bmatrix}_0 = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|\zeta\|^2=\eta} h\Big(\sum_{k=1}^{n}(-1)^{k-1}s_k ds_{[k]}\Big)\wedge d\zeta \tag{2.16}$$

for $\eta > 0$ small enough. From now on, we fix $\eta = \eta_0$ so that 0 is the only common zero of $(f_1, \dots, f_n)$ as well as of $(g_1, \dots, q_n)$ in the closed ball $\{\|\zeta\| \leq \eta_0\}$. We now remark, if

$$\tilde{s}_j := \frac{\overline{g}_j}{|g_1|^2+\cdots+|g_n|^2}\,,\ j=1,\dots,n\,,$$

outside the origin, that, in a neighborhood V of the sphere $\{\|\zeta\| = \eta_0\}$,

$$\begin{aligned} 1 \equiv \sum_{j=1}^{n}\tilde{s}_j g_j \ &\equiv\ \sum_{j=1}^{n}\tilde{s}_j\Big(\sum_{k=1}^{n}a_{j,k}f_k\Big)\\ &\equiv\ \sum_{k=1}^{n}\Big(\sum_{j=1}^{n}a_{j,k}\tilde{s}_j\Big)f_k\\ &\equiv\ \sum_{k=1}^{n}S_kf_k\,,\end{aligned}$$

where

$$S_j := \sum_{j=1}^{n}a_{k,j}\tilde{s}_k\,,\ j=1,\dots,n$$

and

$$\langle S\,,\, f\rangle \equiv \langle s\,,\, f\rangle \equiv 1\,.$$

For any $t \in [0,1]$, one can see that

$$\sum_{j=1}^{n}((1-t)s_j + tS_j)f_j = \langle (1-t)s + tS\,,\, f\rangle \equiv 1$$

in V. From this, follows immediately that, in $V\times[0,1]$,

$$d_t\Big[\Big(\sum_{k=1}^{n}(-1)^{k-1}((1-t)s_k+tS_k)d[(1-t)s+tS]_{[k]}\Big)\wedge d\zeta\Big] \equiv 0\,,$$

so that

$$\frac{d}{dt}\Big[\int_{\{|\zeta\|^2=\eta_0\}} h\Big(\sum_{k=1}^{n}(-1)^{k-1}((1-t)s_k+tS_k)d[(1-t)s+tS]_{[k]}\Big)\wedge d\zeta\Big] = 0\,,$$

from which it follows that 2.16 with $\eta = \eta_0$ can be also transformed as

$$\begin{aligned}
\mathrm{Res}\begin{bmatrix} h\,d\zeta \\ f_1\,,\ldots,\,f_n \end{bmatrix}_0 &= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|\zeta\|^2=\eta_0} h\Big(\sum_{k=1}^n (-1)^{k-1}s_k ds_{[k]}\Big)\wedge d\zeta \\
&= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|\zeta\|^2=\eta_0} h\Big(\sum_{k=1}^n (-1)^{k-1}S_k dS_{[k]}\Big)\wedge d\zeta \\
&= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|\zeta\|^2=\eta_0} h\Delta\Big(\sum_{k=1}^n (-1)^{k-1}\tilde{s}_k d\tilde{s}_{[k]}\Big)\wedge d\zeta\,,
\end{aligned} \tag{2.17}$$

where

$$\Delta := \det\Big[a_{j,k}\Big]_{\substack{1\le j\le n\\ 1\le k\le n}}\,.$$

If one conducts the same computations for $(g_1, \ldots, g_n)$ instead of $(f_1, \ldots, f_n)$ and $h\Delta$ instead of Δ, one can see that the pendant of 2.16 for $\eta = \eta_0$ is

$$\mathrm{Res}\begin{bmatrix} h\Delta\,d\zeta \\ g_1\,,\ldots,\,g_n \end{bmatrix}_0 = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|\zeta\|^2=\eta_0} h\Delta\Big(\sum_{k=1}^n (-1)^{k-1}\tilde{s}_k d\tilde{s}_{[k]}\Big)\wedge d\zeta \tag{2.18}$$

Combining 2.17 and 2.18 provides the transformation law. $\diamondsuit$

Remark. The transformation law can also be proved following the algebraic approach presented in section 2.3.2 (see corollary 2.8 in [Li], chapter 2). We will admit here this fact. Since both approaches (the analytic and the algebraic one) coincide when $f = (\zeta_1^\nu, \ldots, \zeta_n^\nu)$, it follows from the transformation law (carried simultaneously in the analytic and in the algebraic contexts) and from the fact that $\mathfrak{M}^\nu \subset (f_1, \ldots, f_n)$ for some ν when $(f_1, \ldots, f_n)$ is $\mathfrak{M}$-primary, that the two approaches (analytic and algebraic), when carried when $R = \mathcal{O}_n$ and $f_1, \ldots, f_n$ is a regular sequence, lead exactly to the same objects.

2.3.4 The local duality theorem and Bergman-Weil developments

It is well known (this is almost a definition of a convergent power series) that any $h \in \mathcal{O}_n$ can be developped (as a convergent power series) about the origin as :

$$h(z) = \sum_{\underline{k}\in\mathbb{N}^n} a_{\underline{k}} z_1^{k_1}\ldots z_n^{k_n}\,,$$

where

$$a_{\underline{k}} = \mathrm{Res}\begin{bmatrix} h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ \zeta_1^{k_1+1}\,,\ldots,\,\zeta_n^{k_n+1} \end{bmatrix}_0\,.$$

If we forget about the convergence of the series and keep only the algebraic flavor of the result, one can state it as follows : for any $N \in \mathbb{N}^*$,

$$h - \sum_{\substack{|\underline{k}|\in\mathbb{N}^n\\ |\underline{k}|\le N}} \mathrm{Res}\begin{bmatrix} h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ \zeta_1^{k_1+1}\,,\ldots,\,\zeta_n^{k_n+1} \end{bmatrix}_0 z^{\underline{k}} \in (z_1, \ldots, z_n)^{N+1}\,,\ \forall N \in \mathbb{N}\,.$$

Such a result holds in any n-dimensional regular local ring $\mathfrak{O}_n$ with residue field $\mathfrak{O}_n/\mathfrak{M}$ (whatever its characteristic is), $z_1, ..., z_n$ been even replaced by any regular sequence $f_1, ..., f_n$.

Proposition 2.3 *Let $f_1, ..., f_n$ be a regular sequence in a n-dimensional regular local ring $\mathfrak{O}_n$ with residue field $\mathfrak{O}_n/\mathfrak{M}$ (of any characteristic) ; assume that*

$$1 \otimes f_j - f_j \otimes 1 = \sum_{k=1}^{n} a_{j,k}.(1 \otimes \zeta_k - \zeta_k \otimes 1)\,,\ j = 1, ..., n\,,$$

where $\zeta_1, ..., \zeta_n$ are representants in $\mathfrak{M}$ for the generators $\overline{\zeta_1}, ..., \overline{\zeta_n}$ of the polynomial graded algebra $\mathfrak{O}_n/\mathfrak{M} + \mathfrak{M}/\mathfrak{M}^2 + \cdots$; let

$$\det[a_{j,k}] = \sum_{l=1}^{M} u_l \otimes v_l\,;$$

then, for any $N \in \mathbb{N}$, for any $h \in \mathfrak{O}_n$,

$$h - \sum_{\substack{\underline{k} \in \mathbb{N}^n \\ k_1+...+k_n \leq N}} \Big(\sum_{l=1}^{M} \mathrm{Res} \begin{bmatrix} hu_l\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1^{k_1+1}, ..., f_n^{k_n+1} \end{bmatrix}_0 v_l \Big) f_1^{k_1} \ldots f_n^{k_n} \in (f_1, ..., f_n)^{N+1}\,.$$

Proof.

1. The analytic case. We will first give the proof in the analytic case $\mathfrak{O}_n = \mathcal{O}_n$. The starting point is to use the classical Cauchy formula in a polydisc ; it is well known that if h is a representative of a germ of holomorphic function about the origin in $\mathbb{C}^n$ and if $\epsilon_1, ..., \epsilon_n$ are small enough, that, for any point in the polydisk

$$\Delta_\epsilon = \{z \in \mathbb{C}^n\,;\, |z_j| < \epsilon_j\,,\ j = 1, ..., n\}\,,$$

one has

$$h(z) = \frac{1}{(2i\pi)^n} \int_{|\zeta_1|=\epsilon_1, ..., |\zeta_n|=\epsilon_n} \frac{h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}\,.$$

We choose $\epsilon_1, ..., \epsilon_n$ small enough so that 0 is the only common zero of $f_1, ..., f_n$ in the closure of Δ_ϵ (which is possible sine 0 is necesseraly an isolated common zero of $f_1, ..., f_n$ which are assumed to form a regular sequence). Because of Stokes's theorem, we can even assert that for $|z_j| < \epsilon_j/2$, $j = 1, ..., n$,

$$h(z) = \frac{1}{(2i\pi)^n} \int_{|\zeta_1 - z_1|=\epsilon_1, ..., |\zeta_n - z_n|=\epsilon_n} \frac{h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}. \tag{2.19}$$

The right-hand side in formula (2.19) can be understood as the total sum of residues in the polydisc Δ_ϵ (respect to the quasi-regular sequence $(\zeta_1 - z_1, ..., \zeta_n - z_n)$) of the differential abelian $(n, 0)$-form $h\, d\zeta_1 \wedge \ldots \wedge d\zeta_n$. Let us write in Δ_ϵ (using for example either Taylor formula or Newton's method of divided differences)

$$f_j(\zeta) - f_j(z) = \sum_{k=1}^{n} a_{j,k}(z, \zeta)\, (\zeta_k - z_k)\,,\ j = 1, ..., n\,.$$

Thanks to a semi-local version of the transformation law (which can be proved exactly as the global version stated in proposition 2.4), one has, for $|z_j| < \eta_j <$

$\epsilon_j/2$ (small enough so that the functions $f_j - f_j(z)$, $j = 1, ..., n$, have exactly $\mu(f)$ common zeroes (counted with multiplicities) in $\overline{\Delta_\epsilon}$, all lying inside Δ_ϵ, the following represention formula :

$$h(z) = \mathrm{Res}\left[\begin{matrix} h(\zeta)\ \det[a_{j,k}](z,\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1(\zeta) - f_1(z), ..., f_n(\zeta) - f_n(z) \end{matrix}\right]_{\Delta_\epsilon} .$$

If $\rho_1, ..., \rho_n$ are small enough, then the set

$$\{z \in \Delta_\epsilon \,;\, |f_j| \leq \rho_j \,,\, j = 1, ..., n\}$$

is connected and included in Δ_ϵ ; moreover (if we refine the choice of $\eta_1, ..., \eta_n$), then, for $|z_j| < \eta_j$, $j = 1, ..., n$, all common zeroes of $f_j - f_j(z)$, $j = 1, ..., n$ in Δ_ϵ lie in the set

$$\{z \in \Delta_\epsilon \,;\, |f_j| < \rho_j \,,\, j = 1, ..., n\} .$$

We therefore have the representation formula (for $|z_j| < \eta_j$, $j = 1, ..., n$),

$$h(z) = \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1(\zeta)|=\rho_1,...,|f_n(\zeta)|=\rho_n \\ \zeta\in\Delta_\epsilon}} \frac{h(\zeta)\ \det[a_{j,k}](z,\zeta)}{(f_1(\zeta) - f_1(z)) \ldots (f_n(\zeta) - f_n(z))} .$$

We may know expand

$$\frac{1}{(f_1(\zeta) - f_1(z)) \ldots (f_n(\zeta) - f_n(z))} = \sum_{\underline{k}\in\mathbb{N}^n} \frac{(f_1(z))^{k_1} ... (f_n(z))^{k_n}}{(f_1(\zeta))^{k_1+1} ... (f_n(\zeta))^{k_n+1}}$$

as a normally convergent geometric series on $\{|f_1(\zeta)| = \rho_1, ..., |f_n(\zeta)| = \rho_n\}$ when $|f_1(z)| < \rho_1/2, ..., |f_n(z)| < \rho_n/2$ (which is achieved provided the η_j are small enough). Thanks to the normal convergence, we get, for such z close to 0, the so-called Bergman-Weil [24] expansion

$$h(z) = \sum_{\underline{k}\in\mathbb{N}^n} \mathrm{Res}\left[\begin{matrix} h(\zeta)\ \det[a_{k,l}](z,\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1^{k_1+1}(\zeta)\,, ...,\ f_n^{k_n+1}(\zeta) \end{matrix}\right]_0 (f_1(z))^{k_1} \ldots (f_n(z))^{k_n} ,$$

the series being normally convergent on any compact set, which proves of course the assertion in the proposition in the case $\mathfrak{O}_n = \mathcal{O}_n$.

2. The general case. Let us prove the result in the general case. Let $\tilde{f}_1, ..., \tilde{f}_n$ defining a regular sequence in $\mathfrak{O}_n$ and $\tilde{a}_{j,k} \in \mathfrak{O}_n \otimes \mathfrak{O}_n$ such that

$$\tilde{f}_j \otimes 1 - 1 \otimes \tilde{f}_j = \sum_{k=1}^{n} \tilde{a}_{j,k}.(\zeta_k \otimes 1 - 1 \otimes \zeta_k)\,,\ j = 1, ..., n\,,$$

and

$$\det[\tilde{a}_{j,k}] = \sum_{l=1}^{\widetilde{M}} \tilde{u}_l \otimes \tilde{v}_l\,.$$

[24] A close version of this formula (not developped) was proposed by André Weil in [Weil] ; the idea of such developments goes back to S.B. Bergman (1936) [Berg], see also for further modern developments the monography of Aizenberg-Yuzhakov [AY].

Let us prove that

$$\widetilde{h} = h - \sum_{l=1}^{\widetilde{M}} \operatorname{Res} \begin{bmatrix} h\,\tilde{u}_l\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ \tilde{f}_1, \ldots, \tilde{f}_n \end{bmatrix} \tilde{v}_l \in (\tilde{f}_1, \ldots, \tilde{f}_n)\,. \tag{2.20}$$

Thanks to the transformation law, for any g in $\mathfrak{O}_n$, for K large enough (such that $\mathfrak{M}^K \subset (\tilde{f}_1, \ldots, \tilde{f}_n)$),

$$\begin{aligned}
& \operatorname{Res} \begin{bmatrix} (h \otimes g).\Big(\sum_{l=1}^{\widetilde{M}} \tilde{u}_l \otimes \tilde{v}_l\Big) \bigwedge_{j=1}^{n} d[\zeta_j \otimes 1] \wedge \bigwedge_{j=1}^{n} d[1 \otimes \zeta_j] \\ \tilde{f}_1 \otimes 1, \ldots, \tilde{f}_n \otimes 1, 1 \otimes \tilde{f}_1, \ldots, 1 \otimes \tilde{f}_n \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (h \otimes g).\Big(\sum_{l=1}^{\widetilde{M}} \tilde{u}_l \otimes \tilde{v}_l\Big) d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \tilde{f} \otimes 1, 1 \otimes \tilde{f} \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (h \otimes g).\Big(\sum_{l=1}^{\widetilde{M}} \tilde{u}_l \otimes \tilde{v}_l\Big) d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \tilde{f} \otimes 1 - 1 \otimes \tilde{f}, 1 \otimes \tilde{f} \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (h \otimes g)\, d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \zeta \otimes 1 - 1 \otimes \zeta, 1 \otimes \tilde{f} \end{bmatrix} = \operatorname{Res} \begin{bmatrix} (h \otimes g)\, d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \zeta \otimes 1 - 1 \otimes \zeta, 1 \otimes \tilde{f} \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (1 \otimes gh) \prod_{j=1}^{n} \Big(\sum_{k=0}^{K-1} (\zeta_j^k \otimes \zeta_j^{K-1-k}\Big) d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \zeta^K \otimes 1 - 1 \otimes \zeta^K, 1 \otimes \tilde{f} \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (1 \otimes gh) \prod_{j=1}^{n} \Big(\sum_{k=0}^{K-1} (\zeta_j^k \otimes \zeta_j^{K-1-k}\Big) d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \zeta^K \otimes 1, 1 \otimes \tilde{f} \end{bmatrix} \\
= \; & \operatorname{Res} \begin{bmatrix} (1 \otimes gh)\, d[\zeta \otimes 1] \wedge d[1 \otimes \zeta] \\ \zeta \otimes 1, 1 \otimes \tilde{f} \end{bmatrix} = \operatorname{Res} \begin{bmatrix} gh \\ \tilde{f} \end{bmatrix}.
\end{aligned}$$

This proves that for any $g \in \mathfrak{O}_n$, one has

$$\operatorname{Res} \begin{bmatrix} g \Big(h - \sum_{l=1}^{\widetilde{M}} \operatorname{Res} \begin{bmatrix} h\tilde{u}_l d\zeta \\ \tilde{f} \end{bmatrix} \tilde{v}_l\Big)\, d\zeta \\ \tilde{f} \end{bmatrix} = 0\,.$$

Following the algebraic construction of the residue symbol developped in section , we conclude that for any $g \in \mathfrak{O}_n$, the trace of the multiplication operator

$$\dot{r} \in \frac{\mathfrak{O}_n}{(\tilde{f}_1, \ldots, \tilde{f}_n)} \longmapsto r\widetilde{h} \mod (\tilde{f}_1, \ldots, \tilde{f}_n)$$

is the zero operator, which proves the assertion (2.20). It also concludes the proof of the proposition when $N = 0$. In order to obtain it for arbitrary N, we need to apply the above result with $\tilde{f}_j = f_j^{N+1}$, $j = 1, \ldots, n$ and use the relations

$$\tilde{f}_j \otimes 1 - 1 \otimes \tilde{f}_j = \Big(\sum_{k=0}^{N} f_j^k \otimes f_j^{N-k}\Big) \Big(\sum_{k=1}^{n} a_{j,k}\, (\zeta_k \otimes 1 - 1 \otimes \zeta_k)\Big)\,,\ j = 1, \ldots, n\,.$$

The assertion of the proposition follows from the above reasoning applied with such $\tilde{f}_j$'s. $\diamondsuit$

The particular case $N = 1$, which asserts that, for any $h \in \mathfrak{o}_n$, one has

$$h \equiv \sum_{l=1}^{M} \mathrm{Res} \begin{bmatrix} h\, u_l\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ \tilde{f}_1, ..., \tilde{f}_n \end{bmatrix} v_l \mod (f_1, ..., f_n)$$

is know as *Kronecker Trace formula.* It has a crucial corollary[25] :

Corollaire 2.1 (the local duality theorem) *Let $f_1, ..., f_n$ be a regular sequence in a regular local ring $\mathfrak{O}_n$; then an element $h \in \mathfrak{O}_n$ lies in $(f_1, ..., f_n)$ if and only if*

$$\forall g \in \mathfrak{O}_n\,,\ \mathrm{Res} \begin{bmatrix} gh\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1, ..., f_n \end{bmatrix} = 0\,,$$

where $\overline{\zeta_1}, ..., \overline{\zeta_n}$ are elements in $\mathfrak{M}/\mathfrak{M}^2$ which generate the polynomial algebra

$$\frac{\mathfrak{O}_n}{\mathfrak{M}}[X_1, ..., X_n] = \mathfrak{O}_n/\mathfrak{M} + \mathfrak{M}/\mathfrak{M}^2 + \cdots\,.$$

2.4 Global residues in $\mathbb{K}[X_1, ..., X_n]$

2.4.1 The global transformation law

Besides the local version of the transformation law, we may state in $\mathbb{K}[X_1, ..., X_n]$ a global version as follows :

Proposition 2.4 (Global Transformation Law). *Let two quasi-regular sequences $(P_1, ..., P_n)$ and $(Q_1, ..., Q_n)$ in $\mathbb{K}[X_1, ..., X_n]$ ($\mathbb{K}$ being a commutative field) such that there exists polynomials $A_{j,k}$, $1 \leq j, k \leq n$, with*

$$Q_j = \sum_{k=1}^{n} A_{j,k} P_k\,,\ j = 1, ..., n\,.$$

Then, for any $Q \in \mathbb{K}[X_1, ..., X_n]$,

$$\begin{aligned} \mathrm{Res} \begin{bmatrix} Q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ P_1\,, ...,\ P_n \end{bmatrix} &:= \sum_{\alpha \in V(P_1,...,P_n)} \mathrm{Res} \begin{bmatrix} Q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ P_1\,, ...,\ P_n \end{bmatrix}_\alpha \\ &= \mathrm{Res} \begin{bmatrix} Q(X)\ \det[A_{j,k}](X)\, dX_1 \wedge \ldots \wedge dX_n \\ Q_1\,, ...,\ Q_n \end{bmatrix} \\ &:= \sum_{\beta \in V(Q_1,...,Q_n)} \mathrm{Res} \begin{bmatrix} Q(X)\ \det[A_{j,k}(X)]\, dX_1 \wedge \ldots \wedge dX_n \\ Q_1\,, ...,\ Q_n \end{bmatrix}_\beta\,. \end{aligned} \tag{2.21}$$

Proof. Since $(P_1, ..., P_n)$ and $(Q_1, ..., Q_n)$ are quasi-regular, the images $\phi(P)$ (*resp.* $\phi(Q)$) in any localisation at a prime ideal $\mathfrak{P}$ in $\overline{\mathbb{K}}[X_1, ..., X_n]$ which contains $V(P)$ (resp. $V(Q)$) define regular sequences, so that we can define local residues respect to these localisations. If $\alpha \in V(P_1, ..., P_n) \cap V(Q_1, ..., Q_n)$, it follows from the local transformation law (theorem 2.2) in the corresponding local ring instead of $\mathcal{O}_n$, one has

$$\mathrm{Res} \begin{bmatrix} Q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ P_1\,, ...,\ P_n \end{bmatrix}_\alpha = \mathrm{Res} \begin{bmatrix} Q(X)\ \det[A_{j,k}(X)]\, dX_1 \wedge \ldots \wedge dX_n \\ Q_1\,, ...,\ Q_n \end{bmatrix}_\beta\,.$$

[25] In fact we used this so-called "corollary" in our algebraic proof, since such statement appears in fact as an avatar of the algebraic construction developped in section 2.3.4.

If $\beta \in V(Q_1, ..., Q_n) \setminus V(P_1, ..., P_n)$, one can see from Cramer's rule that

$$\phi(\det[A_{j,k}]) \in (\phi(P_1), ..., \phi(P_n)),$$

which implies that

$$\operatorname{Res}\begin{bmatrix} Q(X)\ \det[A_{j,k}(X)]\, dX_1 \wedge \ldots \wedge dX_n \\ Q_1, ..., Q_n \end{bmatrix}_\beta = 0\,.$$

Therefore, the two sums in (2.21) are equal and the global Transformation Law is proved. $\diamondsuit$

2.4.2 Jacobi's theorem

A consequence of the global transformation law (proposition 2.4) is an important result due to C. Jacobi [Jac], who proved it under some minor additional assumptions[26].

Theorem 2.3 (Jacobi's residue theorem) *Let $P_1, ..., P_n$ be n polynomials in $\mathbb{K}[X_1, ..., X_n]$, where $\mathbb{K}$ is a commutative field, such that the homogeneous parts of higher degree $p_1, ..., p_n$ of $P_1, ..., P_n$ define a $(X_1, ..., X_n)$-primary ideal in the polynomial ring $\mathbb{K}[X_1, ..., X_n]$. Then $P_1, ..., P_n$ define a quasi regular sequence in $\mathbb{K}[X_1, ..., X_n]$ and, for any $Q \in \mathbb{K}[X_1, ..., X_n]$,*

$$\deg Q < \sum_{j=1}^{n} (\deg P_j - 1) \Longrightarrow \operatorname{Res}\begin{bmatrix} Q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ P_1, ..., P_n \end{bmatrix} = 0.$$

Proof. Let $|\ |$ be a non trivial absolute value on $\mathbb{K}$ and $\overline{\mathbb{K}}$ an integral closure. Since the homogeneous parts $p_1, ..., p_n$ of $P_1, ..., P_n$ define a $(X_1, ..., X_n)$-primary ideal in $\mathbb{K}[X_1, ..., X_n]$, there exists a strictly positive constant κ such that

$$\forall x \in \overline{\mathbb{K}}^n \setminus \{(0, ..., 0)\}\,,\ \sum_{j=1}^{n} \frac{|p_j(x)|}{\|x\|^{\deg P_j}} \geq \kappa$$

(here $\|x\| = |x_1| + \cdots + |x_n|$ for example). As a consequence, there exists $K > 0$, such that

$$\forall x \in \overline{\mathbb{K}}^n\,,\ \|x\| \geq K \Longrightarrow \sum_{j=1}^{n} \frac{|P_j(x)|}{\|x\|^{\deg P_j}} \geq \frac{\kappa}{2}\,;$$

this implies that the affine algebraic variety

$$V(P_1, ..., P_n) = \{x \in \overline{K}^n\,;\ P_1(x) = \cdots = P_n(x) = 0\}$$

lies entirely in $\{x \in \overline{K}\,;\ \|x\| < K\}$ and therefore is zero - dimensional [27], which proves that $(P_1, ..., P_n)$ defines a quasi-regular sequence in $\mathbb{K}[X_1, ..., X_n]$.

[26] In the context $\mathbb{K} = \mathbb{C}$, Jacobi was assuming in addition tranversality conditions respect to the hypersurfaces $\{P_j = 0\}$ in $\mathbb{C}^n$.

[27] If not, the Krull dimension of the homogeneous ideal generated by the homogeneisations ${}^hP_1, ..., {}^hP_n$ would have Krull dimension strictly bigger than 1 in $\mathbb{K}[X_0, ..., X_n]$, which would imply that for any homogeneous linear form L in $X_0, ..., X_n$, the Krull dimension of $({}^hP_1, ..., {}^hP_n, L)$ would be strictly positive !

Suppose that card $V(P_1,...,P_n) = N$. For each $j = 1,...,n$, one can find a monic polynomial $R_j \in \mathbb{K}[X_j]$ with degree N in the radical of $(P_1,...,P_n)$. Thanks to Hilbert's nullstellensatz, there exists an integer M and polynomials $A_{j,k}$, $1 \le j,k \le n$ such that

$$R_j^M(X_j) = \sum_{k=1}^n A_{j,k}(X)P_k(X)\,, j = 1,...,n\,.$$

Since $\{p_1 = \cdots = p_n = 0\} = \{0\}$ and $(P_1,...,P_n)$ is quasi-regular in $\mathbb{K}[X_1,...,X_n]$, the homogeneous polynomials ${}^hP_1,...,{}^hP_n$ define a regular sequence in the polynomial ring $\mathbb{K}[X_0,...,X_n]_{\rm hom}$, which implies that the primary decomposition of the ideal $({}^hP_1,...,{}^hP_n)$ does not involve embedded primes (thanks to Macaulay unmixed theorem, see for example [Mat], theorem 17.3, page 134), in particular $(X_0,...,X_n)$ cannot be an embedded prime in this primary decomposition. Therefore,

$$R_j^M(X_j) \in ({}^hP_1,...,{}^hP_n)\,,\ j = 1,...,n\,,$$

which implies that one can choose the $A_{j,k}$, $1 \le j,k \le n$, such that

$$\deg(A_{j,k}\,P_k) \le \deg R_j^M = MN\,,\ 1 \le j,k \le n\,.$$

We now use the global transformation law (proposition 2.4), which asserts that for any polynomial Q in $\mathbb{K}[X_1,...,X_n]$, one has

$$\mathrm{Res}\begin{bmatrix} Q(X)\,dX_1 \wedge \ldots \wedge dX_n \\ P_1(X)\,,...,\,P_n(X)\end{bmatrix} = \mathrm{Res}\begin{bmatrix} Q(X)\ \det[A_{j,k}(X)]\,dX_1 \wedge \ldots \wedge dX_n \\ R_1^M(X_1)\,,...,\,R_n^M(X_n)\end{bmatrix} \quad (2.22)$$

In order to prove Jacobi's result, it is enough to understand what happens when $n = 1$. Let $P(X) = a_0X^d + a_1X^{d-1} + ... + a_d$ $(a_0 \neq 0)$ and

$$R(X) = \sum_{j=0}^{\deg Q - 1} \alpha_j X^j$$

be the remainder in the euclidean division of Q by P in $\mathbb{K}[X_1,...,X_n]$. As seen in the previous section, one has

$$\mathrm{Res}\begin{bmatrix} Q(X)\,dX \\ P(X)\end{bmatrix} = \frac{\alpha_{\deg Q - 1}}{a_0}\,,$$

which is zero when $\deg Q < \deg P - 1$; this implies Jacobi's result when $n = 1$. Since one has, for each $(k_1,...,k_n) \in \mathbb{N}^n$,

$$\mathrm{Res}\begin{bmatrix} X_1^{k_1}\ldots X_n^{k_n}\,dX_1 \wedge \ldots \wedge dX_n \\ R_1^M(X_1)\,,...,\,R_n^M(X_n)\end{bmatrix} = \prod_{j=1}^n \mathrm{Res}\begin{bmatrix} X_j^{k_j}\,dX_j \\ R_j^M(X_j)\end{bmatrix}\,,$$

the right-hand term in (2.22) vanishes as soon as

$$\deg(Q\det[A_{j,k}]) < \sum_{j=1}^n (M\deg R_j - 1)\,;$$

since

$$\deg(\det[A_{j,k}]) + \sum_{j=1}^n \deg P_j \le nM\deg R_j\,,$$

a sufficient condition on Q which ensures the vanishing of such right-hand side is

$$\deg Q < \sum_{j=1}^n (\deg P_j - 1)\,,$$

which concludes the proof of Jacobi's theorem. $\diamondsuit$

2.4.3 Max Nœther theorem

We may now formulate (in a complete explicit way) an important algebraic result by Max Nœther ([Nœ], see also [Sco]) which has been "revisited" by italian geometers (Severi, Segre) along the last century (see for example [Sem]).

Theorem 2.4 (Max Nœther's theorem) *Let $\mathbb{K}$ be a commutative field and P_j, $j = 1, ..., n$, a sequence of polynomials in $\mathbb{K}[X_1, ..., X_n]$ which define a regular sequence and are such the homogeneous parts $p_1, ..., p_n$ of higher degrees define a $(X_1, ..., X_n)$-primary ideal in $\mathbb{K}[X_1, ..., X_n]$. Then $(P_1, ..., P_n)$ is a quasi-regular sequence which generates a proper ideal. Moreover, let*

$$P_j(X) - P_j(Y) = \sum_{k=1}^{n} a_{j,k}(X, Y)\,(X_k - Y_k)\,,\ j = 1, ..., n\,,$$

where the $a_{j,k} \in \mathbb{K}[X, Y]$ may be for example constructed thanks to Newton's divided differences method. Then, for any $Q \in (P_1, ..., P_n)$, the formal identity

$$Q(Y) = \sum_{\underline{k}\in\mathbb{N}^n\setminus\{0,...,0\}} \mathrm{Res}\begin{bmatrix} Q(X)\ \det[a_{j,k}(X, Y)]\,dX \\ P_1^{k_1+1}(X), ..., P_n^{k_n+1}(X)\end{bmatrix} P_1^{k_1}(Y)\ldots P_n^{k_n}(Y)$$

truncates exactly in order to provide a polynomial identity

$$Q(Y) = \sum_{j=1}^{n} A_j(Y)P_j(Y)\,,$$

where

$$\max_j[\deg P_jQ_j] = \deg Q\,.$$

Proof. If $(P_1, ..., P_n)$ had non common zero in $\overline{K}^n$, one would have, since ${}^hP_1, ..., {}^hP_n$ define a regular sequence in the homogeneous polynomial ring $\mathbb{K}[X_0, ..., X_j]$ (see the proof of Jacobi's theorem 2.3 above) ,

$$X_j = \sum_{k=1}^{n} \widetilde{Q}_{j,k}(X_0, ..., X_n)\,{}^hP_k(X_0, ..., X_n)\,,\ j = 1, ..., n\,.$$

This implies that one can write, after deshomogoneization (take $X_0 = 1$)

$$X_j = \sum_{k=1}^{n} Q_{j,k}P_k\,,$$

where $\max\deg(Q_{j,k}P_k) = 1$; all P_j are affine polynomials and $P_1, ..., P_n$ have a common zero since the homogeneous parts of higher degree correspond to an invertible $n \times n$ matrix of elements in $\mathbb{K}$. Therefore, this leads to a contradiction and $(P_1, ..., P_n)$ define necesseray a proper ideal in $\mathbb{K}[X_1, ..., X_n]$.

It follows from elementary properties of global residues and of the global transformation law (proposition 2.4 applied in $\mathbb{K}(Y)[X]$) that

$$\begin{aligned} Q(Y) &= \mathrm{Res}\begin{bmatrix} Q(X)\,dX_1\wedge\ldots\wedge dX_n \\ X_1 - Y_1, ..., X_n - Y_n\end{bmatrix} \\ &= \mathrm{Res}\begin{bmatrix} Q(X)\ \det[a_{j,k}]dX_1\wedge\ldots\wedge dX_n \\ P_1(X) - P_1(Y), ..., P_n(X) - P_n(Y)\end{bmatrix}. \end{aligned}$$

If we localise at some maximal ideal $\mathfrak{M}$ in $\overline{\mathbb{K}}[X_1, ..., X_n]$ (corresponding to a common zero $\alpha \in \overline{K}^n$ for $P_1, ..., P_n$), we get about such point α the Bergman-Weil development

$$\begin{aligned}\phi(Q)(z) &= \sum_{\underline{k}\in\mathbb{N}^n\setminus\{(0,...,0)\}} \operatorname{Res}\begin{bmatrix}\Phi(Q)\,\phi(\det[a_{j,k}](X,z))\,dX_1\wedge\ldots\wedge dX_n\\ P_1^{k_1+1}(X),...,P_n^{k_n+1}(X)\end{bmatrix}_\alpha\\ &\quad\times(\phi(P_1)(z))^{k_1}\ldots(\phi(P_n)(z))^{k_n}\,. \qquad (2.23)\end{aligned}$$

But Cramer's rule (see the argument used to deduce the global transformation law from the local one as in the proof of proposition 2.4), one has

$$\begin{aligned}&\operatorname{Res}\begin{bmatrix}\Phi(Q)\,\phi(\det[a_{j,k}](X,z))\,dX_1\wedge\ldots\wedge dX_n\\ P_1^{k_1+1}(X),...,P_n^{k_n+1}(X)\end{bmatrix}_\alpha\\ =\;&\operatorname{Res}\begin{bmatrix}\Phi(Q)\,\phi[\det[a_{j,k}](X,z)\,dX_1\wedge\ldots\wedge dX_n\\ P_1^{k_1+1}(X),...,P_n^{k_n+1}(X)\end{bmatrix}.\end{aligned}$$

Moreover, one has

$$\operatorname{Res}\begin{bmatrix}\Phi(Q)\,X_1^{\beta_1}\ldots X_n^{\beta_n}\,dX_1\wedge\ldots\wedge dX_n\\ P_1^{k_1+1}(X),...,P_n^{k_n+1}(X)\end{bmatrix}=0$$

as soon as

$$(k_1+1)\deg P_1\cdots+(k_n+1)\deg P_n>\deg Q+\beta_1+\cdots+\beta_n+n\,,$$

or

$$k_1\deg P_1+\ldots+k_n\deg P_n+\sum_{j=1}^n(\deg P_j-1)>\deg Q+\beta_1+\cdots+\beta_n\,.$$

As the total degree of $\det[a_{j,k}(X,Y)]$ is precisely $\sum_{j=1}^n(\deg P_j-1)$, the right-hand side of (2.23) truncates exactly as

$$\phi\Big(\sum_{j=1}^n A_jP_j\Big)$$

where $\max_j(\deg A_jP_j)=\deg Q$. The local algebraic identity

$$\phi(Q-\sum_{j=1}^n A_jP_j)=0$$

implies (because of a flatness argument) the global algebraic polynomial identity

$$Q=\sum_{j=1}^n A_jP_j\,.$$

This ends the proof of Max Nœther's result. $\diamondsuit$

2.4.4 The Bézout identity in a particular case

We combine in this section the different tools involved in the previous sections in order to prove the following version of Bézout identity :

Proposition 2.5 *Let $\mathbb{K}$ be a commutative field and $P_1, ..., P_{n+1}$ $n+1$ polynomials in n variables such that the homogeneous parts $p_1, ..., p_n$ of $P_1, ..., P_n$ define a $(X_1, ..., X_n)$-primary ideal in $\mathbb{K}[X_1, ..., X_n]$ and $(P_1, ..., P_{n+1}) = \mathbb{K}[X_1, ..., X_n]$. Let $a_{j,k}$, $j = 1, ..., n+1, k = 1, ..., n$ be $(n+1) \times n$ polynomials in $\mathbb{K}[X, Y]$ such that*

$$P_j(X) - P_j(Y) = \sum_{k=1}^{n} a_{j,k}(X,Y)\,(X_k - Y_k)\,,\ j = 1, ..., n+1\,.$$

The following Bézout identity holds :

$$1 = \mathrm{Res}\begin{bmatrix} \begin{vmatrix} a_{1,1}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,1}(X,\mathbf{Y}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,n}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,n}(X,\mathbf{Y}) \\ \mathbf{P_1}(\mathbf{Y}) & \dots & \dots & \mathbf{P_{n+1}}(\mathbf{Y}) \end{vmatrix} \dfrac{dX_1 \wedge \ldots \wedge dX_n}{P_{n+1}(X)} \\ \\ P_1(X), ..., P_n(X) \end{bmatrix}. \tag{2.24}$$

Proof. Following the proof of Nœther's theorem 2.4, the Cauchy global identity

$$1 = \mathrm{Res}\begin{bmatrix} dX_1 \wedge \ldots \wedge dX_n \\ X_1 - \mathbf{Y_1}, ..., X_n - \mathbf{Y_n} \end{bmatrix}$$

can be transformed, thanks to the global transformation law (proposition 2.4) combined with Jacobi's theorem (2.3), into the polynomial identity

$$\begin{aligned} 1 &= \mathrm{Res}\begin{bmatrix} \det[a_{j,k}(X,Y)]_{1\le j,k\le n}\, dX_1 \wedge \ldots \wedge dX_n \\ \\ P_1(X), ..., P_n(X) \end{bmatrix} \\ &= \mathrm{Res}\begin{bmatrix} P_{n+1}(X)\det[a_{j,k}(X,\mathbf{Y})]_{1\le j,k\le n} \dfrac{dX_1 \wedge \ldots \wedge dX_n}{P_{n+1}(X)} \\ \\ P_1(X), ..., P_n(X) \end{bmatrix} \\ &= \mathrm{Res}\begin{bmatrix} \begin{vmatrix} a_{1,1}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,1}(X,\mathbf{Y}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,n}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,n}(X,\mathbf{Y}) \\ 0 & \dots & 0 & P_{n+1}(X) \end{vmatrix} \dfrac{dX_1 \wedge \ldots \wedge dX_n}{P_{n+1}(X)} \\ P_1(X), ..., P_n(X) \end{bmatrix} \\ &= \mathrm{Res}\begin{bmatrix} \begin{vmatrix} a_{1,1}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,1}(X,\mathbf{Y}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,n}(X,\mathbf{Y}) & \dots & \dots & a_{n+1,n}(X,\mathbf{Y}) \\ \mathbf{P_1}(\mathbf{Y}) - P_1(X) & \dots & \mathbf{P_n}(\mathbf{Y}) - P_n(X) & \mathbf{P_{n+1}}(\mathbf{Y}) \end{vmatrix} \dfrac{dX}{P_{n+1}(X)} \\ \\ P_1(X), ..., P_n(X) \end{bmatrix}. \end{aligned}$$

We conclude with the fact that any global residue respect to $P_1, ..., P_n$ is annihilated by each P_j, $j = 1, ..., n$. $\diamondsuit$

Chapter 3

Lesson 3 : about integral closure

3.1 Some equivalent definitions

In this section $\mathbb{A}$ denotes a commutative nœtherian domain, I an ideal in $\mathbb{A}$. The fraction field $\mathrm{Frac}\,(\mathbb{A})$ may have arbitrary characteristic. The main reference used in this chapter is the survey paper by K. E. Smith [Smith] (also useful to provide accurate references).

Definition 3.1 *The integral closure $\overline{I}$ of an ideal I is the set of elements $h \in \mathbb{A}$ which satisfy a homogeneous relation of integral dependency*

$$h^N + a_1 h^{N-1} + \cdots + a_N \equiv 0\,, \tag{3.1}$$

where $a_k \in I^k$, $k = 1, ..., N$ and $N = N(h) \in \mathbb{N}^*$.

Remark. Let $\mathrm{grad}\, I$ be the graded $\mathbb{A}[X]$-algebra

$$\bigoplus_{n \geq 0} I^n X^n\,;$$

an element h of $\mathbb{A}$ lies in $\overline{I}$ if and only if hX satisfies in $\mathbb{A}[X]$ a monic equation

$$(hX)^N + U_1 (hX)^{N-1} + \cdots + U_N\,,\ U_1, ..., U_N \in \mathrm{grad}\, I\,.$$

This caracterization (just multiply by X relation (3.1) is the most convenient in order to check (which is not completely trivial following definition 3.1) that $\overline{I}$ is an ideal.

Another equivalent caracterisation of the integral closure is the *valuative criterion* (see [ZS], lemma p. 354) :

Proposition 3.1 *Given a nœtherian commutative domain $\mathbb{A}$ and an ideal I in $\mathbb{A}$, an element $h \in \overline{I}$ if and only if $h \in IV$ for all discrete valuation rings V lying between $\mathbb{A}$ and its fraction field* $\mathrm{Frac}\,(A)$.

Example. When $\mathbb{A} = \mathcal{O}_n$ and $h \in \overline{I}$ if and only if for any germ $t \mapsto \gamma(t)$ of curve such that $\gamma(0) = 0$, one has

$$\mathrm{val}(h \circ \gamma\ ; 0) \geq \min_{a \in I} \mathrm{val}(a \circ \gamma\ ; 0)\,.$$

For example, if $f \in \mathfrak{M}$, f belongs to the integral closure of the jacobian ideal generated by the germs of $\frac{\partial f}{\partial z_j}$, $j = 1, ..., n$: this follows from Leibniz rule :

$$(f \circ \gamma)'(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(\gamma(t))\, \gamma_j'(t)$$

for any germ of curve γ which passes trough the origin ; therefore

$$\mathrm{val}(f\circ\gamma\,;0)\geq \mathrm{val}\,((f\circ\gamma)\,;0)\geq \min_j \mathrm{val}\,(\partial_{z_j}\circ\gamma\,;0)\,,$$

so that the valuative criterion is fulfilled.

Example. When $\mathbb{A}=\mathcal{O}_n$ and I is a monomial ideal generated by the monomials $\zeta^{\alpha_1},...,\zeta^{\alpha_M}$, $M\in\mathbb{N}^*$, $(\alpha_1,...,\alpha_N$ being elements in $\mathbb{N}^n\setminus\{(0,...,0)\})$, $\overline{I}$ is generated by all monomials z^β where

$$\beta\in \text{convex enveloppe}\Big(\bigcup_{j=1}^{M}[\alpha_j+([0,+\infty[)^n]\Big)\,,$$

that is β belongs to the convex enveloppe of the *staircase* of I.

As a consequence of the valuative criterion, here is also an interesting caracterization of the integral closure of an ideal :

Proposition 3.2 *Let $\mathbb{A}$ be a commutative nœtherian domain and I an ideal in $\mathbb{A}$. An element $h\in\mathbb{A}$ lies in $\overline{I}$ if and only if there is some $c=c(h)\in\mathbb{A}\setminus\{0\}$ such that for infinitely many integers $n\in\mathbb{N}$, one has $ch^n\in I^n$.*

3.2 Tight closure in positive characteristic

In this section, we consider that the commutative nœtherian domain $\mathbb{A}$ is such that its fraction field Frac $(\mathbb{A})$ has positive characteristic $p>0$[1].

Definition 3.2 *Let $\mathbb{A}$ be a commutative nœtherian domain with fraction field of characteristic $p>0$ and $I=(f_1,...,f_m)$ some ideal in $\mathbb{A}$. The tight closure of I is the ideal in $\mathbb{A}$ which consists of elements $h\in\mathbb{A}$ such that there exists $c=c(h)\in\mathbb{A}\setminus\{0\}$ such that for all e sufficiently large,*

$$ch^{p^e}\in(f_1^{p^e},...,f_m^{p^e})\,. \tag{3.2}$$

Remark. Taking the p^e-roots, formula 3.2 leads to

$$c^{1/p^e}\,h\in I\cdot\mathbb{A}^{1/p^e}\,,$$

which means, since c^{1/p^e} "tends" to 1 when e tends to infinity (just take valuatons to quantify that), that h is indeed "almost" in I (here comes the reason for the terminology "*tight*" closure).

Because of the caracterisation given in proposition 3.2, there is a relation between the two notions of integral closure and tight closure, namely we have the :

Proposition 3.3 *Let $\mathbb{A}$ be a commutative nœtherian domain with fraction field of positive characteristic p and $I=(f_1,...,f_m)$ an ideal in $\mathbb{A}$. Then, one has $\overline{I^m}\subset I^*$ if I^* denotes the tight closure of I in $\mathbb{A}$.*

Proof. Because of the caracterization given in proposition 3.2, if $h\in\overline{I^m}$, there exists $c=c(h)\in\mathbb{A}\setminus\{0\}$ such that, for any $e\in\mathbb{N}$ sufficiently large,

$$ch^{p^e}\in I^{mp^e}\,;$$

[1] In fact, the notion of *tight closure* we introduce in this section could be introduced as well provided $\mathbb{A}$ is a commutative nœtherian ring containing a field with characteristic $p>0$ ($\mathbb{A}$ is then called of "equi-characteristic").

but if $k_1, ..., k_m$ are m positive integers such that $k_1 + \ldots + k_m \geq mp^e$, at least one of the k_j is greater than p^e, so that $ch^{p^e} \in I^{mp^e}$ implies

$$ch^{p^e} \in (f_1^{p^e}, ..., f_m^{p^e}) .$$

Since this holds for any $e >> 0$, h lies in the tight closure I^*. $\diamondsuit$

We now consider the particular situation when $\mathbb{A} = \mathfrak{O}_n$ is a regular local ring with dimension n such that the residue field $\mathfrak{O}_n/\mathfrak{M}$ is a perfect[2] field with positive characteristic p.

Proposition 3.4 *Let $\mathfrak{O}_n$ be a commutative n-dimensional regular local ring with perfect residue field $\mathfrak{k} = \mathfrak{O}_n/\mathfrak{M}$ with characteristic $p > 0$. Then, any ideal in $\mathfrak{O}_n$ is tightly closed, that is satisfies $I = I^*$.*

Proof. We need to show that if $I = (f_1, ..., f_m)$, any element in I^* lies in I. Let $h \in I^*$. There exists $c = c(h) \in \mathfrak{O}_n \setminus \{0\}$ such that

$$ch^{p^e} = \sum_{j=1}^{m} a_{e,j} f_j^{p^e} \tag{3.3}$$

for some $a_{e,j}$ in $\mathfrak{O}_n$. Since $c \neq 0$, there is some $e_c \in \mathbb{N}$ such that, for $e > e_c$, c does not belong to the ideal generated by $\zeta_1^{p^e}, ..., \zeta_n^{p^e}$, where $\overline{\zeta}_1, ..., \overline{\zeta}_n$ are generators (in $\mathfrak{M}/\mathfrak{M}^2$) for the polynomial n-dimensional $\mathfrak{k}$-graded algebra $\mathfrak{O}_n/\mathfrak{M} + \mathfrak{M}/\mathfrak{M}^2 + ...$ Since $\mathfrak{O}_n$ is regular and $\mathfrak{k}$ is perfect, the Frobenius map $h \mapsto h^p$ is finite and $\mathfrak{O}_n$ is a free module over each of the $\mathfrak{O}_n^{p^e}$ for any $e \in \mathbb{N}$, so that we get, for $e > e_c$, a $\mathfrak{O}_n^{p^e}$-linear map ϕ_e from $\mathfrak{O}_n$ into $\mathfrak{O}_n^{p^e}$ such that $\phi_e(c) = 1$. Then, it follows from (3.3), taking the action of ϕ_e on both sides, that

$$h^{p^e} = \sum_{j=1}^{m} \phi_e(a_{e,j}) f_j^{p^e} ;$$

from that we deduce, taking p^e-roots, that h lies in I ; note that here of course, binomial identities of the form

$$(u_1 + \cdots + u_m)^{p^e} = u_1^{p^e} + \ldots + u_m^{p^e}$$

have played an essential role (here is the great advantage of the non zero characteristic setting). $\diamondsuit$

Combining propositions 3.3 and 3.4, we get the following result, known (here in the context of positive characteristic) as Briançon-Skoda theorem[3] :

Proposition 3.5 *Let I be an ideal generated by m elements in a regular local ring $\mathfrak{O}_n$ with perfect residue field $\mathfrak{k}$ with characteristic $p > 0$. Then, one has the inclusion $\overline{I^m} \subset I$.*

[2] "*Perfect*" means (when k is a field with positive characteristic p) that for any a in the field, the equation $z^p = a$ admits at least one root in k.

[3] The first proof of this result was obtained in the characteristic zero case (which is indeed the most delicate !) by Joël Briançon and Henri Skoda using complex analytic L^2 methods (due to Lars Hörmander) in 1974 [BriS] ; then a proof in a more general context (including the positive characteristic setting) was given by J. Lipman and A. Sathaye [LS], at the same time than a proof based on duality and residue calculus was proposed by J. Lipman and B. Teissier [LT]. The case of positive characterisic was extensively studied by Craig Huneke, Melvin Hochster (see [HH] and the notes on the web site of Melvin Hochster : http://www.math.lsa.umich.edu/∼hochster), Karen Smith,...

3.3 Residue calculus and integral closure

Briançon-Skoda's theorem in the n-dimensional regular local ring $\mathcal{O}_n$ can be stated as follows :

Theorem 3.1 (Briançon-Skoda theorem, analytic version) *Let I an ideal in $\mathcal{O}_n$ generated by m elements ; let $\mu = \min(m,n)$. Then, for any $k \in \mathbb{N}^*$, one has $\overline{I^{\mu-1+k}} \subset I^k$.*

Proof. We will give here a proof when I is a proper $\mathfrak{M}$-primary ideal (and therefore $\mu = n$) which is generated by a regular sequence $f_1, ..., f_n$. The general case follows from two observations :

- by Krull's theorem, any ideal I in $\mathcal{O}_n$ can be expressed as
$$I = \bigcap_{k \geq 1} (I + \mathfrak{M}^k)$$
since $\bigcap_{k\geq 1} \mathfrak{M}^k = \{0\}$;
- when I is a proper primary ideal generated by $m \geq n$ elements $f_1, ..., f_m$, then any system $(\tilde{f}_1, ..., \tilde{f}_n)$ of n generic linear combinations (with complex coefficients) of $f_1, ..., f_m$, defines a regular sequence in $\mathcal{O}_n$ and is such that the integral closures of $(\tilde{f}_1, ..., \tilde{f}_n)$ and $(f_1, ..., f_m)$ coïncide.

If $(f_1, ..., f_n)$ is a regular sequence in $\mathcal{O}_n$, then[4], one has, for any $k \geq 1$,

$$(f_1, ..., f_n)^k = \bigcap_{\substack{\underline{l} \in \mathbb{N}^n \\ l_1 + \cdots + l_n = k-1+n}} (f_1^{l_1}, ..., f_n^{l_n}) \tag{3.4}$$

(in order to see that, just write down the quasi-regularity conditions at any order). If h lies in the integral closure of I^{n-1+k}, then, it follows easily from the integral dependence relation

$$h^N + u_{k,1} h^{N-1} + \cdots + u_{k,N} = 0\,,\ \ u_{k,j} \in I^{j(n-1+k)}$$

(because of its homogeneity) that

$$|h(\zeta)| \leq C \|f\|^{n-1+k}$$

for some strictly positive constant C in a neighborhood of the origin[5]. Because of the local duality theorem (corollary 2.1), in order to check that such h belongs to I^k, it is enough, in view of 3.4, to check that for any $g \in \mathcal{O}_n$, for any $\underline{l} \in ((\mathbb{N})^*)^n$ such that $l_1 + \cdots + l_n = k - 1 + n$, one has

$$\mathrm{Res} \begin{bmatrix} g(\zeta)\, h(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \\ f_1^{l_1}, ..., f_n^{l_n} \end{bmatrix}_0 = 0\,.$$

[4] As pointed out by Melvin Hochster in the Appendix of [LT].

[5] In fact, if $I = (f_1, ..., f_m)$ is an ideal in $\mathcal{O}_n$, the condition $|h| \leq C\|f\|$ in a neighborhood of the origin caracterizes the membership of h to the integral closure of $(f_1, ..., f_m)$; the proof (see [LeT]), which is pretty difficult, goes through the valuative criterion and uses resolution of singularities in characteristic zero [Hir] ; finding explicitely the integral dependence relation starting with the condition $|h| \leq C\|f\|$ remains an open question.

It is rather surprizing here that for the first time, we are going to exploit the integral symbol involved in the definition of the local residue :

$$\mathrm{Res}\begin{bmatrix} g(\zeta)\,h(\zeta)\,d\zeta_1\wedge\ldots\wedge d\zeta_n \\ f_1^{l_1},\ldots,f_n^{l_n}\end{bmatrix}_0 = \frac{1}{(2i\pi)^n}\int_{\Gamma_{\vec{\epsilon}}(f)}\frac{g(\zeta)h(\zeta)\,d\zeta}{f_1^{l_1}(\zeta)\ldots f_n^{l_n}(\zeta)}\,,$$

where

$$\Gamma_{\vec{\epsilon}}(f) := \{|f_1| = \epsilon_1,\ldots,|f_n| = \epsilon_n\}$$

and $(\epsilon_1,\ldots,\epsilon_n)$ are choosen (outside a set of Lebesgue measure zero in $]0,\eta]^n$ for η small enough) so that $\Gamma_{\vec{\epsilon}}(f)$ is a real n-dimensional cycle with smooth support. Moreover, a classical result in differential geometry (the *coaerea formula*, see for example [Fed], Theorem 3.2.11, p. 248) ensures that one can find $\epsilon_1\sim\epsilon,\ldots,\epsilon_n\sim\epsilon$, so that, when ϵ tends to zero, the n-dimensional volume of the support of $\Gamma_{\vec{\epsilon}}(f)$ tends to 0 [6]. Estimating integrals, we get that

$$\Big|\frac{1}{(2i\pi)^n}\int_{\Gamma_{\vec{\epsilon}}(f)}\frac{g(\zeta)h(\zeta)\,d\zeta}{f_1^{l_1}(\zeta)\ldots f_n^{l_n}(\zeta)}\Big| \leq C\|g\|_\infty\frac{\epsilon^{k+n-1}}{\epsilon_1^{l_1}\ldots\epsilon_n^{l_n}}\,\mathrm{vol}\,(\Gamma_{\vec{\epsilon}}(f)) \overset{\epsilon\to 0}{\longrightarrow} 0$$

since $\epsilon_1^{l_1}\ldots\epsilon_n^{l_n}\sim\epsilon^{l_1+\ldots+l_n}=\epsilon^{k-n+1}$. Because the integral does not depend of ϵ (thanks to Stokes's formula), the residue symbol equals 0 and we conclude applying the duality theorem that $h\in(f_1^{l_1},\ldots,f_n^{l_n})$ for all such multi-indices $\underline{l}$, which concludes the proof of the theorem. $\diamondsuit$

When $\mathfrak{O}_n$ is a regular n-dimensional local ring (whatever the characteristic of the residue field $\mathfrak{k}$ is), one can provide a characterisation of the membership to integral closure thanks to residue symbols ; namely we have the :

Proposition 3.6 *Let $\mathfrak{O}_n$ be a n-dimensional regular local ring, $\overline{\zeta}_1,\ldots,\overline{\zeta}_n$ being a set of generators in $\mathfrak{M}/\mathfrak{M}^2$ for the graded $\mathfrak{k}$-n dimensional algebra $\mathfrak{O}_n/\mathfrak{M}+\mathfrak{M}/\mathfrak{M}^2+\ldots$, and $f_1,\ldots,f_n$ a regular sequence in $\mathfrak{O}_n$. Then an element $h\in\mathfrak{O}_n$ belongs to the integral closure $\overline{(f_1,\ldots,f_n)}$ if and only if for any $g\in\mathfrak{O}_n$, the formal power series belonging to $\mathfrak{k}[[u]]$:*

$$\widehat{F}_{h,g} := \sum_{\underline{k}\in\mathbb{N}^n}\mathrm{Res}\begin{bmatrix} g h^{k_1+\ldots+k_n}\,d\zeta \\ f_1^{k_1+1},\ldots,f_n^{k_n+1}\end{bmatrix} u_1^{k_1}\ldots u_n^{k_n}$$

corresponds to the development about the origin of some element in $\mathfrak{k}(u)$ which has no pole at $u=0$ and degree (maximum of the degrees of numerator and denominator in a reduced expression) bounded independently of g, the denominator being independent of g.

Proof. From the relation of integral dependency

$$h^N+\sum_{j=1}^{N}\Big(\sum_{\substack{\underline{q}\in\mathbb{N}^n\\ q_1+\cdots+q_n=j}} a_{j,\underline{q}}f_1^{q_1}\ldots f_n^{q_n}\Big)\,h^{N-j}=0\,,$$

one can see, when h belongs to the integral closure of $(f_1,\ldots,f_n)$, that the coefficients of $\widehat{F}_{h,g}$ obey a difference equation, which shows indeed that $\widehat{F}_{h,g}$ corresponds to a

[6] In fact, the *coarea formula* asserts that the $L^1(]0,\eta]^n)$-norm of the n-dimensional Lebesgue measure of the trace of the support of $\Gamma_{\vec{\epsilon}}(f)$ with the euclidean ball $B(0,\rho)=\{\|z\|\leq\rho\}$ (as a function of $\vec{\epsilon}$) is bounded by $\kappa\rho^{2n}\|df\|^n_{B(0,\rho)}$ and therefore goes to zero when ρ tends to zero.

rational fraction in $\mathfrak{k}(u)$, whose degree is bounded by $2N$. Conversely, let us assume $\widehat{F}_{h,g}$ corresponds to such a rational function, with denominator

$$1 + \sum_{\substack{\underline{q}\in\mathbb{N}^n \\ q_1+\ldots+q_n\leq N}} \alpha_{\underline{q}} u_1^{q_1} \ldots u_n^{q_n}$$

For any $\underline{l} \in \mathbb{N}^n$ with $l_1 + \ldots + l_n = 2N$, one has

$$\mathrm{Res}\begin{bmatrix} g\Big(h^{2N} + \sum\limits_{j=1}^{N} (\sum\limits_{\substack{\underline{q}\in\mathbb{N}^n \\ q_1+\ldots+q_n=j}} \alpha_{\underline{q}} f_1^{q_1} \ldots f_n^{q_n}) h^{2N-k}\Big)\, d\zeta \\ f_1^{l_1+1}, \ldots, f_n^{l_n+1} \end{bmatrix} = 0\,,$$

which implies thanks to the local duality result (corollary 2.1) that for any such $\underline{l}$,

$$h^{2N} + \sum_{j=1}^{N} (\sum_{\substack{\underline{q}\in\mathbb{N}^n \\ q_1+\ldots+q_n=j}} \alpha_{\underline{q}} f_1^{q_1} \ldots f_n^{q_n}) h^{2N-k} \in (f_1^{l_1+1}, \ldots, f_n^{l_n+1})\,.$$

In view of (3.4) it follows that

$$h^{2N} + \sum_{j=1}^{N} (\sum_{\substack{\underline{q}\in\mathbb{N}^n \\ q_1+\ldots+q_n=j}} \alpha_{\underline{q}} f_1^{q_1} \ldots f_n^{q_n}) h^{2N-k} \in (f_1, \ldots, f_n)^{2N+1}\,,$$

which provides the integral dependency relation we need in order to conclude that h belongs to the integral closure of $(f_1, \ldots, f_n)$. $\diamondsuit$

Given an ideal in a n-dimensional regular local ring $\mathfrak{O}_n$, the smallest integer m such that there exists elements $f_1, \ldots, f_m$ in I such that I and $(f_1, \ldots, f_m)$ have the same integral closure in $\mathfrak{O}_n$ is called the *analytic spread* of I (see [NR]). It will have another interpretation when we will introduce the graded ring

$$G_I(\mathfrak{O}_n) := \bigoplus_{k=0}^{\infty} \frac{I^k}{I^{k+1}}\,,$$

since the analytic spread of I will coïncide with the dimension (as a $\mathfrak{O}_n/\mathfrak{M}$-vectorial space) of

$$G_I(\mathfrak{O}_n) \otimes_{\mathfrak{O}_n} \frac{\mathfrak{O}_n}{\mathfrak{M}} = \frac{G_I(\mathfrak{O}_n)}{\mathfrak{M}\, G_I(\mathfrak{O}_n)}\,.$$

3.4 Integral closure and membership problems

We will see in the next chapters that the geometric Bézout theorem (which governs geometric intersection theory) governs in fact the effective resolution of Hilbert's Nullstellensatz ; namely, if $P_1, \ldots, P_m$ are m polynomials in $\mathbb{K}[X_1, \ldots, X_n]$ defining $\mathbb{K}[X_1, \ldots, X_n]$ as an ideal and having respective degrees $D_1 \geq D_2 \ldots \geq D_m$, one can write an explicit Bézout identity

$$1 = 1^l = \sum_{j=1}^{m} A_j P_j\,,$$

where

$$\max_j \deg(A_j P_j) \leq l D_1 D_2 ... D_{\inf(n,m)} ,$$

$l \leq n+1$ being precisely the analytic spread of the ideal $({}^h P_1, ..., {}^h P_m)$ in the local ring $\mathcal{O}_{n+1}(z_0, ..., z_n)$ (see [Hi] for such a result and updated references) [7].

On the other hand, the *membership problem* remains with its intrinsic algebraic complexity : given polynomials $P_1, ..., P_m$ with degree at most D and Q a polynomial wchich is known to belong to the ideal $(P_1, ..., P_m)$, there is in general no hope to expect an effective membership formula :

$$Q = \sum_{j=1}^{m} A_j P_j ,$$

with $\max \deg A_j P_j \leq \deg Q + \kappa_n D^{2^n}$. In fact, in a famous paper [MM], E. Mayr and A. Meyer were able to construct (as an automat), for any integer $D \geq 1$, for any integer $k \in \mathbb{N}^*$, $10k+1$ binomials $P_{k,0}, ..., P_{k,10k}$ in $10k$ variables such that $X_1 \in (P_{k,0}, ..., P_{k,10k+1})$ and, for any relation

$$X_1 = \sum_{j=0}^{10k} A_{k,j} P_{k,j} ,$$

one has necesseraly

$$\max_j \deg(A_{k,j} P_{k,j}) \geq (D-2)^{2^{k-1}} .$$

On the other hand, $I = (P_1, ..., P_m)$ being an ideal in $\mathbb{K}[X_1, ..., X_n]$, consider the set of polynomials $Q \in \mathbb{K}[X_1, ..., X_n]$ such that, if one takes the localization at any maximal ideal $\mathfrak{M}$, the image of Q lies in the integral closure (in the local ring $\mathbb{K}[X_1, ..., X_n]_{\mathfrak{M}}$) of the ideal generated by the images of $P_1, ..., P_m$. This is an ideal $\overline{I} \subset \mathbb{K}[X_1, ..., X_n]$ which lies in between I and its radical. We will see in the following courses that analysis may recover some algebraic information relative to $\overline{I}$ (while geometry may only recover information on $\sqrt{I}$ or eventually about multiplicities attached to <u>isolated</u> components in the primary decomposition of I). For example, if $Q \in \overline{I}$, one can recover an explicit formula

$$Q^{\inf(n+1,m)} = \sum_{j=1}^{m} A_j P_j ,$$

with

$$\max_j \deg(A_j P_j) \leq \inf(n+1, m)(\deg Q + D^n) ,$$

where $D = \max \deg P_j$ (this is a result by M. Hickel in [Hi]). Clearly, effectiveness of Briançon-Skoda's theorem is governed by the Bézout estimates involved in geometric intersection theory.

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[7] W.D. Brownawell in [Brow1] was the first to realize Bézout geometric estimates govern the effective resolution of Hilbert's nullstellensatz ; the proof was clarified and settled in geometric terms by J. Kollár [Ko1], then in [Ko2] ; more clarification came from the work of L. Ein and R. Lazarsfeld [EinL] and M. Hickel [Hi] we quote here. Very recently, Z. Jelonek, using a fundamental result by O. Perron [Perr], gave a quite elementary proof of J. Kollár's result [Jel] ; nevertheless, the intertwining with residue theory and Briançon-Skoda's result is not yet completely understood (and remains crucial when working in an arithmetic setting instead of just a geometric one).

Chapter 4

Lesson 4 : Lelong-Poincaré and Green relations

4.1 Volume and degree of a projective algebraic set

In this section, our setting will be the projective setting $\mathbb{P}^n(\mathbb{C})$. On $\mathbb{P}^n(\mathbb{C})$, a $(1,1)$ closed form plays an important role, the form

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log[|z_0|^2 + \cdots + |z_n|^2] = dd^c \log \|Z\|^2\,,$$

where $Z := [z_0 : z_1 : \cdots : z_n]$; such a form is globally defined since

$$dd^c \log |f|^2 \equiv 0$$

when f is a non-vanishing holomophic function. Such a form ω is a positive form and an easy computation shows that

$$\int_{\mathbb{P}^n(\mathbb{C})} \omega^n = \int_{\mathbb{C}^n} (dd^c \log[1 + |\zeta_1|^2 + ... + |\zeta_n|^2])^n = 1\,.$$

(one can make easily the computation when $n = 1$ using spherical coordinates).

Let

$$V = \{Z \in \mathbb{P}^n(\mathbb{C})\,;\, P_1(Z) = \ldots = P_M(Z) = 0\}$$

be an algebraic set in $\mathbb{P}^n(\mathbb{C})$, defined as the zero set $V(P)$ of a collection of M homogeneous polynomials $(P_1, ..., P_m)$. One says that V has pure codimension equal to k $(1 \leq k \leq n+1)$ if and only if the Krull dimension of all isolated primes in the primary decomposition of $\mathfrak{I} = (P_1, ..., P_M)$ in $\mathbb{C}[X_0, ..., X_n]$ equals $n-k+1$ (the case $k = n+1$ means that $V(P)$ is empty).

If $\mathfrak{I}$ is a prime ideal in $\mathbb{C}[X_0, ..., X_n]$ generated by $(F_1, ..., F_\nu)$ (with Krull dimension $p+1 \in \{1, ..., n\}$), then the ideal generated by $F_1, ..., F_\nu$, together with all $n-p$ minors of the jacobian matrix $D(F_1, ..., F_\nu)/D(z_0, ..., z_n)$, defines a proper subset of $V(\mathfrak{I})$ (see [Ha1], chapter 1, section 5) ; this implies that, when V has pure codimension equal to $k \in \{1, ..., n\}$, then V is a smooth $2(n-k)$-real submanifold (*resp.* $(n-k)$-complex submanifold) of the real $2n$-manifold $\mathbb{P}^n(\mathbb{C})$ (*resp.* the n-complex manifold $\mathbb{P}^n(\mathbb{C})$) outside a singular subalgebraic variety $\operatorname{Sing} V$ which has dimension strictly smaller than $n-k$. Therefore, the integration on such V of the (k,k)-positive differential form ω^k makes sense and this provides the following important definition :

Definition 4.1 ((and Wirtinger's theorem, see [Stolz]) *The degree of an algebraic subset (of pure codimension $k \in \{1, ..., n\}$) of $\mathbb{P}^n(\mathbb{C})$ is defined as the volume*

$$\deg V(P) = \int_V \omega^{n-k}.$$

It is a positive integer which equals the number of intersection points between V and the projective linear subset

$$L_U = \{\langle U_1, Z\rangle = ... = \langle U_{n-k}, Z\rangle = 0\}$$

(where $\langle U_j, Z\rangle = u_{j0}z_0 + ... + u_{jn}z_n$, $[u_{j0} : ... : u_{jn}] \in \mathbb{P}^n(\mathbb{C})$, $j = 1, ..., n-k$) when $U_1, ..., U_{n-k}$ are generic.

Example. When $V = V(P)$, P being a homogeneous irreducible polynomial with degree D, then $\deg V(P) = D$.

4.2 Lelong number of a positive current

Let U be an open subset in $\mathbb{C}^n$ and $k \in \{1, ..., n\}$. A (k,k) current

$$T = \sum_{1\leq l_1<l_2<\cdots<l_k\leq n} \ \sum_{1\leq m_1<m_2<\cdots<m_k\leq n} T_{l,m} \bigwedge_{\lambda=1}^{k} dz_{l_\lambda} \wedge \bigwedge_{\mu=1}^{k} d\overline{z}_{m_\mu}$$

(where the $T_{l,m}$ are distributions in U) is *positive* if and only if the action of T on any $i^{(n-k)^2}\alpha\wedge\overline{\alpha}$, where α is a $(n-k,0)$- smooth form with compact support gives a positive number. Positivity implies that the distributions $i^{k^2}T_{l,m}$ are in fact positive measures.

Example. If V is a purely dimensional closed analytic set in U with codimension k (which means that V is locally defined as the zero set of holomorphic functions $f_1, ..., f_m$ such that all isolated primes in the primary decomposition of $(f_1, ..., f_m)$ in $\mathcal{O}_{z_0}$, $z_0 \in \{f_1 = ... = f_m = 0\}$ have all Krull dimension $n-k$), then the integration current on V, which associates to any $(n-k, n-k)$-smooth form with compact support

$$\int_V \varphi = \int_{\mathrm{Reg}\, V} \varphi$$

is a positive (k,k)-current. The reason why the integration on V makes sense is that it makes sense outside the singular points of V (which form a closed analytic subset of V with dimension strictly smaller than $n-k$) and that the restriction of this integration current on $U \setminus \mathrm{Sing}\,(V)$ is a positive (k,k)-current satisfying $\partial T = \overline{\partial} T = 0$; such a current as a unique extension to U as a (k,k)-current with the same properties (see for example [Lel], appendix).

If T is a (k,k)-positive d-closed current in U and a a point in the support of T, one can show that, for $r > 0$ small enough, the function

$$r \longmapsto \int_{\|z-a\|\leq r} T \wedge (dd^c \log|z-a|^2)^{n-k} = \frac{(n-k)!\,\sigma_T(B(a,r))}{r^{2(n-k)}\pi^{n-k}},$$

where $\sigma_T(B(a,r))$ denotes the positive measure

$$\sigma_T = \frac{1}{(n-k)!} T \wedge (\frac{i}{2}\partial\overline{\partial}\|z\|^2)^{n-k} \,,$$

is an increasing function of r ; its limit when r tends to zero is a positive real number which is called the *Lelong number* of the positive closed (k,k)-current T at the point a (see [Lel] or [De1], section 6).

Example. When T is the integration current on a closed analytic set V of pure codimension k, then the Lelong number of T at $a \in V$ is a positive integer which equals the number of sheets of the covering

$$\pi \ : V \longrightarrow \mathbb{C}^{n-k}$$

when π is a generic linear projection from $\mathbb{C}^n$ to $\mathbb{C}^{n-k}$ such that the restriction $\pi_{|V} \ : V \longrightarrow \mathbb{C}^{n-k}$ is a proper map in a neighborhood of a. For example, if $V := \{(z,w) \in \mathbb{C}^2 \,;\, z^2 - w^3 = 0\}$, the Lelong number of V at the origin equals 2. If a is a regular point of a closed analytic subset V with pure codimension k, then the Lelong number of the integration current on V at a equals 1.

4.3 Lelong-Poincaré, Monge-Ampère equations

Let U be an open connected set in $\mathbb{C}^n$. When f is a holomorphic function in U, then $dd^c \log|f|^2$ defines a positive $(1,1)$-current which is d-closed. The Lelong number at a point $a \in V(f) = \{f = 0\}$ equals the multiplicity of a as a zero of f, that is the valuation of

$$u \longmapsto f(a+u)$$

at the origin. If $V(f)$ is irreducible (as a closed analytic set in U), then the Lelong number of $dd^c \log|f|^2$ remains constant (equal to $\mu \in \mathbb{N}$) on $\operatorname{Reg} V(f)$ and the formula

$$dd^c \log|f|^2 = \mu[V(f)] \,, \tag{4.1}$$

where $[V(f)]$ is the integration current on $V(f)$, is (in the particular case of principal ideals, we will extend it later to quasi-regular sequences) the *Lelong-Poincaré* equation.

It also has a global version. If $P = P_1^{\mu_1}...P_M^{\mu_M}$ is a homogeneous polynomial in $n+1$ variables with irreducible factors $P_1,...,P_M$, then one has then the formula in $\mathbb{P}^n(\mathbb{C})$:

$$dd^c\Big[-\log \frac{|P(Z)|^2}{\|Z\|^{2\deg P}}\Big] + \sum_{j=1}^{M} \mu_j [V(P_j)] = \deg P \times \omega(Z) \,, \tag{4.2}$$

which appears as the *global version of Lelong-Poincaré equation.* The current

$$\sum_{j=1}^{M} \mu_j [V(P_j)]$$

can be interpreted as the integration current associated to the effective geometric cycle

$$C := \sum_{j=1}^{M} \mu_j \, C_j \,,$$

where C_j is the irreducible algebraic set $\{P_j = 0\}$.

When $f_1, ..., f_k$ define a quasi regular sequence in the ring $H(U)$ of holomorphic functions in n variables in a connected open subset $U \subset \mathbb{C}^n$, then one can show that for any smooth test form φ of type $(n-k, n-k)$ with compact support in U,

$$\lim_{\tau \longrightarrow 0_+} (dd^c \log[|f_1|^2 + \ldots + |f_k|^2 + \tau])^k = \sum_{j=1}^{M} \mu_j \int_{V_j} \varphi \,,$$

where $V_1, ..., V_M$ are the closed irreducible analytic sets which correspond to the (necesserally) isolated primes in the primary decomposition of $(f_1, ..., f_k)$, μ_j being defined as the intersection multiplicity $\mu(f, u)$ of the regular sequence

$$\Big(f_1, ..., f_k, \sum_{l=1}^{n} u_{1,l}(z_l - a_l), ..., \sum_{l=1}^{n} u_{n-k,l}(z_l - a_l)\Big)$$

(for $u_{j,l}$, $j = 1, ..., n-k$, $l = 1, ..., n$, generic) in the local ring $\mathcal{O}_a$, $a = (a_1, ..., a_n)$ being an arbitrary point in $\operatorname{Reg} V_j$. Such a result can be summarized as

$$(dd^c \log[\|f\|^2])^k = [V(f_1, ..., f_k)]_{\text{mult}} \,, \tag{4.3}$$

where $[V(f_1, ..., f_k)]_{\text{mult}}$ denotes the integration current (with multiplicities being taken into account) on the effective cycle

$$\sum_{j=1}^{M} \mu_j V_j$$

attached to the quasi-regular sequence $(f_1, ..., f_k)$; in fact here, one can write

$$(f_1, ..., f_k) = \bigcap_{j=1}^{M} \mathfrak{Q}_j \,,$$

where $\mathfrak{Q}_j$ is $\mathfrak{P}_j$-primary, $V_j = V(\mathfrak{P}_j)$, and μ_j is realized as

$$\mu_j = \inf\{\mu \,;\, \mathfrak{P}_j^{\mu} \subset \mathfrak{Q}_j\} \,.$$

Formula (4.3) is the *Monge-Ampère formula for quasi-regular sequences*, which fits with *Lelong-Poincaré formula* (4.1) in the codimension one case.

Let $P_1, ..., P_k$ be k homogeneous polynomials in $n+1$ variables, with respective degrees $D_1, ..., D_k$, which define a regular sequence in the homogeneous polynomial ring $\mathbb{C}[X_0, ..., X_n]$; then, if $\widehat{D}_j = \prod_{l \neq j} D_l$, $j = 1, ..., k$ and $D := D_1 \cdots D_k$, one has, from Monge-Ampère equation (4.3), the formula

$$\begin{aligned} dd^c\Big[-\log\Big(\frac{\sum_{j=1}^{k} |P_j(Z)|^{2\widehat{D}_j}}{\|Z\|^{2D}}\Big) \times \Big(\sum_{l=0}^{k-1} D^{-l} \omega^{k-1-l} \wedge \Big(dd^c\Big[\log \sum_{j=1}^{k} |P_j|^{2\widehat{D}_j}\Big]\Big)^l\Big)(Z)\Big] \\ + [V(P_1, ..., P_k)]_{\text{mult}} = D\omega^k(Z) = \Big(\deg[V(P_1, ..., P_k)]_{\text{mult}}\Big)\, \omega^k(Z) \,, \end{aligned} \tag{4.4}$$

which appears as a generalization of (4.2) for regular sequences in $\mathbb{C}[X_0, ..., X_n]$.

There is also an extension of Lelong-Poincaré formula to the case of quasi-regular sequences : namely, if $f_1, ..., f_k$ are holomorphic functions in n variables defining a quasi-regular sequence in an open connected set in $\mathbb{C}^n$, one has

$$\lim_{\substack{\tau_j \longrightarrow 0_+ \\ j=1,...,k}} \left(\bigwedge_{j=1}^{k} dd^c \log[|f_j|^2 + \tau_j] \right) = [V(f_1, ..., f_k)]_{\mathrm{mult}}$$

in the (weak) sense of currents.

4.4 The notion of height for an arithmetic cycle

Let $P_1, ..., P_M$ be M homogeneous polynomials with integer coefficients defining a projective algebraic set with pure codimension k in $\mathbb{P}^n(\mathbb{C})$. Besides the geometric notion of degree, which has been introduced in the previous section, one would like to "quantify" the arithmetic complexity of the arithmetic cycle $[(P_1, ..., P_M)]^{\mathrm{arith}}_{\mathrm{mult}}$ in the projective $(n+1)$-dimensional scheme Proj $(\mathbb{Z}[X_0, ..., X_n])$.

In order to do that, we introduce a linear projective subspace of codimension $n+1-k$, defined by the equations

$$\langle U_j \, , \, Z \rangle = \sum_{l=0}^{n} u_{j,l} z_j = 0 \, , \; j = 0, ..., n-k \, ,$$

where the coefficients $u_{j,l}$, $j = 0, ..., n+1-k$, $l = 0, ..., n$, are generic in $\mathbb{Z}$. The generator a_u (in $\mathbb{N}^*$) of the ideal of all $a \in \mathbb{Z}$ such that one has

$$a.(X_0, ..., X_n)^q \subset \bigoplus_{j=1}^{M} \mathbb{Z}[X_0, ..., X_n] \, P_j + \bigoplus_{j=0}^{n+1-k} \mathbb{Z}[X_0, ..., X_n] \, \langle U_j \, , \, Z \rangle$$

for some $q \in \mathbb{N}^*$ defines a zero arithmetic cycle $[a_u]$; if

$$a_u = \prod_{p \ \mathrm{prime}} p^{v_{a_u}(p)} \, ,$$

the *logarithmic height* of the zero cycle

$$[a_u] := \sum_{p \ \mathrm{prime}} v_{a_u}(p).\{p\} \tag{4.5}$$

is defined as

$$h([a_u]) := \sum_{p} v_{a_u}(p) \, \log \, p \, . \tag{4.6}$$

Unfortunately, this height $h([a_u])$ <u>is not independent</u> of $U_1, ..., U_{n-k+1}$, even if the coefficients u_{jl} are generic, so that $h([a_u])$ cannot be used as an intrinsic measure of the arithmetic height of the arithmetic cycle $[(P_1, ..., P_M)]^{\mathrm{arith}}_{\mathrm{mult}}$ (which makes the difference with the definition of the degree of the geometric cycle $[(P_1, ..., P_M)]_{\mathrm{mult}}$).

In order to correct such a definition, we need to recall Jensen's formula in one complex variable, which asserts that, if

$$P(X) = a_0 \prod_{j=1}^{d} (X - a_j) \, ,$$

where all a_j are such that $|a_j| > 1$, then

$$\log |a_0| + \sum_{j=1}^{d} \log |\alpha_j| - \frac{1}{2}\Big(\frac{1}{2i\pi}\int_0^1 \log |P(e^{i\theta})|^2 \, d\theta\Big) = 0 \tag{4.7}$$

(one can check easily the formula when $d = 1$, as a consequence of Poisson formula, and then just add) ; such a formula can be interpreted as follows : the "arithmetic" contribution to the loarithmic height

$$\log |a_0| + \sum_{j=1}^{d} \log |\alpha_j|$$

needs to be "balanced" by the "analytic" contribution

$$-\frac{1}{2}\Big(\frac{1}{2i\pi}\int_0^1 \log |P(e^{i\theta})|^2 \, d\theta\Big) ,$$

which involves (introducing homogeneous coordinates)

$$-\frac{1}{2}\log \frac{|{}^hP(z_0, z_1)|^2}{\|Z\|^{2d}} = \frac{1}{2} G_{[V(P)]_{\mathrm{mult}}} ,$$

where $G_{[V(P)]_{\mathrm{mult}}}$ is a solution of the so-called *Green equation* :

$$dd^c G_{[V(P)]_{\mathrm{mult}}} + [V(P)]_{\mathrm{mult}} = \deg P \times \omega \, .$$

The "balance" between the arithmetic and the analytic contributions which expresses Jensen's formula (4.7) can also be viewed as an *avatar* of the well known *product formula* : when a/b is a rational number, one has

$$\prod_{p \ \mathrm{prime}} |a/b|_p = \frac{1}{|a/b|_\infty}$$

if $|a/b|_p$ denotes the p-adic non archimediaa absolute value of a/b and $|a/b|_\infty$ the archimedian one ; this "balance" phenomenm between arithmetic and algebra (which involves if one pretends for example to define an intrinsic notion of arithmetic logarithmic height in accordance with the geometric notion of degree) a necessary complementarity between arithmetic objects (arithmetic cycles) and analytic ones (Green currents attached to the correspondibg geometric cycles).

The following definition was proposed in [Falt] and [GS] :

Definition 4.2 *Let Z be an arithmetic cycle (with pure dimension $n + 1 - k$ in* $\mathrm{Proj}\,(\mathbb{Z}[X_0, ..., X_n])$*) and $Z_{\mathbb{C}}$ the corresponding purely $(n - k)$ dimensional complex geometric cycle in $\mathbb{P}^n(\mathbb{C})$. A normalized Green current attached to Z is a $(k-1, k-1)$-current on $\mathbb{P}^n(\mathbb{C})$ with singular support the support $|Z_{\mathbb{C}}|$ of the geometric cycle Z_C, with integrable singularities of logarithmic type, such that*

$$dd^c G + [Z_{\mathbb{C}}] = \deg Z_{\mathbb{C}} \times \omega^k \, .$$

Examples. Formulaes (4.2) and (4.4) provide Green currents when the geometric cycle $Z_{\mathbb{C}}$ is defined as the cycle attached to a quasi-regular sequence of homogeneous polynomials in $\mathbb{Z}[X_0, ..., X_n]$.

The fundamental idea which supports recent developments of arithmetic intersection theory (following [GS] and [BGS]) is based on the pairing (Z, G_Z) between an arithmetic cycle Z and such a normalized Green current attached to $Z_{\mathbb{C}}$. Formally, intersecting cycles amounts to formally define the product of two such pairs

$$(Z_1, G_{Z_1}) \bullet (Z_2, G_{Z_2})$$

when Z_1 and Z_2 are two arithmetic cycles whose supports intersect properly (that is the codimension of $|Z_1| \cap |Z_2|$ equals the sum of the codimensions of $|Z_1|$ and $|Z_2|$). Formally, one defines the intersection product as

$$(Z_1, G_{Z_1}) \bullet (Z_2, G_{Z_2}) = \Big(Z_1 \bullet Z_2 \ , \ [Z_{2,\mathbb{C}}] \wedge G_{Z_1} + \deg Z_{1,\mathbb{C}} \ \omega^{\operatorname{codim} Z_{1,\mathbb{C}}} \wedge G_{Z_2} \Big) \, .$$

One can check that

$$dd^c \Big[[Z_{2,\mathbb{C}}] \wedge G_{Z_1} + \deg Z_{1,\mathbb{C}} \ \omega^{\operatorname{codim} Z_{2,\mathbb{C}}} \Big] = [Z_{2,\mathbb{C}}] \wedge \Big(\deg Z_{1,\mathbb{C}} \ \omega^{\operatorname{codim} Z_{1,\mathbb{C}}} - [Z_{1,\mathbb{C}}] \Big)$$
$$+ \deg Z_{1,\mathbb{C}} \ \omega^{\operatorname{codim} Z_{1,\mathbb{C}}} \wedge \Big(\deg Z_{2,\mathbb{C}} \ \omega^{\operatorname{codim} Z_{2,\mathbb{C}}} - [Z_{2,\mathbb{C}}] \Big) \, ,$$

which fits with the fact that

$$\deg(Z_{1,\mathbb{C}} \bullet Z_{2,\mathbb{C}}) = \deg Z_{1,\mathbb{C}} \times \deg Z_{2,\mathbb{C}}$$

by Bézout theorem (the intersection of the geometric cycles being a proper one) and the codimension of the support of this intersection geometric cycle is precisely the sum of the codimensions of $Z_{1,\mathbb{C}}$ and $Z_{2,\mathbb{C}}$ (for the same reason).

The main technical difficulty here is to define the product of currents $[Z_{2,\mathbb{C}}] \wedge G_{Z_1}$, which can de done thanks to the fact that the supports of $Z_{\mathbb{C},1}$ and $Z_{\mathbb{C},2}$ intersect properly and the singularities of G_{Z_1} are of logarithmic type (of course, one needs to use here Hironaka's theorem about resolution of singularities in the characteristic zero setting [Hir]) ; there are also alternate arguments based on the concept of holomicity and precise description of the wave front sets of the two currents $[Z_{2,\mathbb{C}}]$ and G_{Z_1} which allow the multiplication of such currents thanks to the idea of analytic continuation, in the spirit of Atiyah [Ati], Gelfand [GelfS] and J. Bernstein [Bern], see for example [Bj3].

Following Jensen's formula and the "balance" it imposes between the arithmetic and the analytic contributions, the natural definition of the logarithmic height of a arithmetic cycle Z of codimension k is, when $U_0, ..., U_{n+1-k}$ are generic in $\mathbb{Z}^{n+1}$,

$$h(Z) = h([a_u]) + \frac{1}{2} \int_{\mathbb{P}^n(\mathbb{C})} G_{Z,u} \, ; \tag{4.8}$$

here the arithmetic 0-cycle $[a_u]$ and its logarithmic height $h([a_u])$ have been defined in (4.5, 4.6) and $G_{Z,u}$ is a (n,n)-Green current for such a 0-cycle $[a_u]$ which is obtained as

$$[\Pi_u] \wedge G_Z + \deg Z_{\mathbb{C}} \ \omega^k \wedge L_u^{(k)} \, ,$$

where Π_u is the $(k-1)$-dimensional projective plane

$$\Pi_u := \{Z \in \mathbb{P}^n(\mathbb{C})\,;\, \langle U_0, Z\rangle = \cdots = \langle U_{n-k}, Z\rangle = 0\}\,,$$

L_u is a $(n-k, n-k)$-current solution of

$$dd^c L_u + [\Pi_u] = \omega^{n+1-k}\,,$$

and G_Z a Green current solution of

$$dd^c G_Z + [Z_{\mathbb{C}}] = \deg Z_{\mathbb{C}} \times \omega^k\,.$$

The construction of $L_u^{(k)}$ follows from example from (4.4) (since the $\langle U_j, Z\rangle$, $j = 0, ..., n-k$ define a regular sequence for generic U_j) ; the computation of

$$\int_{\mathbb{P}^n(\mathbb{C})} \omega^k \wedge L_u^{(k)} = \sum_{p=k}^{n}\sum_{j=1}^{p}\frac{1}{j}$$

can be found for example in [Stol]. So we get, from (4.8),

$$h(Z) = h([a_u]) + \frac{1}{2}\int_{\Pi_u} G_Z + \frac{\deg Z_{\mathbb{C}}}{2}\sum_{p=k}^{n}\sum_{j=1}^{p}\frac{1}{j}$$

as the natural definition of the logarithmic height ; this definition is independent of $U_0, ..., U_{n-k}$ provided the coefficients $u_{j,l}$, $j = 0, ..., n-k$, $l = 0, ..., n$, are generic.

We just mention here the basis result about arithmetic intersection theory, namely the arithmetic Bézout theorem, which can be stated as follows :

Theorem 4.1 [BGS] *Let Z_1 and Z_2 two arithmetic cycles which intersect properly in* Proj $\mathbb{Z}[X_0, ..., X_n]$. *Then*

$$\begin{aligned} h(Z_1 \bullet Z_2) \leq \deg Z_{1,\mathbb{C}} \times h(Z_2) + \deg Z_{2,\mathbb{C}} \times h(Z_1) \\ + \kappa(\operatorname{codim} Z_1, \operatorname{codim} Z_2) \times (\deg Z_{1,\mathbb{C}}\, \deg Z_{2,\mathbb{C}})\,. \end{aligned}$$

Sharp height estimates for the arithmetic nullstellentatz have been obtained in [BY1], [BY2] and finally [KPR] (where the optimal bounds are found). The results which are obtained show that the arithmetic Nullstellesatz is governed by arithmetic Bézout theorem, which means that, even from the arithmetic point of view, Hilbert's Nullstellensatz remains a geometric problem ; basically, if $P_1, ..., P_M$ are M polynomials in n variables with coefficients in $\mathbb{Z}$ and without common zeroes in $\mathbb{C}^n$, then one can write a Bézout identity

$$a = \sum_{j=1}^{M} A_j(X) P_j(X)\,,$$

where $A_1, ..., A_M$ are in $\mathbb{Z}[X_1, ..., X_n]$, $a \in \mathbb{Z}^*$ and

$$\max(\log|a|, \log|\mathrm{coeff.}\,(A_j)|) \leq \kappa(n) h D^{n+1}\,,$$

where $D = \max(\deg P_j)$ and $h = \max(\log|\mathrm{coeff.}\,(P_j)|)$.

Multivariate residue calculus, which plays a role in the "splitting" of Lelong-Poincaré formula, provides a tool (based on the use of Trace formula, Transformation Law, and Briançon-Skoda's theorem, presented in this course) in order to make such Bézout formulaes explicit and help to a better understanding of arithmetic *division theory* besides *arithmetic intersection theory.*

Unfortunately, we have to stop at this point this introductive course. It is clear that, like in multivariate residue theory, algebraic (or arithmetic) and analytic tools have to complement each other in *the arithmetic intersection theory* developed from Arakelov's original ideas (see [L]) for which we tried to sketch a brief presentation here.