Construction of Monotone Hamiltonians for First Order HJ Equations on Unstructured Meshes

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Overview

- What is the problem ?
- Viscosity solutions and boundary conditions
- Numerical approximation

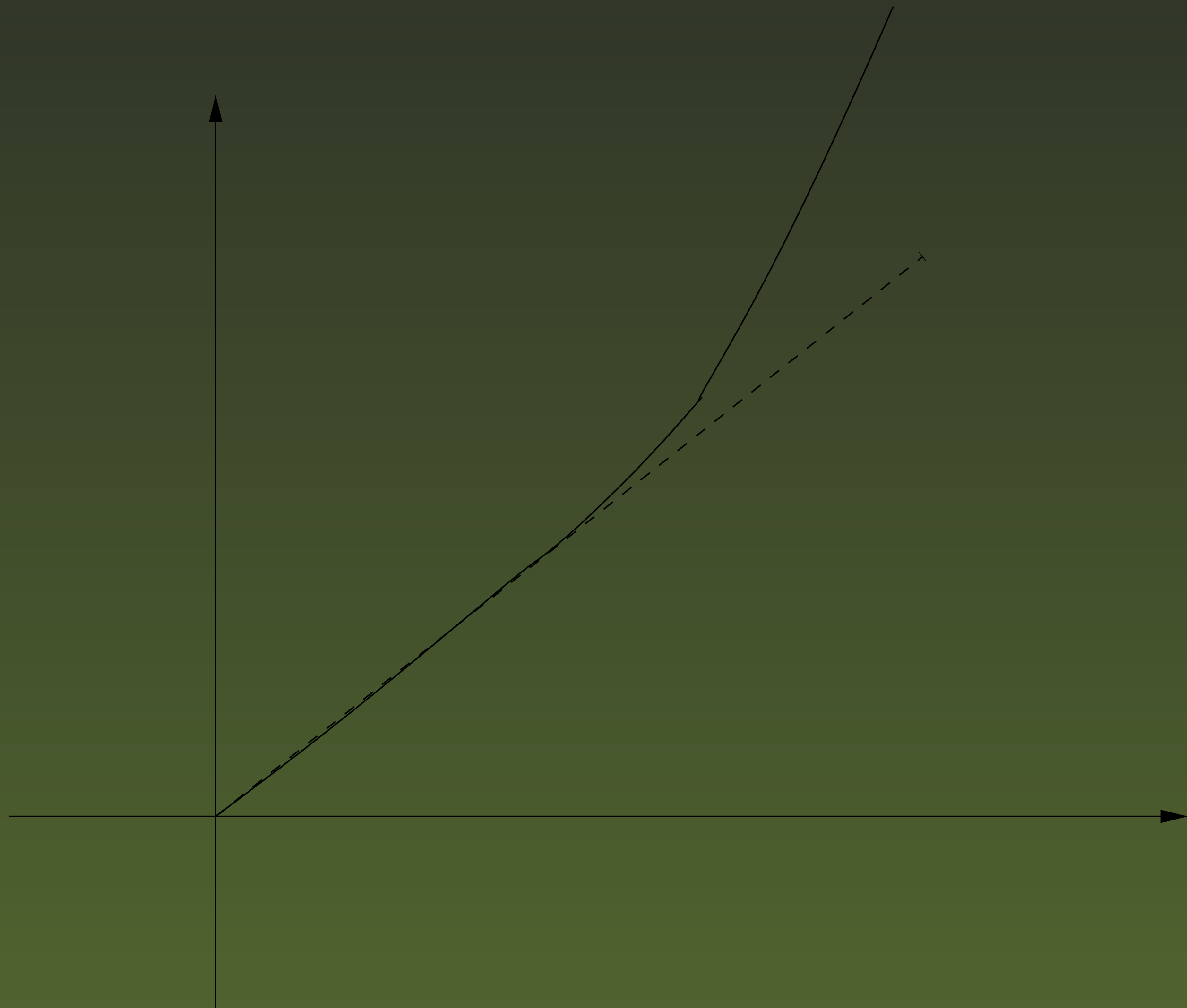
The problem

Take

$$\begin{aligned} |u'| - 1 &= \varepsilon u_{xx} && x \in [0,1], \\ u(0) &= 0 && u(1) = 2 > 1 \end{aligned}$$

Solution

$$u_\varepsilon(x) = x + \frac{\exp\left(\frac{x-1}{\varepsilon}\right) - \exp\left(-\frac{1}{\varepsilon}\right)}{1 - \exp\left(-\frac{1}{\varepsilon}\right)}$$



Boundary condition

$$\begin{aligned} H(x, u(x), \nabla u(x)) &= 0 \quad x \in \Omega \\ F(x, u(x), \nabla u(x) &= 0 \quad\ \ x \in \partial\Omega \end{aligned}$$

Define

$$G^*(x, t, p) = \begin{cases} H(x, t, p) & \text{if } x \in \Omega \\ \max(F(x, t, p), H(x, t, p)) & \text{if } x \in \partial\Omega \end{cases}$$

$$G_*(x, t, p) = \begin{cases} H(x, t, p) & \text{if } x \in \Omega \\ \min(F(x, t, p), H(x, t, p)) & \text{if } x \in \partial\Omega \end{cases}$$

Viscosity sub-solution

u super semi–continuous function (semi–continue supérieurement)

$\forall \phi \in C^2(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local maximum of $u - \phi$, then

$$G_*(x_0, u_(x_0), \nabla\phi(x_0)) \leq 0.$$

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$\forall \phi \in C^2(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local minimum of $u - \phi$, then

$$G^*(x_0, u_(x_0), \nabla\phi(x_0)) \geq 0.$$

Dirichlet conditions

$$G_*(x,t,p) = \begin{cases} H(x,t,p) & \text{if } x \in \Omega \\ \max(H(x,t,p), u(x) - g(x)) & \text{if } x \in \partial\Omega \end{cases}$$

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Numerical approximation

$$\begin{array}{ll} H(x,u(x),\nabla u(x)) = 0 & x \in \Omega \\ F(x,u(x),\nabla u(x) = 0 & x \in \partial\Omega \end{array}$$

$F(x,t,p) = t - g(x)$ (Dirichlet) or

$F(x,t,p) = p \cdot \vec{n} - g(x)$ Neuman

Numerics

$$\mathcal{S}(\rho, x_i, u(x_i), u) = \begin{cases} \mathcal{S}_H(x_i, u(x_i), \{u_j; j \in \mathcal{V}_i\}) = 0 \\ x_i \in \Omega \\ \max(\mathcal{S}_{H_b}(x_i, u(x_i), \{u_j; j \in \mathcal{V}_i\}, \\ \mathcal{S}_F(x_i, u(x_i), \{u_j; j \in \mathcal{V}_i\}) = 0 \\ x_i \in \partial\Omega \end{cases}$$

Under which conditions, does the scheme converge to "the" solution of the problem.

General convergence result

Barles and Souganidis result for $S(\rho, x, t, u)$

1. Monotonicity: if $u \geq v$, for all $\rho \geq 0$, $x \in \Omega$, $t \in \mathbb{R}$ and $u, v \in L^\infty(\Omega)$, then $S(\rho, x, t, u) \leq S(\rho, x, t, v)$.
2. Stability: for all $\rho > 0$, there exists a solution $u^\rho \in L^\infty(\Omega)$ to (1) with a bound independant of ρ,
3. Consistency: for all $x \in \Omega$ and $\phi \in C_b^\infty(\Omega)$,

$$\limsup_{\rho\to 0, y\to x, \xi\to 0} S(\rho, y, \phi(y) + \xi, \phi + \xi) \leq G_*(x, \phi(x), D\phi(x))$$

$$\liminf_{\rho\to 0, y\to x, \xi\to 0} S(\rho, y, \phi(y) + \xi, \phi + \xi) \geq G^*(x, \phi(x), D\phi(x))$$

4. Strong uniqueness principle: if $u \in L^\infty(\Omega)$ is a subsolution and $v \in L^\infty(\Omega)$ is a supersolution, then $u \leq v$ on Ω.

Barles and Souganidis

Theorem 0.1 (Barles, Souganidis). *Assuming the monotonicity, the consistency, the stability and the strong uniqueness properties, then, when $\rho \to 0$, the solution u^ρ of converges locally uniformly to the unique continuous viscosity solution of $H(x, u(x), \nabla u) = 0$.*

Application

Assume that

1. $H_b \leq H$,
2. S_H, S_{H_b} and S_F are monotone and stable,
3. for all $\phi \in C_b^\infty(\Omega)$, we have, For any $x \in \Omega$ or in a neighborhodd of $\partial\Omega$

$$\lim_{\rho \to 0, y \to x, \xi \to 0} S_H(\rho, y, \varphi(y) + \xi, \varphi + \xi) = H(x, \varphi(x), D\varphi(x))$$

$$\lim_{\rho \to 0, y \to x, \xi \to 0} S_{H_b}(\rho, y, \varphi(y) + \xi, \varphi + \xi) = H_b(x, \varphi(x), D\varphi(x))$$

$$\lim_{\rho \to 0, y \to x, \xi \to 0} S_F(\rho, y, \varphi(y) + \xi, \varphi + \xi) = F(x, \varphi(x), D\varphi(x))$$

4. The equation (**??**) has a uniqueness principle,

Application

Then the family u^ρ defined by (**??**) converges locally uniformly to the solution of (**??**) in Ω. We have the same result for (**??**) provided the condition 1 is replaced by $H \leq H_b$.

Visual proof

Convex Hamiltonians + Dirichlet conditions

$$H(x,u,p) = \sup_{v\in V}\{-b(x,v)\,.\,p + \lambda u - f(x,v)\}$$

+ Dirichlet

$$0 = u(x) - \inf_{v(.)}\Big[\int_0^{\min(T,\tau)} f(y_x(t),v(t))e^{-\lambda t}dt$$

$$+ 1_{\{T<\tau\}}u(y_x(T))e^{-\lambda T} + 1_{\{T\geq\tau\}}\varphi(y_x(\tau))e^{-\lambda\tau}\Big]$$

Examples

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- etc

Numerical examples

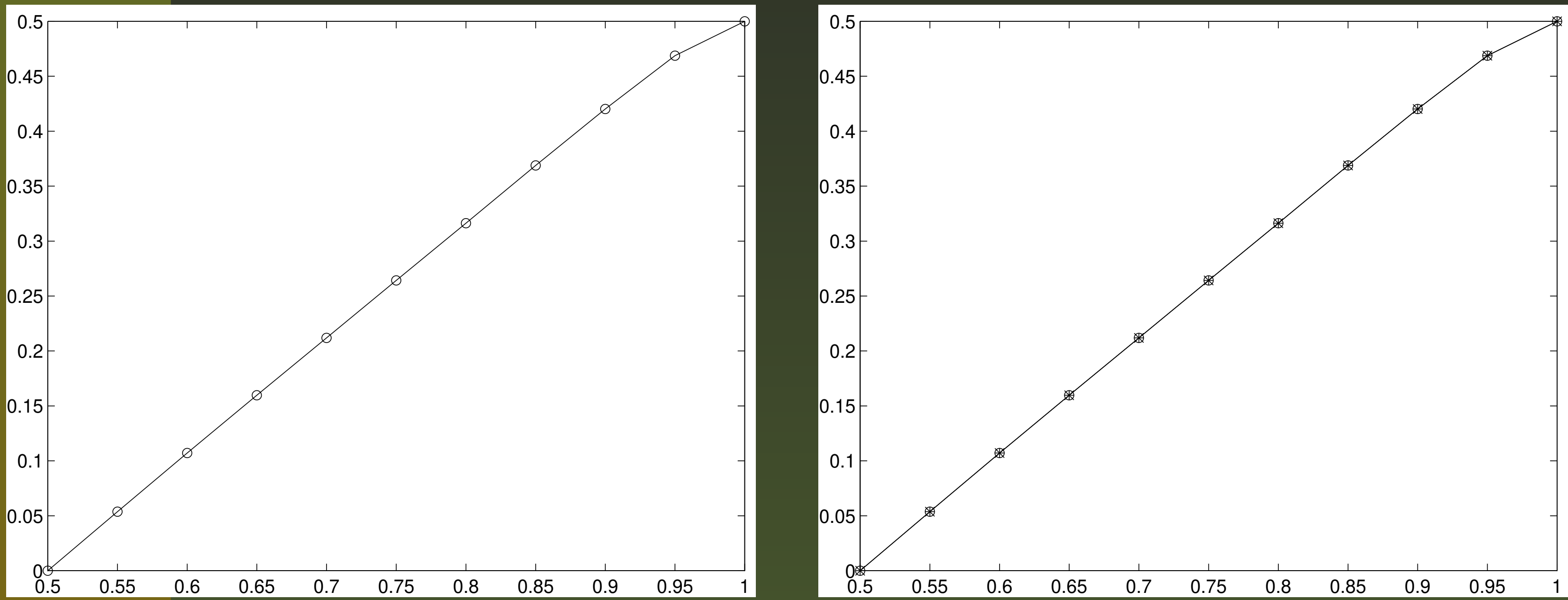
$$\begin{cases} ||\nabla u|| = 1 & \text{in } \Omega \\ u = 0 & \text{for } ||x|| = 0.5 \\ u = C & \text{for } ||x|| = 1. \end{cases}$$

The constant C takes the values displayed in Table ??

Case	1	2	3	4
C	0.5	10	0.25	-11

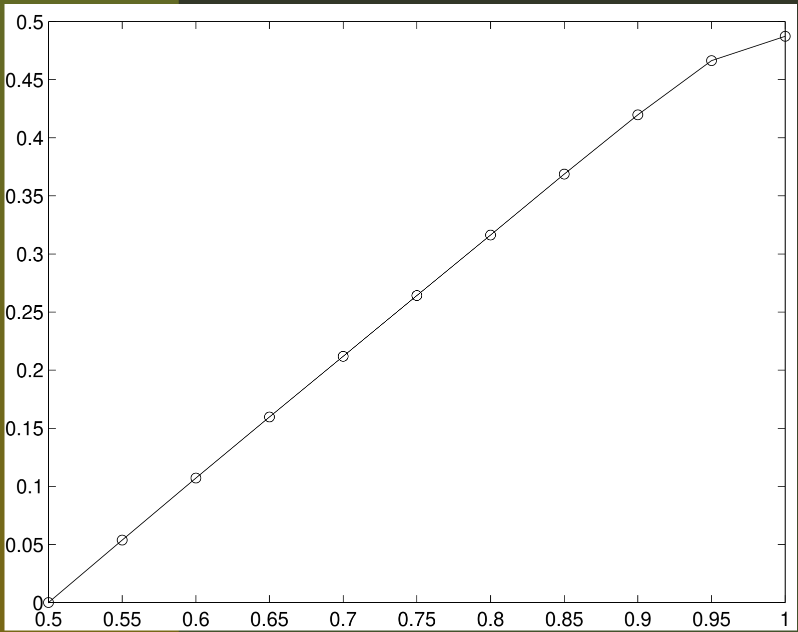
Table 0: Boundary condition for the Dirichlet b.c.s

Examples

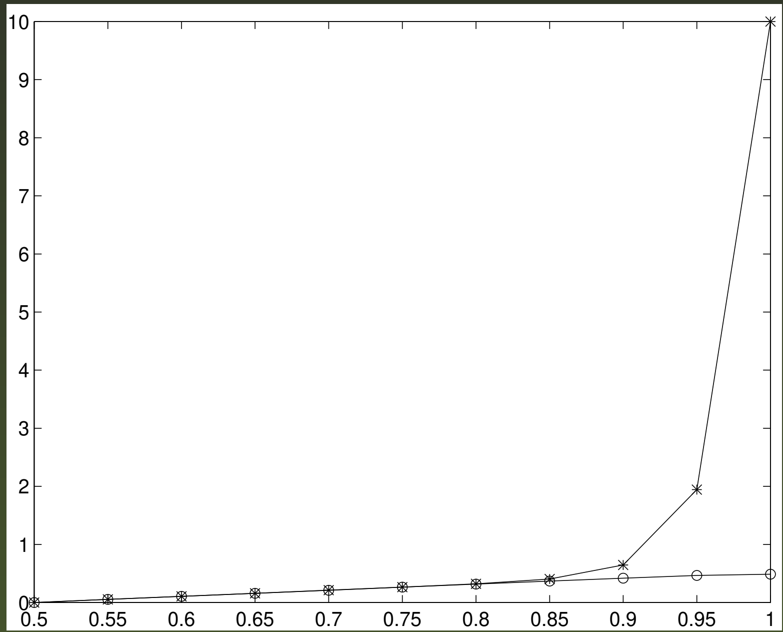


(a) (b)

Figure 0: Dirichlet, case A, (a) weak conditions, (b) comparaison weak ($\circ$) strong ($*$)

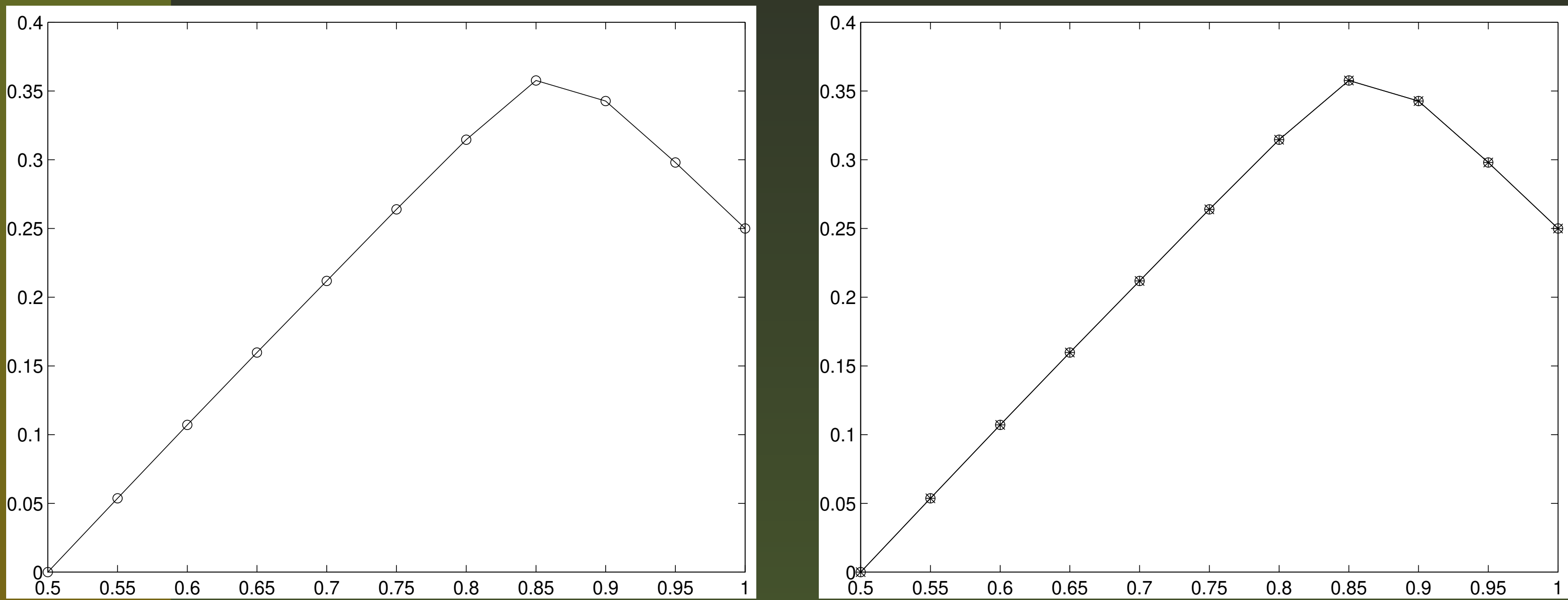


(a)



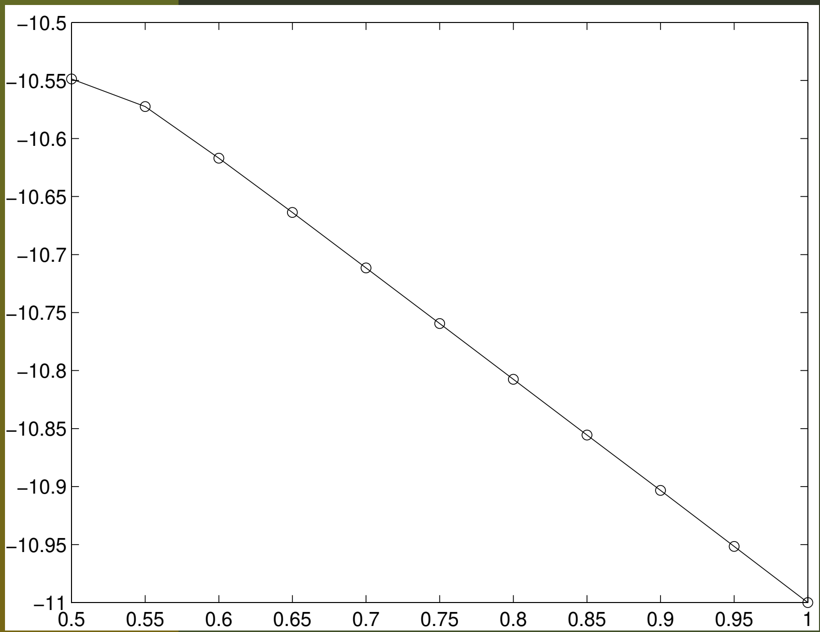
(b)

Figure 0: Dirichlet, case B, (a) weak conditions, (b) comparaison weak ($\circ$) strong ($*$)

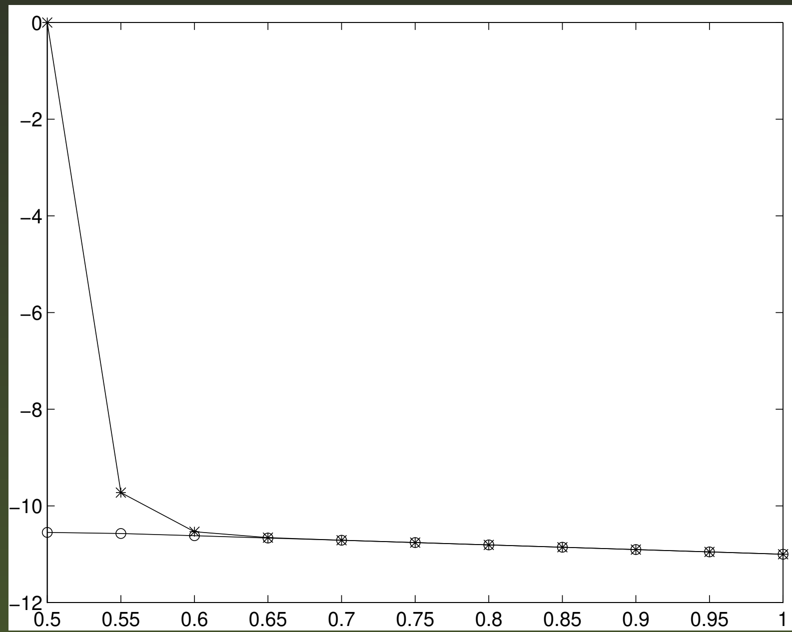


(a) (b)

Figure 0: Dirichlet, case C, (a) weak conditions, (b) comparaison weak ($\circ$) strong ($*$)



(a)



(b)

Figure 0: Dirichlet, case D, (a) weak conditions, (b) comparaison weak ($\circ$) strong ($*$)

Improved convergence result

For convex Hamiltonians, take the Godunov boundary Hamitonian.

No boundary layers : control of ∇u.

Soner Bcs

Dirichlet Bcs

Non convex Hamiltonians

Situation much les clear