Construction of accurate Hamiltonians for First Order HJ Equations on Unstructured Meshes

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Scheme

$$u_t + H(x, u, \nabla u) = 0 \qquad u(x, 0) = u_0(x)$$

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$\dfrac{du}{dt} + \mathcal{H}(x_i, u(x_i), \{u_j\}_{j \in \mathcal{V}_i})$ and Runge Kutta type integration

Overview

1. ENO type scheme (WENO)
2. Tuning of artificial dissipation
3. Schemes for steady problems

ENO – WENO type

- $\dfrac{du}{dt} + \mathcal{H}(x_i, u(x_i), \{u_j\}_{j\in\mathcal{V}_i}) \to$ Runge Kutta (Shu–Osher for preservation of stability properties of $\mathcal{H}$)

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- Take first order scheme

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\begin{aligned}
\mathcal{H}_i =& \mathcal{H}(x_i, u(x_i), \{u_j\}_{j\in\mathcal{V}_i}) \\
\equiv& \mathcal{H}(x_i, u(x_i), \nabla u_{T_1}, \cdots, \nabla u_{T_k})
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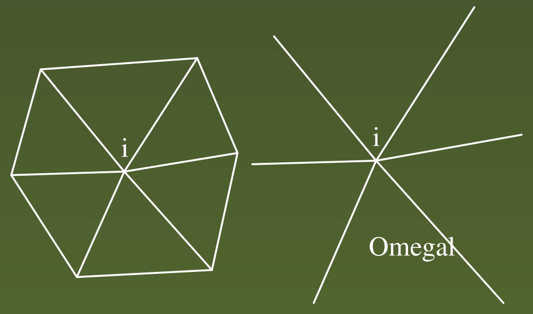


Algorithm

- need to define a family of stencils,
- compute the one that “minimise” oscillation

High order approximation of ∇u_T

- start from node values (Lagrange)
- k-th order approximation of ∇u_T : polynomial of degree $k+1$,
- need $(k+1) \times \big((k+1)+1\big)/2$ points
- construction of stencils with $(k+1) \times \big((k+1)+1\big)/2$
- need M_i inside $(k+1) \times \big((k+1)+1\big)/2 - 1$ additional points

Construction of potential stencils

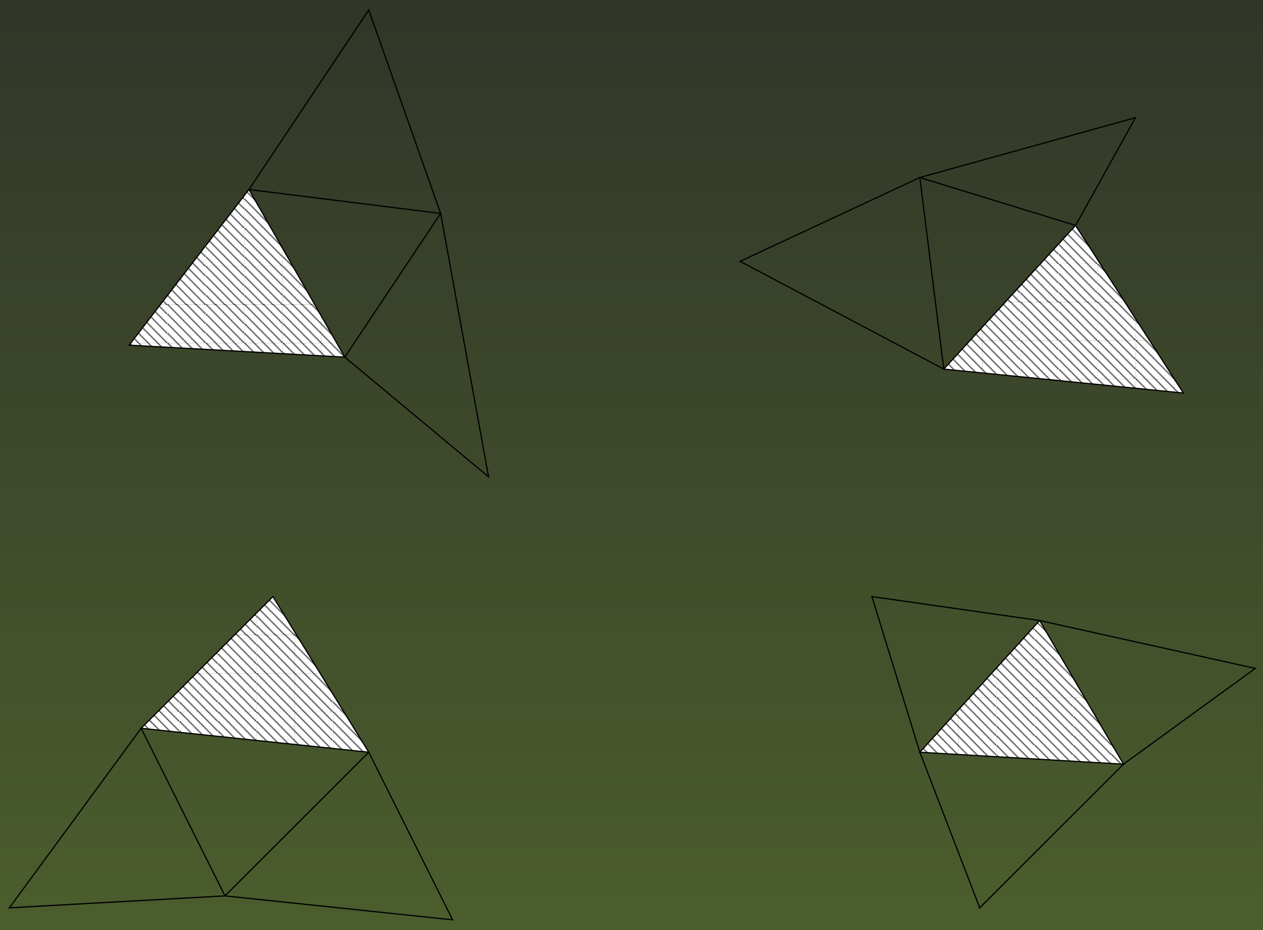
- All directions needed,
- As few stencils as possible,
- What is oscillation

Hierarchical algorithm

Step 1
- Consider all triangles that surround M_i (about 6 for regular meshes): ∇u_T
- Choose T_{min} that minimize $||\nabla u_T||$

Step 2 : T_{min} known construct 3 more points

Hierarchical algorithm, 2



Centered

Facts and justifications

- Compute stencils : not always possible (think of 3 aligned points)
- In practice always a solution
- P P^k interpolation for f smooth

$$\max_{x \in \mathcal{K}} \left|\left| f^{(r)} - P^{(r)} \right|\right|_\infty \leq C M_{p+1} \frac{h^{p+1}}{\rho^r}$$

- f not smooth in the stencil of P

$$P = \sum_{l=0}^{r} \sum_{i+j=r} a_{ij}(x - x_0)^i (y - y_0)^j$$

$$\sum_{i+j=r} |a_{ij}| \geq C \times \text{jump of } f^{(r)} \times h^{p-r}$$

Numerical results

$$H(p, x, y) = ||p|| - n(x, y)$$

$$n(x,y) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{4}\left(1 - \cos 2\pi d\right) & \text{if } d^2 = \left(\frac{x}{0.8}\right)^2 + \left(\frac{x}{0.2}\right)^2 \leq 1 \\ 1 & \text{else} \end{cases}$$

Problems

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- Non compact stencil,
- not obvious to increase order (even for WENO)
- Not obvious for 3D, computational not very efficient

Construction of Compact schemes, problems

- “ Discontinuous” Galerkin–like schemes,
- How to have “provable” convergent schemes (no Lax Wendroff like theorem) ? i.e How to have automatic “entropy” conditions
- Do this directly in the scheme, i.e. start from a viscosity solution framework. Different from Shu et al.

Solution 1

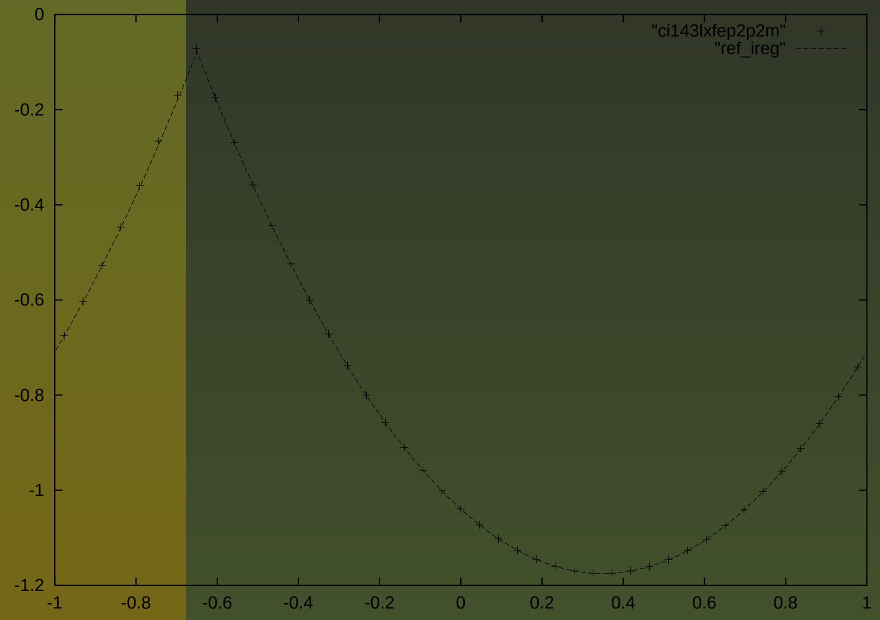
S. Augoula's thesis.
Available at
http://www.math.u-bordeaux.fr/∼abgrall/these.tar.gz
J. Scientific Computing, 2000, vol **2**

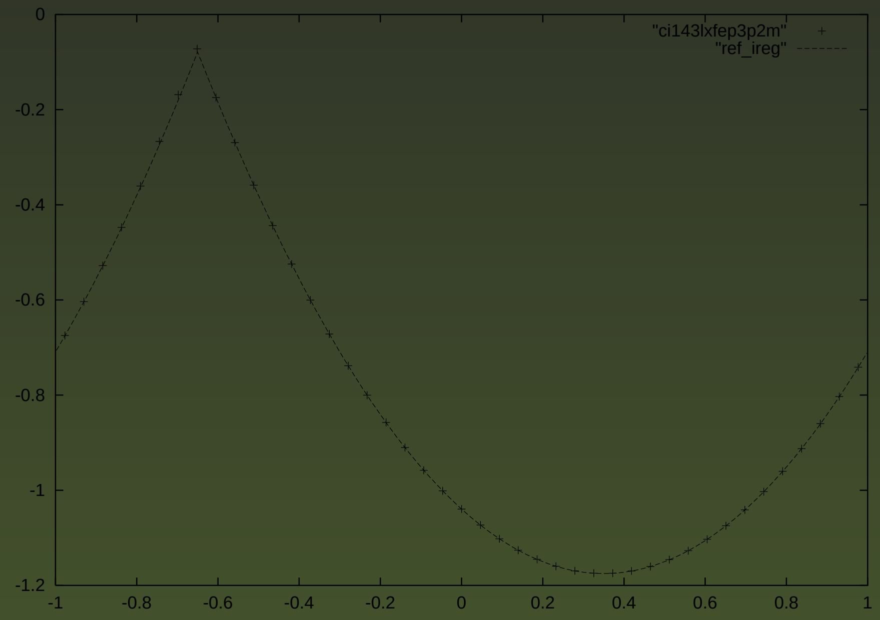
Idea

- Use $u(x,t) = \inf_y \left(u_0(y) + tH^*\left(\frac{x-y}{t}\right)\right)$ (convex/concave H), and u_0 computed from P^2 interpolation)
- Use

$$u_i^{n+1} = u_i^n - \Delta t \mathcal{H}_i$$
$$\mathcal{H}_i = \overline{H} - \frac{\epsilon(\rho)}{\rho}\int_{C_\rho} \nabla u \cdot \vec{n}$$

1 D





second order

Third order

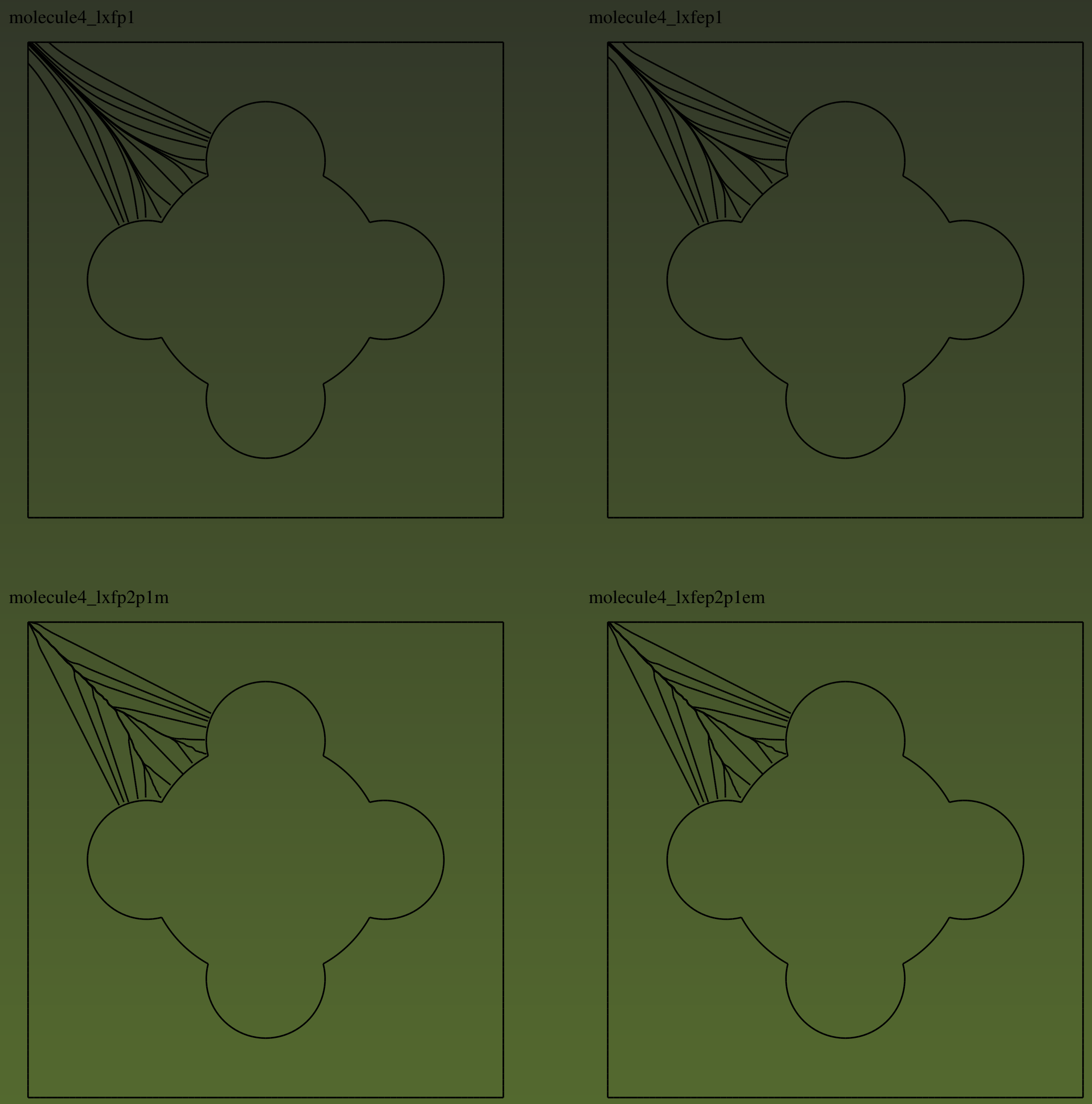
$$u_t + \frac{(u_x)^2}{2} = 0$$

$$u(x, 0 = -\cos(\pi x)$$

Errors

	P^1		P^2		P^3	
N	L^∞ error	Order	L^∞ error	Order	L^∞ error	Order
10	0.24E+00	-	0.94E-02	-	0.71E-03	-
20	0.13E+00	0.88	0.27E-02	1.80	0.16E-03	2.15
40	0.66E-01	0.98	0.68E-03	1.99	0.29E-04	2.46
80	0.34E-01	0.96	0.17E-03	2.00	0.47E-05	2.62
160	0.17E-01	1.00	0.44E-04	1.95	0.71E-06	2.73

Two dimensional example



Solution 2 , steady problems

- Compute $\mathcal{H}_i \simeq H(x, u(x), \nabla u)$ as

$$\mathcal{H}_i = \ell \big(\mathcal{H}_i\big)^L + (1-\ell)\big(\mathcal{H}_i\big)^H$$

- $\big(\mathcal{H}_i\big)^{(L)}$ first order, monotone, approximation of $H(x, u(x), \nabla u)$
- $\big(\mathcal{H}_i\big)^H$ high order approximation of $H(x, u(x), \nabla u)$

Properties of ℓ, 1

- Steady problems $H(x, u(x), \nabla u(x)) = 0$
- u smooth $\left(\mathcal{H}_i\right)^H = O(h^p)$, $p \geq 2$
- $\ell(u) = O(h^p)$

then

$$\mathcal{H}_i = O(h^p)$$

Properties of ℓ, 2

Iterative scheme

$$u_i^{n+1} = u_i^n - \Delta t \mathcal{H}_i$$

- $\left(\mathcal{H}_i\right)^{(1)}$ monotone under $\Delta t \leq Ch$
- $\left(\mathcal{H}_i\right)^{(H)}$ L^p stable for $\Delta t \leq C'h$
- $\ell \in [0,1]$ same stability range for blended scheme

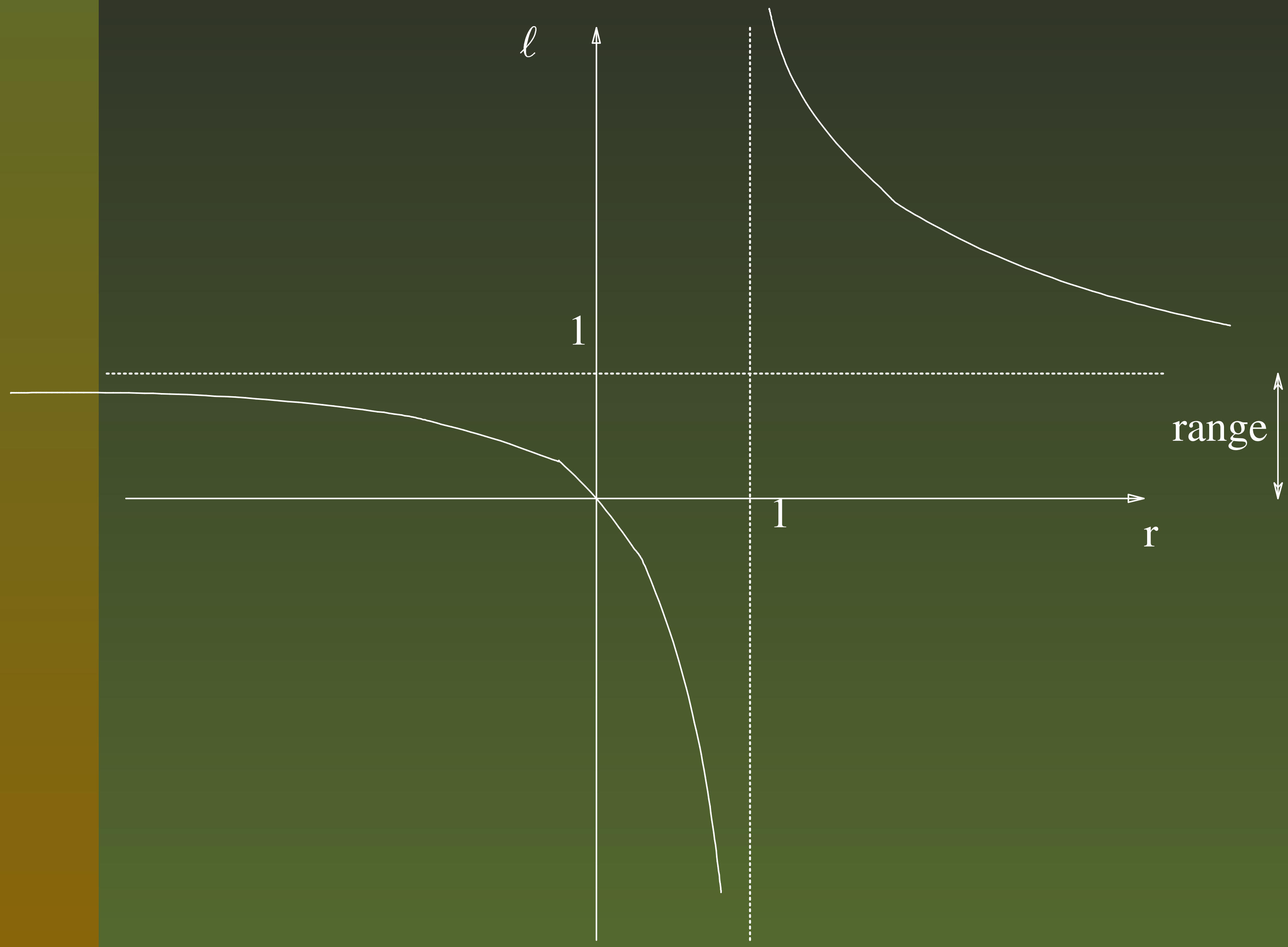
Compute ℓ

$$\ell\left(\mathcal{H}_i\right)^L + (1-\ell)\left(\mathcal{H}_i\right)^H = \left(\ell + (1-\ell)r\right)\left(\mathcal{H}_i\right)^L$$

$$r = \frac{\mathcal{H}^{(H)}}{\mathcal{H}^{(L)}}$$

so ask

$$\ell + (1-\ell)r \geq 0$$


ℓ
1
range
1
r

High order accuracy

$$\ell\big(\mathcal{H}_i\big)^L + (1-\ell)\big(\mathcal{H}_i\big)^H = \left(\frac{\ell}{r} + (1-\ell)\right)\big(\mathcal{H}_i\big)^{(H)}$$

$$r = \frac{\mathcal{H}^{(H)}}{\mathcal{H}^{(L)}}$$

If $\ell \in [0,1]$ and $\left|\frac{\ell}{r}\right| \leq C$ for some $C > 0$, then

$$\left|\ell\big(\mathcal{H}_i\big)^L + (1-\ell)\big(\mathcal{H}_i\big)^H\right| \leq C\big|\mathcal{H}^{(H)}\big|.$$

and high order accuracy.

Solution

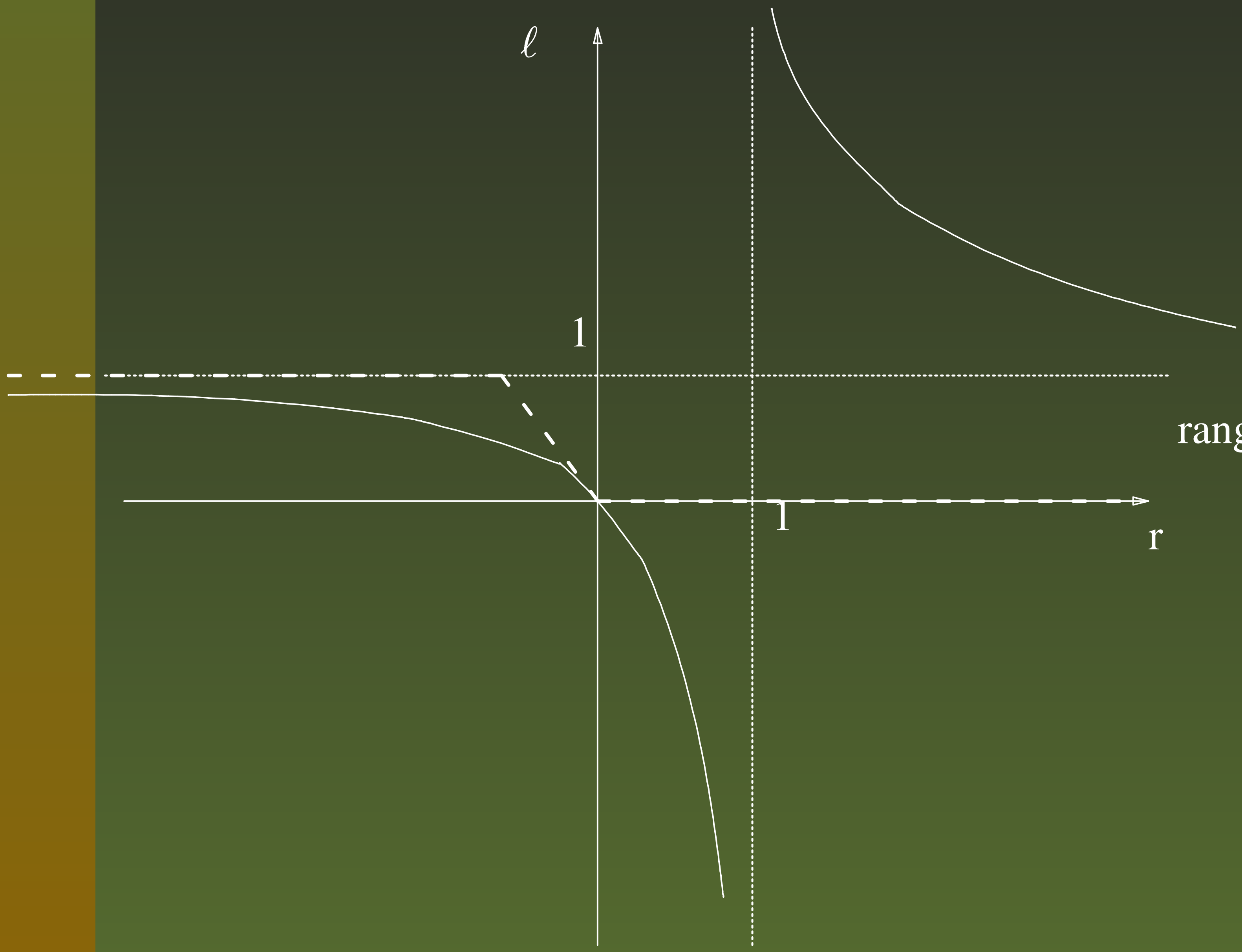
- Hyperbola $\ell + (1-\ell)r = 0$ slope 1 at $(0,0)$,
- $C \geq 1$

$$\ell = \begin{cases} 0 & \text{if } r > 0 \\ \min(1, C|r|) & \text{if } r \leq 0 \end{cases}$$

- or

$$\ell = \min(1, C|r|), \qquad C \geq 1$$

Minmod like



Pseudo proof for convergence for $H(x, u(x), \nabla u) = 0$

Barles and Souganidis result for

$$S(\rho, x, t, u) \equiv \mathcal{H}_\rho(x, u(x), \{u_j\}_{j \in \mathcal{V}_i}) = 0 \qquad (1)$$

1. Monotonicity: if $u \geq v$, for all $\rho \geq 0$, $x \in \Omega$, $t \in \mathbb{R}$ and $u, v \in L^\infty(\Omega)$, then $S(\rho, x, t, u) \leq S(\rho, x, t, v)$.
2. Stability: for all $\rho > 0$, there exists a solution $u^\rho \in L^\infty(\Omega)$ to (1) with a bound independant of ρ,
3. Consistency: for all $x \in \Omega$ and $\phi \in C_b^\infty(\Omega)$,

   $$\limsup_{\rho \to 0, y \to x, \xi \to 0} S(\rho, y, \phi(y) + \xi, \phi + \xi) \leq H(x, \phi(x), D\phi(x))$$
   $$\liminf_{\rho \to 0, y \to x, \xi \to 0} S(\rho, y, \phi(y) + \xi, \phi + \xi) \geq H(x, \phi(x), D\phi(x))$$

4. Strong uniqueness principle: if $u \in L^\infty(\Omega)$ is a subsolution and $v \in L^\infty(\Omega)$ is a supersolution, then $u \leq v$ on Ω.

Barles and Souganidis

Theorem 0.0 (Barles, Souganidis). *Assuming the monotonicity, the consistency, the stability and the strong uniqueness properties, then, when $\rho \to 0$, the solution u^ρ of converges locally uniformly to the unique continuous viscosity solution of $H(x, u(x), \nabla u) = 0$.*

Pseudo–proof, 1

- Assume $\mathcal{H}^{(L)}$ monotone
- ℓ as before
- assume iterative convergence reached

Pseudo–proof, 2

$\ell \in [0,1]$ L^∞ : bound on the solution (CFL condition)

Take $\ell = \min(1, |r|)$, same CFL number as low order scheme.

Pseudo–proof, 3

$u - \varphi$ maximum at x_0 : $u(x_0) \geq \varphi(x_0)$

$$0 = \ell \mathcal{H}^{(L)} + (1-\ell)\mathcal{H}^{(H)} = \Big(\ell + (1-\ell)r\Big)(u)\mathcal{H}^{(L)}(u)$$
$$\geq \Big(\ell + (1-\ell)r\Big)(u) \times \mathcal{H}^{(L)}(\varphi)$$

so $\mathcal{H}^{(L)}(\varphi) \leq 0$ if $\big(\ell + (1-\ell)r\big)(u) \neq 0$
i.e. $\mathcal{H}^{(H)}(u) \neq 0$
then $u = \lim u^h$ super–solution of $H(x, u(x), \nabla u) = 0$

Example

$$
\begin{array}{ll}
||\nabla u|| - 1 = 0 & x \in \Omega \\
u = 0 & x \in \Gamma \subset \partial\Omega
\end{array}
$$

Example

- Low order scheme : Godunov scheme
- High order scheme. Take P^2

$$\mathcal{H}_i^{(H)} = \max_{\text{angular sectors}} \mathcal{K}_i^{(k)}$$

$$\mathcal{K}_i^{(k)} = \begin{cases} \nabla u_{\text{ang sec}}(x_i) & \text{if } \nabla u_{\text{ang sec}}(x_i) \in -\Big(\text{angular} \\ \max(0, \dfrac{u_p - u_i}{pi}) & \text{else} \end{cases}$$