Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 294 (2004) 438–461
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Existence and stability for some partial neutral functional differential equations with infinite delay

Mostafa Adimy,[a,*] Hassane Bouzahir,[b,1] and Khalil Ezzinbi[c]

[a] *Département de Mathématiques Appliquées, IPRA, Université de Pau, Avenue de l'université, 64000 Pau, France*
[b] *Université Ibn Zohr, Ecole Nationale des Sciences Appliquées, B.P. 33/S Agadir, Morocco*
[c] *Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390 Marrakesh, Morocco*

Received 21 June 2001

Submitted by C.E. Wayne

Abstract

In this paper, we study a class of partial neutral functional differential equations with infinite delay. We suppose that the linear part is not necessarily densely defined but satisfies the resolvent estimates of the Hille–Yosida theorem. We give some sufficient conditions ensuring the existence, uniqueness and regularity of solutions. A principle of linearized stability is also established in the autonomous case. To illustrate our abstract results, we conclude this work by an example.

1. Introduction

In their study of a ring array of identical resistively coupled transmission lines, Wu and Xia [32,33] showed that the corresponding system of hyperbolic equations is equivalent to a partial neutral functional differential–difference equation (PNFDDE) defined on the unit circle S^1. They considered equations of the form

* Corresponding author.
E-mail addresses: mostafa.adimy@univ-pau.fr (M. Adimy), bouzahir@esta.ac.ma (H. Bouzahir), ezzinbi@ucam.ac.ma (K. Ezzinbi).
[1] The second author was supported by a grant from the "Action Intégrée" MA/212/00 between FSSM-Morocco and IRD-France.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.02.033

$$\frac{\partial}{\partial t}\big[x(\xi,t)-qx(\xi,t-r)\big]=k\frac{\partial^2}{\partial\xi^2}\big[x(\xi,t)-qx(\xi,t-r)\big]+f\big(x_t(\xi,.)\big) \tag{1}$$

for $t \geqslant 0$, where $\xi \in S^1$, $x_t(\xi,\theta)=x(\xi,t+\theta)$, $-r \leqslant \theta \leqslant 0$, $t \geqslant 0$, k is a positive constant, f is a continuous function and $0 \leqslant q < 1$.

Motivated by this work, Hale considered in [20] and [21] a more general class of partial neutral functional differential equations (PNFDEs) of the form

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{D}v_t = k\frac{\partial^2}{\partial x^2}\mathcal{D}v_t + f(v_t), \quad t \geqslant 0,\\ v_0=\varphi\in\mathcal{C}:=C([-r,0];C(S^1,\mathbb{R})),\end{cases} \tag{2}$$

with k a positive constant, $(\mathcal{D}\phi)(s):=\phi(0)(s)-\int_{-r}^{0}[d\eta(\theta)]\phi(\theta)(s)$ for $s\in S^1$ and $\phi\in\mathcal{C}$, where η is of bounded variation and nonatomic at 0; that is, there exists a continuous nondecreasing function $\delta:[0,r]\to[0,+\infty)$ such that $\delta(0)=0$ and

$$\left|\int_{-\epsilon}^{0}\big[d\eta(\theta)\big]\psi(\theta)\right| \leqslant \delta(\epsilon)\sup_{-\epsilon\leqslant\theta\leqslant 0}\big|\psi(\theta)\big|, \quad \psi\in\mathcal{C},\ \epsilon\in[0,r],$$

with $|.|$ a norm in $C(S^1,\mathbb{R})$. The Laplace operator $A=k(\partial^2/\partial x^2)$ with domain $C^2(S^1,\mathbb{R})$ is an infinitesimal generator of a C_0-semigroup of bounded linear operators on $C(S^1,\mathbb{R})$. Hale presented the basic theory of existence, uniqueness and properties of the solution operator associated to Eq. (2). The book by Wu [31] contains a detailed analysis of the results obtained in [20,21,32,33].

In [8–12], we considered a general equation of the type (2), but with finite delay,

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{D}x_t = A\mathcal{D}x_t + F(x_t), \quad t \geqslant 0,\\ x_0=\varphi\in\mathcal{C},\end{cases} \tag{3}$$

where A is a nondensely defined linear operator that satisfies the Hille–Yosida condition on a Banach space $(E,|.|)$, $\mathcal{C}$ is the space of all continuous functions on $[-r,0]$ with values in E, provided with the uniform norm topology. $\mathcal{D}:\mathcal{C}\to E$ is a bounded linear operator given by

$$\mathcal{D}\phi := \phi(0) - \int_{-r}^{0}\big[d\eta(\theta)\big]\phi(\theta),$$

where η is of bounded variation on $[-r,0]$ and nonatomic at 0. F is a continuous function from $\mathcal{C}$ into E. We established several results concerning the existence and regularity of solutions. We also obtained some results concerning the stability and the asymptotic behavior of the solution semigroup.

In [17], Desch et al. studied an abstract functional differential equations of neutral type with infinite delay. They proved that the model proposed by Coleman and Gurtin [15], Gurtin and Pipkin [19], and Miller [28] can be regarded as the following abstract functional differential equation of neutral type with infinite delay:

$$\frac{d}{dt}\left[x(t)+\int_{-\infty}^{t}K(t-s)x(s)\,ds\right]$$
$$=A\left[x(t)+\int_{-\infty}^{t}K(t-s)x(s)\,ds\right]+F(t,x_t), \tag{4}$$

where the operator A is a generator of a C_0-semigroup on a Banach space.

Furthermore, Hernandez and Henriquez [24,25] established some results concerning the existence, uniqueness and qualitative properties of the solution operator of the following general PNFDE with infinite delay:

$$\begin{cases} \frac{d}{dt}[x(t)-G(t,x_t)] = Ax(t)+F(t,x_t), & t \geqslant 0, \\ x_0=\varphi \in \mathcal{B}, \end{cases} \tag{5}$$

where A generates an analytic semigroup on a Banach space E, $\mathcal{B}$ is the phase space of functions mapping $(-\infty,0]$ into E, which will be specified later, G and F are continuous functions from $[0,+\infty)\times\mathcal{B}$ into E and for each $x:(-\infty,b]\to E$, $b>0$, and $t\in[0,b]$, x_t represents, as usual, the mapping defined from $(-\infty,0]$ into E by

$$x_t(\theta)=x(t+\theta) \quad \text{for } \theta\in(-\infty,0].$$

In [3] and [4], we have extended many similar results of [24] and [25] to the case where A is a Hille–Yosida operator not necessarily densely defined on E, but only in the case $G=0$. We have obtained also some results on the local existence and stability.

In this paper, we prove that the same results can be reproduced in the neutral case with infinite delay. We consider the following general class of nonlinear partial neutral functional differential equations with infinite delay:

$$\begin{cases} \frac{d}{dt}[x(t)-G(t,x_t)] = A[x(t)-G(t,x_t)]+F(t,x_t), & t \geqslant 0, \\ x_0=\varphi \in \mathcal{B}. \end{cases} \tag{6}$$

One can consider the following more general system:

$$\begin{cases} \frac{d}{dt}[x(t)-G_1(t,x_t)] = A[x(t)-G_2(t,x_t)]+F(t,x_t), & t \geqslant 0, \\ x_0=\varphi \in \mathcal{B}, \end{cases} \tag{7}$$

with two distinct functions G_1 and G_2. But we need, for problem (7) to be well posed, the following assumption: $\mathrm{Range}(G_2-G_1)\subseteq D(A)$ and $A(G_2-G_1)$ is a continuous function on E. This assumption permits to write Eqs. (5) and (7) as an equation of type (6). We suppose that A satisfies the Hille–Yosida condition (H1) (with nondense domain).

Note that there are many examples where evolution equations are not densely defined. One can refer to [16] or to [8] for references and discussion on this subject. In particular, nondensity occurs in many situations due to restrictions on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or due to boundary conditions (for example, the space C^1 with null value on the boundary is non dense in the space of continuous functions). Our idea to use a nondense operator has been successful for functional differential equations with finite and infinite delay, and for partial neutral functional differential equations with finite delay (see [1–12]). Our objective is to extend this idea to Eq. (6).

The paper is organized as follows. In Section 2, we recall some results that will be used in this work. In Section 3, we first prove the existence and uniqueness of integral solutions. Then, the integral solutions are shown to be strict under more restrictive assumptions. In Section 4, we state some properties of the solution operator associated to the autonomous case of Eq. (6). Also, we investigate the stability near an equilibrium. Mainly, we prove that the equilibrium of the solution semigroup associated to the autonomous case is locally exponentially stable when its linearized solution semigroup around this equilibrium is exponentially stable. Finally, to illustrate our results, we give in Section 5, an example which is a special nonlinear case of Eq. (4).

2. Preliminary results and definitions

We first study the existence and uniqueness of solutions of Eq. (6). Throughout this paper, we suppose that $(E, |.|)$ is a Banach space and $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed abstract linear space of functions mapping $(-\infty, 0]$ into E, and satisfies the following fundamental axioms, which have been introduced first in [22] and widely discussed in [26].

- (A) There exist a positive constant H and functions $K(.)$, $M(.): \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathcal{B}$ and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:
  - (i) $x_t \in \mathcal{B}$,
  - (ii) $|x(t)| \leqslant H\|x_t\|_{\mathcal{B}}$, which is equivalent to
  - (ii′) $|\varphi(0)| \leqslant H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \leqslant K(t-\sigma)\sup_{\sigma \leqslant s \leqslant t}|x(s)| + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$.
- (A1) For the function $x(.)$ in (A), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.
- (B) The space $\mathcal{B}$ is complete.

We assume also that the operator A satisfies the Hille–Yosida condition:

- (H1) There exist two constants $\bar{M} \geqslant 1$ and $\bar{\omega} \in \mathbb{R}$ such that $(\bar{\omega}, +\infty) \subset \rho(A)$ and $\sup\{\|(\lambda - \bar{\omega})^n R(\lambda, A)^n\|: \lambda > \bar{\omega},\ n \in \mathbb{N}\} \leqslant \bar{M}$, where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$.

We start by introducing the following definitions.

Definition 1. Let $\varphi \in \mathcal{B}$. We say that a function $x : (-\infty, a] \to E$, $a > 0$, is an integral solution of Eq. (6) in $(-\infty, a]$ if the following conditions hold:

- (i) x is continuous on $[0, a]$;
- (ii) $x(t) = \varphi(t)$, $-\infty < t \leqslant 0$;

(iii) $\int_0^t (x(s) - G(s, x_s))\,ds \in D(A)$ for $t \in [0, a]$;
(iv) $x(t) = G(t, x_t) + \varphi(0) - G(0, \varphi) + A\int_0^t (x(s) - G(s, x_s))\,ds + \int_0^t F(s, x_s)\,ds$ for $0 \leqslant t \leqslant a$.

Definition 2. Let $\varphi \in \mathcal{B}$. We say that a function $x : (-\infty, a] \to E$ is a strict solution of Eq. (6) if the following conditions hold:

(i) $t \mapsto x(t) - G(t, x_t) \in C^1([0, a]; E) \cap C([0, a]; D(A))$;
(ii) x satisfies Eq. (6) on $(-\infty, a]$.

From the closedness property of the operator A, we can prove the following statements.

Lemma 3.

(i) *If x is an integral solution of Eq.* (6) *on $(-\infty, a]$, then for all $t \in [0, a]$, $x(t) - G(t, x_t) \in \overline{D(A)}$. In particular $\varphi(0) - G(0, \varphi) \in \overline{D(A)}$.*
(ii) *If x is an integral solution of Eq.* (6) *on $(-\infty, a]$, such that $t \mapsto x(t) - G(t, x_t)$ belongs to $C^1([0, a]; E)$ or $C([0, a]; D(A))$, then x is a strict solution.*

Proof. Let x be an integral solution of Eq. (6). To prove (i), it suffices to remark that for all $t \in [0, a]$,

$$x(t) - G(t, x_t) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \big(x(s) - G(s, x_s)\big)\,ds$$

and

$$\int_t^{t+h} \big(x(s) - G(s, x_s)\big)\,ds \in D(A) \quad \text{for } h > 0 \text{ and } t + h \leqslant a.$$

We will prove now (ii). By definition, for all $t \in [0, a]$ and $h > 0$ such that $t + h \leqslant a$,

$$A\frac{1}{h} \int_t^{t+h} \big(x(s) - G(s, x_s)\big)\,ds = \frac{1}{h}\big\{x(t+h) - G(t+h, x_{t+h}) - x(t) + G(t, x_t)\big\} - \frac{1}{h} \int_t^{t+h} F(s, x_s)\,ds.$$

If $x(s) - G(s, x_s)$ is differentiable, since F is continuous, then the right side of the above equality tends, as h tends to 0^+, to

$$\frac{d}{dt}\big(x(t) - G(t, x_t)\big) - F(t, x_t)$$

and

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\big(x(s)-G(s,x_s)\big)\,ds = x(t)-G(t,x_t).$$

From the closedness of the operator A, we get that $x(t)-G(t,x_t)\in D(A)$ and

$$A\big(x(t)-G(t,x_t)\big)=\frac{d}{dt}\big(x(t)-G(t,x_t)\big)-F(t,x_t)\quad \text{for } t\in[0,a].$$

Then, x is a strict solution.

On the other hand, suppose that $t\mapsto x(t)-G(t,x_t)$ belongs to $C([0,a];D(A))$. Again by definition, for all $t\in[0,a]$ and $h>0$, we have

$$\frac{1}{h}\big\{x(t+h)-G(t+h,x_{t+h})-x(t)+G(t,x_t)\big\}$$
$$=\frac{1}{h}\int_t^{t+h}A\big(x(s)-G(s,x_s)\big)\,ds+\frac{1}{h}\int_t^{t+h}F(s,x_s)\,ds.$$

Since $A(x(s)-G(s,x_s))$ and F are continuous, the right side of the above equality tends, as h tends to 0^+, to $A(x(t)-G(t,x_t))+F(t,x_t)$. Which implies that $x(t)-G(t,x_t)$ is differentiable at the right in t and satisfies

$$\frac{d^+}{dt}\big(x(t)-G(t,x_t)\big)=A\big(x(t)-G(t,x_t)\big)+F(t,x_t).$$

It is well known that if the right derivative is continuous, then the C^1 property holds. We conclude that $t\mapsto x(t)-G(t,x_t)$ is continuously differentiable on $[0,a]$ and satisfies

$$\frac{d}{dt}\big(x(t)-G(t,x_t)\big)=A\big(x(t)-G(t,x_t)\big)+F(t,x_t).$$

This completes the proof of the lemma. □

We know from [27] that under condition (H1), A is the generator of a locally Lipschitz continuous integrated semigroup $(S(t))_{t\geqslant 0}$ on E. In addition, the derivative $(S'(t))_{t\geqslant 0}$ of $(S(t))_{t\geqslant 0}$ generates a C_0-semigroup on $\overline{D(A)}$ such that

$$\big|S'(t)x\big|\leqslant \bar{M}e^{\bar{\omega}t}|x|\quad \text{for all } t\geqslant 0 \text{ and } x\in\overline{D(A)}.$$

We need to recall some general properties of the integrated semigroup $(S(t))_{t\geqslant 0}$.

Proposition 4 [13]. *For all $x\in E$ and $t\geqslant 0$,*

$$\int_0^t S(s)x\,ds\in D(A)\quad \textit{and}\quad S(t)x=A\left(\int_0^t S(s)x\,ds\right)+tx.$$

Moreover, for all $x\in D(A)$ and $t\geqslant 0$,

$$S(t)x\in D(A),\qquad AS(t)x=S(t)Ax\quad \textit{and}\quad S(t)x=\int_0^t S(s)Ax\,ds+tx.$$

Corollary 5 [13]. *For all $x \in E$ and $t \geqslant 0$, one has $S(t)x \in \overline{D(A)}$. Moreover, for a given $x \in E$, $S(\cdot)x$ is right-side differentiable in $t \geqslant 0$ if and only if $S(t)x \in D(A)$, and in that case we have $S'(t)x = AS(t)x + x$.*

Proposition 6 [14]. *Let $f:[0,a] \to E$, $a > 0$, be a Bochner-integrable function. Then, the function $B:[0,a] \to E$ defined by $B(t) = \int_0^t S(t-s)f(s)\,ds$, is continuously differentiable on $[0,a]$ and satisfies, for $t \in [0,a]$, $|B'(t)| \leqslant \bar{M}\int_0^t e^{\bar{\omega}(t-s)}|f(s)|\,ds$.*

3. Existence and regularity of solutions

To obtain the global existence and uniqueness of the integral solutions, we make the following hypothesis.

(H2) $G:[0,+\infty) \times \mathcal{B} \to E$ is continuous and there exists $\alpha_0 > 0$ satisfying $\alpha_0 K(0) < 1$, such that

$$\big|G(t,\varphi_1) - G(t,\varphi_2)\big| \leqslant \alpha_0\|\varphi_1 - \varphi_2\|_{\mathcal{B}} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B} \text{ and } t \geqslant 0.$$

(H3) $F:[0,+\infty) \times \mathcal{B} \to E$ is continuous and there exists $\beta_0 > 0$ such that

$$\big|F(t,\varphi_1) - F(t,\varphi_2)\big| \leqslant \beta_0\|\varphi_1 - \varphi_2\|_{\mathcal{B}} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B} \text{ and } t \geqslant 0.$$

Consider the mapping $\mathcal{G}:[0,+\infty) \times \mathcal{B} \to E$ defined by

$$\mathcal{G}(t,\varphi) = \varphi(0) - G(t,\varphi), \quad (t,\varphi) \in [0,+\infty) \times \mathcal{B}.$$

Before stating our results, we first rewrite Eq. (6) in an integrated form. Let $\varphi \in \mathcal{B}$ such that $\mathcal{G}(0,\varphi) \in \overline{D(A)}$. From the integrated semigroup theory, it is well known that a function $x:(-\infty,a] \to E$, $a > 0$, is an integral solution of Eq. (6) if and only if x is a solution of the following equation:

$$\begin{cases} \mathcal{G}(t,x_t) = S'(t)\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,x_s)\,ds, & t \geqslant 0, \\ x_0 = \varphi. \end{cases} \tag{8}$$

Note that Corollary 5 and Proposition 6 imply, respectively, that $S(t)\mathcal{G}(0,\varphi)$ and $\int_0^t S(t-s)F(s,x_s)\,ds$ are differentiable with respect to t.

Theorem 7. *Assume that the conditions* (H1)–(H3) *are satisfied. Then, for given $\varphi \in \mathcal{B}$ such that $\mathcal{G}(0,\varphi) \in \overline{D(A)}$, Eq.* (6) *has a unique global integral solution $x(.,\varphi)$ defined on $(-\infty,+\infty)$.*

Proof. Let $a > 0$ and $C([0,a];E)$ be the space of continuous functions from $[0,a]$ into E, provided with the uniform norm topology. Let $\varphi \in \mathcal{B}$ such that $\mathcal{G}(0,\varphi) \in \overline{D(A)}$. Consider the nonempty closed subset of $C([0,a];E)$ defined by

$$Z_a(\varphi) := \big\{z \in C\big([0,a];E\big):\ z(0) = \varphi(0)\big\}.$$

For $z \in Z_a(\varphi)$, we define $\tilde{z}:(-\infty,a] \to E$ by

$$\tilde{z}(t) = \begin{cases} z(t), & t \in [0,a], \\ \varphi(t), & t \leqslant 0. \end{cases}$$

Set $K_a := \max_{0\leqslant t\leqslant a} K(t)$. By virtue of condition (H3) and axiom (A1), the mapping $s \mapsto F(s,\tilde{z}_s)$ is continuous on $[0,a]$. Then, Proposition 6 implies that the mapping $t \mapsto \int_0^t S(t-s)F(s,\tilde{z}_s)ds$ is continuously differentiable on $[0,a]$.

Consider the operator $J : Z_a(\varphi) \to Z_a(\varphi)$ defined by

$$(Jz)(t) := G(t,\tilde{z}_t) + S'(t)\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,\tilde{z}_s)\,ds.$$

Without loss of generality, we suppose that $\bar{\omega} \geqslant 0$. Using the hypothesis, axiom (A)(iii) and Proposition 6, we can see that for every $z^1, z^2 \in Z_a(\varphi)$ and $t \in [0,a]$,

$$\left|(Jz^1)(t) - (Jz^2)(t)\right| \leqslant (\alpha_0 + \beta_0 \bar{M} e^{\bar{\omega}a} a) K_a \|z^1 - z^2\|_\infty.$$

Since K is continuous and $\alpha_0 K(0) < 1$, then we can choose $a > 0$ small enough such that $(\alpha_0 + \beta_0 \bar{M} e^{\bar{\omega}a} a) K_a < 1$.

Then, J is a strict contraction in $Z_a(\varphi)$, and the fixed point of J gives a unique integral solution $x(.,\varphi)$ on $(-\infty, a]$.

A similar argument can be used in $[0, na]$, $n \geqslant 2$, to see that the integral solution exists uniquely in $(-\infty, +\infty)$. This ends the proof. □

The following theorem asserts that, under more restrictive conditions, the integral solution is a strict one. In order to compute the integral in $\mathcal{B}$ from the integral in E, we suppose that $\mathcal{B}$ satisfies one of the following axioms.

(C1) If $(\phi_n)_{n\geqslant 0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\phi_n)_{n\geqslant 0}$ converges compactly to ϕ on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

(D) For a sequence $(\varphi_n)_{n\geqslant 0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}} \to 0$ as $n \to \infty$, then for each $\theta \in (-\infty, 0]$, $|\varphi_n(\theta)| \to 0$ as $n \to \infty$.

Remark that axiom (D) implies that the space $\mathcal{B}$ is normed.

Lemma 8 [29]. *Let $\mathcal{B}$ be a normed space which satisfies axiom* (C1) *and $f : [0,a] \to \mathcal{B}$, $a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t,\theta) \in [0,a] \times (-\infty, 0]$. Then,*

$$\left[\int_0^a f(t)\,dt\right](\theta) = \int_0^a f(t)(\theta)\,dt, \quad \theta \in (-\infty, 0].$$

We obtain a similar result by using axiom (D).

Lemma 9. *Assume that $\mathcal{B}$ satisfies axiom* (D) *and $f : [0,a] \to \mathcal{B}$ is a continuous function. Then, for all $\theta \in (-\infty, 0]$, the function $f(.)(\theta)$ is continuous on $[0,a]$ and satisfies*

$$\left[\int_0^a f(t)\,dt\right](\theta) = \int_0^a f(t)(\theta)\,dt, \quad \theta \in (-\infty, 0].$$

Proof. We have

$$\int_0^a f(t)\,dt = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f\left(\frac{ka}{n}\right) \quad \text{in } \mathcal{B}.$$

Using axiom (D), we get

$$\left[\int_0^a f(t)\,dt\right](\theta) = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f\left(\frac{kn}{a}\right)(\theta), \quad \theta \in (-\infty, 0].$$

On the other hand, the same axiom implies that the function $f(.)(\theta)$ is continuous on $[0, a]$. Then, it is integrable and satisfies

$$\int_0^a f(t)(\theta)\,dt = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} f\left(\frac{kn}{a}\right)(\theta), \quad \theta \in (-\infty, 0].$$

This ends the proof of the lemma. □

For the regularity of the integral solutions, we add the following assumption.

(H4) G and F are continuously differentiable and their partial derivatives are locally Lipschitzian with respect to the second argument in the sense that; for any compact set $Q \subset [0, +\infty) \times \mathcal{B}$, there exists a constant $\beta_1 > 0$ such that

$$\begin{cases} \|D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)\| \leqslant \beta_1 \|\varphi - \psi\|_{\mathcal{B}}, \\ \|D_t F(t,\varphi) - D_t F(t,\psi)\| \leqslant \beta_1 \|\varphi - \psi\|_{\mathcal{B}}, \\ \|D_\varphi G(t,\varphi) - D_\varphi G(t,\psi)\| \leqslant \beta_1 \|\varphi - \psi\|_{\mathcal{B}}, \\ \|D_t G(t,\varphi) - D_t G(t,\psi)\| \leqslant \beta_1 \|\varphi - \psi\|_{\mathcal{B}}, \end{cases} \tag{9}$$

for all (t,φ), $(t,\psi) \in Q$ and $t \geqslant 0$, where $D_t F$, $D_\varphi F$, $D_t G$ and $D_\varphi G$ denote the derivatives with respect to t and φ.

Since for all $a > 0$ and any functions x and y verifying the conditions in axiom (A), the sets $\{(s, x_s)\colon s \in [0, a]\}$ and $\{(s, y_s)\colon s \in [0, a]\}$ are in a compact set of $[0, a] \times \mathcal{B}$, condition (H4) implies that

$$\begin{cases} \|D_\varphi F(s,x_s) - D_\varphi F(s,y_s)\| \leqslant \beta_1 \|x_s - y_s\|_{\mathcal{B}}, \\ \|D_t F(s,x_s) - D_t F(s,y_s)\| \leqslant \beta_1 \|x_s - y_s\|_{\mathcal{B}}, \\ \|D_\varphi G(s,x_s) - D_\varphi G(s,y_s)\| \leqslant \beta_1 \|x_s - y_s\|_{\mathcal{B}}, \\ \|D_t G(s,x_s) - D_t G(s,y_s)\| \leqslant \beta_1 \|x_s - y_s\|_{\mathcal{B}}, \end{cases}$$

for all $s \in [0, a]$ and any functions x and y as in axiom (A).

Theorem 10. *Assume that $\mathcal{B}$ is normed and satisfies axiom* (C1) *or axiom* (D) *and the conditions* (H1)–(H4) *hold. Then, for each continuously differentiable function $\varphi \in \mathcal{B}$ such that*

$$\begin{aligned}&\varphi' \in \mathcal{B}, \quad \mathcal{G}(0,\varphi) \in D(A), \quad D_\varphi \mathcal{G}(0,\varphi)\varphi' + D_t\mathcal{G}(0,\varphi) \in \overline{D(A)},\\ &D_\varphi \mathcal{G}(0,\varphi)\varphi' + D_t\mathcal{G}(0,\varphi) = A\mathcal{G}(0,\varphi) + F(0,\varphi),\end{aligned} \tag{10}$$

the integral solution of Eq. (6) *given by Theorem* 7 *is a strict solution.*

Proof. Let $a > 0$. By Theorem 7, we know that Eq. (6) has a unique integral solution $x := x(.,\varphi)$ which is the unique solution of

$$\mathcal{G}(t,x_t) = S'(t)\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,x_s)\,ds \quad \text{for } t \in [0,a]. \tag{11}$$

By Lemma 3, it suffices to show that x is continuously differentiable on $[0,a]$. From Corollary 5, the assumption $\mathcal{G}(0,\varphi) \in D(A)$ implies that

$$S'(t)\mathcal{G}(0,\varphi) = S(t)A\mathcal{G}(0,\varphi) + \mathcal{G}(0,\varphi).$$

Then, Eq. (11) can be written as

$$\mathcal{G}(t,x_t) = \mathcal{G}(0,\varphi) + S(t)A\mathcal{G}(0,\varphi) + \frac{d}{dt}\int_0^t S(t-s)F(s,x_s)\,ds. \tag{12}$$

Consider the following problem:

$$\begin{cases} \frac{d}{dt}[D_\varphi\mathcal{G}(t,x_t)y_t + D_t\mathcal{G}(t,x_t)] \\ \quad = A[D_\varphi\mathcal{G}(t,x_t)y_t + D_t\mathcal{G}(t,x_t)] \\ \qquad + D_tF(t,x_t) + D_\varphi F(t,x_t)y_t, \quad t \in [0,a], \\ y_0 = \varphi'. \end{cases} \tag{13}$$

Assumptions (H2) and (H3) imply, respectively, that,

$$\|D_\varphi G(t,\psi)\| \leqslant \alpha_0 \quad \text{and} \quad \|D_\varphi F(t,\psi)\| \leqslant \beta_0 \quad \text{for all } \psi \in \mathcal{B} \text{ and } t \geqslant 0.$$

Then, using the same reasoning as in the proof of Theorem 7, one can show that Eq. (13) has a unique integral solution y on $(-\infty,a]$. Let $w : (-\infty,a] \to E$ be the function defined by

$$w(t) = \begin{cases} \varphi(t) & \text{for } t \in (-\infty,0], \\ \varphi(0) + \int_0^t y(s)\,ds & \text{for } t \in [0,a]. \end{cases}$$

Then, using Lemma 8 or Lemma 9, we can see that

$$w_t = \varphi + \int_0^t y_s\,ds \quad \text{for } t \in [0,a]. \tag{14}$$

Next, we show that $x = w$ on $(-\infty, a]$. As in the proof of the last theorem, we proceed by steps, that is, we first take $a > 0$ small enough such that $\alpha_0 K_a < 1$, and we use the same argument to see the similar result on $[a, 2a], \ldots, [na, (n+1)a]$ for any $n \geqslant 2$.

Using the integrated form of Eq. (13) and the expressions satisfied by φ, we obtain for $t \in [0, a]$,

$$\int_0^t D_\varphi \mathcal{G}(s, x_s) y_s \, ds = -\int_0^t D_t \mathcal{G}(s, x_s) \, ds + S(t)\big(A\mathcal{G}(0, \varphi) + F(0, \varphi)\big) + \int_0^t S(t-s)\big(D_t F(s, x_s) + D_\varphi F(s, x_s) y_s\big) \, ds. \tag{15}$$

On the other hand, from (14), the function $t \mapsto w_t$ is continuously differentiable. It follows that, for $t \in [0, a]$,

$$\frac{d}{dt}\int_0^t S(t-s) F(s, w_s) \, ds = S(t) F(0, \varphi) + \int_0^t S(t-s)\big(D_t F(s, w_s) + D_\varphi F(s, w_s) y_s\big) \, ds.$$

Consequently,

$$S(t) F(0, \varphi) = \frac{d}{dt}\int_0^t S(t-s) F(s, w_s) \, ds - \int_0^t S(t-s)\big(D_t F(s, w_s) + D_\varphi F(s, w_s) y_s\big) \, ds. \tag{16}$$

Consider the functions z^1 and z^2 defined on $[0, a]$ by

$$z^1(t) = \mathcal{G}(t, x_t) \quad \text{and} \quad z^2(t) = \mathcal{G}(t, w_t).$$

Using expression (12), we get

$$z^1(t) = \mathcal{G}(0, \varphi) + S(t) A\mathcal{G}(0, \varphi) + \frac{d}{dt}\int_0^t S(t-s) F(s, x_s) \, ds \tag{17}$$

and

$$z^2(t) - \mathcal{G}(0, \varphi) = \int_0^t \frac{d}{ds}\mathcal{G}(s, w_s) \, ds = \int_0^t \big(D_t \mathcal{G}(s, w_s) + D_\varphi \mathcal{G}(s, w_s) y_s\big) \, ds.$$

Then, we obtain from (15),

$$
\begin{aligned}
z^2(t) = {} & \int_0^t \big(D_t\mathcal{G}(s, w_s) - D_t\mathcal{G}(s, x_s)\big)\,ds \\
& + \int_0^t \big(D_\varphi\mathcal{G}(s, w_s) - D_\varphi\mathcal{G}(s, x_s)\big)y_s\,ds + S(t)\big(A\mathcal{G}(0, \varphi) + F(0, \varphi)\big) \\
& + \mathcal{G}(0, \varphi) + \int_0^t S(t-s)\big(D_t F(s, x_s) + D_\varphi F(s, x_s)y_s\big)\,ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
z^1(t) - z^2(t) = {} & \frac{d}{dt}\int_0^t S(t-s)F(s, x_s)\,ds - S(t)F(0, \varphi) \\
& - \int_0^t \big(D_t\mathcal{G}(s, w_s) - D_t\mathcal{G}(s, x_s)\big)\,ds \\
& - \int_0^t \big(D_\varphi\mathcal{G}(s, w_s) - D_\varphi\mathcal{G}(s, x_s)\big)y_s\,ds \\
& - \int_0^t S(t-s)\big(D_t F(s, x_s) + D_\varphi F(s, x_s)y_s\big)\,ds.
\end{aligned}
$$

Expression (16) yields to

$$
\begin{aligned}
z^1(t) - z^2(t) = {} & \frac{d}{dt}\int_0^t S(t-s)\big(F(s, x_s) - F(s, w_s)\big)\,ds \\
& - \int_0^t \big(D_t\mathcal{G}(s, w_s) - D_t\mathcal{G}(s, x_s)\big)\,ds \\
& - \int_0^t \big(D_\varphi\mathcal{G}(s, w_s) - D_\varphi\mathcal{G}(s, x_s)\big)y_s\,ds \\
& + \int_0^t S(t-s)\big(D_t F(s, w_s) - D_t F(s, x_s)\big)\,ds \\
& + \int_0^t S(t-s)\big(D_\varphi F(s, w_s) - D_\varphi F(s, x_s)\big)y_s\,ds.
\end{aligned}
$$

Consequently, we deduce that

$$\left|z^1(t) - z^2(t)\right| \leqslant \sigma(a) \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds,$$

where $\sigma(a) = \bar{M} e^{\bar{\omega} a} \beta_0 + \beta_1 + (H + \beta_1) b_0 + \beta_1 b_0 + b_0^2 \beta_1$ and

$$b_0 = \max\Big\{ \sup_{0 \leqslant s \leqslant a} \big\|S(s)\big\|;\ \sup_{0 \leqslant s \leqslant a} \|y_s\|_{\mathcal{B}} \Big\}.$$

Since $x_0 = w_0 = \varphi$, axiom (A)(iii) implies that

$$\|x_t - w_t\|_{\mathcal{B}} \leqslant K_a \sup_{0 \leqslant s \leqslant t} \left|x(s) - w(s)\right|$$

and

$$\begin{aligned} \left|x(t) - w(t)\right| &\leqslant \alpha_0 \|x_t - w_t\|_{\mathcal{B}} + \sigma(a) \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds \\ &\leqslant \alpha_0 K_a \sup_{0 \leqslant s \leqslant t} \left|x(s) - w(s)\right| + \sigma(a) \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds. \end{aligned}$$

Consequently,

$$\|x_t - w_t\|_{\mathcal{B}} \leqslant K_a \sup_{0 \leqslant s \leqslant t} \left|x(s) - w(s)\right| \leqslant K_a (1 - \alpha_0 K_a)^{-1} \sigma(a) \int_0^t \|x_s - w_s\|_{\mathcal{B}}\, ds.$$

Using the Gronwall's lemma, we conclude that

$$\|x_t - w_t\|_{\mathcal{B}} = 0 \quad \text{for } t \in [0, a].$$

Hence, $x(t) = w(t)$for all $t \in (-\infty, a]$. Repeating the same procedure in $[a, 2a], \ldots, [na, (n+1)a]$, we deduce that $x(t) = w(t)$for all $t \in (-\infty, +\infty)$ and, x is continuously differentiable on $(-\infty, +\infty)$.

Finally, by Lemma 3 we get that x is a strict solution. This completes the proof of the theorem. □

4. The solution semigroup in the autonomous case and the linearized stability principle

In this section, we suppose that F and G are autonomous in t. Then, Eq. (6) becomes

$$\begin{cases} \frac{d}{dt}[x(t) - G(x_t)] = A[x(t) - G(x_t)] + F(x_t), & t \geqslant 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{18}$$

where F and G are Lipschitz continuous on $\mathcal{B}$. Let $\mathcal{G}:\mathcal{B}\to E$ be the operator defined by $\mathcal{G}(\varphi)=\varphi(0)-G(\varphi)$. We verify that the integral solutions of Eq. (18) satisfy the properties of a nonlinear strongly continuous semigroup on the subset of $\mathcal{B}$,

$$\mathcal{Y}:=\{\varphi\in\mathcal{B}\colon\ \mathcal{G}(\varphi)\in\overline{D(A)}\}.$$

We also prove that this semigroup satisfies the translation property and a Lipschitz property.

For each $t\geqslant 0$, define the nonlinear operator $U(t)$ on $\mathcal{Y}$ by

$$U(t)(\varphi)=x_t(.,\varphi),$$

where $x(.,\varphi)$ is the unique integral solution of Eq. (18). Observe that axiom (A1) and Lemma 3 imply that

$$U(t)(\mathcal{Y})\subseteq\mathcal{Y}\quad\text{for all } t\geqslant 0.$$

We have the following result.

Proposition 11. *Assume that conditions* (H1)–(H3) *hold. Then* $(U(t))_{t\geqslant 0}$ *is a nonlinear strongly continuous semigroup on* $\mathcal{Y}$*, that is*

(i) $U(0)=I$,
(ii) $U(t+s)=U(t)U(s)$ *for all* $t,s\geqslant 0$,
(iii) *for all* $\varphi\in\mathcal{Y}$, $U(t)(\varphi)$ *is a continuous function of* $t\geqslant 0$ *with values in* $\mathcal{Y}$.

Moreover,

(vi) *for all* $t\geqslant 0$, $U(t)$ *is continuous from* $\mathcal{Y}$ *into* $\mathcal{Y}$,
(v) $(U(t))_{t\geqslant 0}$ *satisfies, for* $t\geqslant 0$ *and* $\theta\in(-\infty,0]$, *the translation property*

$$\big(U(t)(\varphi)\big)(\theta)=\begin{cases}(U(t+\theta)(\varphi))(0) & \text{if } t+\theta\geqslant 0,\\ \varphi(t+\theta) & \text{if } t+\theta\leqslant 0,\end{cases}$$

(vi) *for all* $T>0$, *there are two functions* $q,r\in L^\infty([0,T],\mathbb{R}^+)$ *such that, for all* φ_1, $\varphi_2\in\mathcal{Y}$,

$$\big\|U(t)(\varphi_1)-U(t)(\varphi_2)\big\|_{\mathcal{B}}\leqslant q(t)e^{r(t)}\|\varphi_1-\varphi_2\|_{\mathcal{B}},\quad t\in[0,T].$$

Proof. The proofs of (i), (ii) and (v) are straightforward. (iii) follows from axiom (A1) and the uniqueness of the integral solutions to Eq. (18). To prove (vi), we suppose without loss of generality that $\bar{\omega}\geqslant 0$. Set, for $\varepsilon>0$, $K_\varepsilon:=\max_{0\leqslant s\leqslant\varepsilon}K(s)$, $M_\varepsilon:=\sup_{0\leqslant s\leqslant\varepsilon}M(s)$, $x^1:=x(.,\varphi_1)$ and $x^2:=x(.,\varphi_2)$. For $t\in[0,\varepsilon]$, we have

$$\begin{aligned}&\big\|U(t)(\varphi_1)-U(t)(\varphi_2)\big\|_{\mathcal{B}}=\big\|x_t^1-x_t^2\big\|_{\mathcal{B}}\\ &\quad\leqslant K(t)\sup_{0\leqslant s\leqslant t}\big|x^1(s)-x^2(s)\big|+M(t)\|\varphi_1-\varphi_2\|_{\mathcal{B}}\\ &\quad\leqslant K_\varepsilon\sup_{0\leqslant s\leqslant t}\big\{\big|G\big(x_s^1\big)-G\big(x_s^2\big)\big|+\big|S'(s)\big(\mathcal{G}(\varphi_1)-\mathcal{G}(\varphi_2)\big)\big|\big\}\end{aligned}$$

$$+ K_\varepsilon \sup_{0\leqslant s\leqslant t}\left|\frac{d}{ds}\int_0^s S(s-\sigma)\big(F\big(x_\sigma^1\big)-F\big(x_\sigma^2\big)\big)\,d\sigma\right| + M_\varepsilon\|\varphi_1-\varphi_2\|_{\mathcal{B}},$$

so

$$\begin{aligned}\big\|U(t)(\varphi_1)-U(t)(\varphi_2)\big\|_{\mathcal{B}} &\leqslant \alpha_0 K_\varepsilon \sup_{0\leqslant s\leqslant t}\big\|x_s^1-x_s^2\big\|_{\mathcal{B}} + \big[K_\varepsilon \bar{M} e^{\bar{\omega}t}(H+\alpha_0)+M_\varepsilon\big]\|\varphi_1-\varphi_2\|_{\mathcal{B}} \\ &\quad + K_\varepsilon \sup_{0\leqslant s\leqslant t}\left|\frac{d}{ds}\int_0^s S(s-\sigma)\big(F\big(x_\sigma^1\big)-F\big(x_\sigma^2\big)\big)d\sigma\right|.\end{aligned}$$

Using Proposition 6, we have for $0\leqslant s\leqslant t$,

$$\left|\frac{d}{ds}\int_0^s S(s-\sigma)\big(F\big(x_\sigma^1\big)-F\big(x_\sigma^2\big)\big)\,d\sigma\right| \leqslant \bar{M}e^{\bar{\omega}t}\beta_0\int_0^s \big\|x_\sigma^1-x_\sigma^2\big\|_{\mathcal{B}}\,d\sigma.$$

Choose $\varepsilon>0$ such that $1-K_\varepsilon\alpha_0>0$. Then, for $t\in[0,\varepsilon]$,

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\big\|x_s^1-x_s^2\big\|_{\mathcal{B}} \leqslant (1-\alpha_0K_\varepsilon)^{-1}\bigg\{&\big[K_\varepsilon\bar{M}e^{\bar{\omega}\varepsilon}(H+\alpha_0)+M_\varepsilon\big]\|\varphi_1-\varphi_2\|_{\mathcal{B}} \\ &+ K_\varepsilon\bar{M}e^{\bar{\omega}\varepsilon}\beta_0\int_0^t \sup_{0\leqslant s\leqslant\sigma}\big\|x_s^1-x_s^2\big\|_{\mathcal{B}}d\sigma\bigg\}.\end{aligned}$$

Using the Gronwall's lemma, we get

$$\sup_{0\leqslant s\leqslant t}\big\|x_s^1-x_s^2\big\|_{\mathcal{B}} \leqslant \upsilon_0(\varepsilon)\|\varphi_1-\varphi_2\|_{\mathcal{B}}, \tag{19}$$

where

$$\upsilon_0(\varepsilon) = (1-\alpha_0K_\varepsilon)^{-1}\big[K_\varepsilon\bar{M}e^{\bar{\omega}\varepsilon}(H+\alpha_0)+M_\varepsilon\big]\exp\big\{(1-\alpha_0K_\varepsilon)^{-1}K_\varepsilon\bar{M}e^{\bar{\omega}\varepsilon}\beta_0\varepsilon\big\}.$$

Repeating similar arguments, we obtain similar estimates for $t\in[n\varepsilon,(n+1)\varepsilon]$ with $n\geqslant 2$. Consequently, (vi) is true. Finally, (iv) is an immediate consequence of (vi). This ends the proof. □

In what follows, we study the stability of an equilibrium of the following autonomous equation:

$$\begin{cases}\frac{d}{dt}(\mathcal{D}x_t+G(x_t)) = A(\mathcal{D}x_t+G(x_t))+F(x_t), & t\geqslant 0,\\ x_0=\varphi\in\mathcal{B},\end{cases} \tag{20}$$

where $\mathcal{D}$ and G satisfy the following condition.

(H5) $\mathcal{D}:\mathcal{B}\to E$ is an operator defined by $\mathcal{D}\varphi=\varphi(0)-\mathcal{D}_0\varphi$ for $\varphi\in\mathcal{B}$, with $\mathcal{D}_0$ a bounded linear operator from $\mathcal{B}$ into E and $G:\mathcal{B}\to E$ is a continuous function such that there exists $\alpha_0>0$ satisfying $(\alpha_0+\|\mathcal{D}_0\|)K(0)<1$,

$$\left|G(\varphi_1) - G(\varphi_2)\right| \leqslant \alpha_0 \|\varphi_1 - \varphi_2\|_{\mathcal{B}} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

For each $u \in E$, we define a constant function $\tilde{u}$ on $(-\infty, 0]$ by $\tilde{u}(\theta) = u$ for all $\theta \in (-\infty, 0]$. By an equilibrium of Eq. (20), we mean a constant function $\tilde{u}$ such that $\tilde{u} \in \mathcal{B}$ and satisfies

$$\mathcal{D}\tilde{u} + G(\tilde{u}) \in D(A) \quad \text{and} \quad A\big(\mathcal{D}\tilde{u} + G(\tilde{u})\big) + F(\tilde{u}) = 0. \tag{21}$$

Set $y(t) := x(t) - \tilde{u}$, then y satisfies, for $t \geqslant 0$, the following equation:

$$\frac{d}{dt}\big(\mathcal{D}(y_t + \tilde{u}) + G(y_t + \tilde{u})\big) = A\big(\mathcal{D}(y_t + \tilde{u}) + G(y_t + \tilde{u})\big) + F(y_t + \tilde{u}).$$

Which is equivalent to

$$\frac{d}{dt}\big(\mathcal{D}y_t + \mathcal{D}\tilde{u} + G(y_t + \tilde{u}) - G(\tilde{u}) + G(\tilde{u})\big)$$
$$= A\big(\mathcal{D}y_t + \mathcal{D}\tilde{u} + G(y_t + \tilde{u}) - G(\tilde{u}) + G(\tilde{u})\big) + F(y_t + \tilde{u}) - F(\tilde{u}) + F(\tilde{u}).$$

Then

$$\frac{d}{dt}\big(\mathcal{D}y_t + G(y_t + \tilde{u}) - G(\tilde{u})\big)$$
$$= A\big(\mathcal{D}y_t + G(y_t + \tilde{u}) - G(\tilde{u})\big) + F(y_t + \tilde{u}) - F(\tilde{u}). \tag{22}$$

Set, for each $\phi \in \mathcal{B}$,

$$\tilde{G}(\phi) = G(\phi + \tilde{u}) - G(\tilde{u}) \quad \text{and} \quad \tilde{F}(\phi) = F(\phi + \tilde{u}) - F(\tilde{u}).$$

Hence, Eq. (22) becomes

$$\frac{d}{dt}\big(\mathcal{D}y_t + \tilde{G}(y_t)\big) = A\big(\mathcal{D}y_t + \tilde{G}(y_t)\big) + \tilde{F}(y_t), \tag{23}$$

with

$$\tilde{G}(0) = \tilde{F}(0) = 0.$$

Consequently, to study the stability of an equilibrium $\tilde{u}$ of Eq. (20) is reduced to study the stability of 0 as an equilibrium of Eq. (23). Then, without loss of generality, we can assume that $\tilde{u} = 0$ and

$$G(0) = F(0) = 0. \tag{24}$$

In that case, condition (21) is reduced to

$$G(0) \in D(A) \quad \text{and} \quad AG(0) + F(0) = 0. \tag{25}$$

We assume that

(H6) F and G are Fréchet-differentiable at 0 and $G'(0) = 0$.

Let $L = F'(0)$. Then, the linearized equation of Eq. (20) around the equilibrium 0 is the following:

$$\begin{cases} \frac{d}{dt}\mathcal{D}x_t = A\mathcal{D}x_t + Lx_t, & t \geqslant 0, \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{26}$$

To define the nonlinear semigroup $(U(t))_{t\geqslant 0}$ associated to Eq. (20) and the linear semigroup $(T(t))_{t\geqslant 0}$ associated to Eq. (26) in the same space

$$\mathcal{B}_{\mathcal{D}} := \{\varphi \in \mathcal{B}\colon \mathcal{D}\varphi \in \overline{D(A)}\},$$

we assume that

(H7) $\operatorname{Range}(G) \subseteq \overline{D(A)}$.

Then, we have the following result.

Theorem 12. *Suppose that the assumptions* (H1), (H3), (H5)–(H7) *and condition* (24) *are satisfied. Then, for $t \geqslant 0$, the Fréchet-derivative at zero of $U(t)$ is $T(t)$.*

The proof of this theorem is based on the following fundamental lemma.

Lemma 13. *Let $H : \mathcal{B} \to E$ be a continuous function such that there exists $\mu_0 > 0$, with $\mu_0 K(0) < 1$, satisfying*

$$|H(\varphi_1) - H(\varphi_2)| \leqslant \mu_0 \|\varphi_1 - \varphi_2\|_{\mathcal{B}} \quad \textit{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

Let $\varphi \in \mathcal{B}$ and $g : [0, +\infty) \to E$ be a continuous function. Suppose that there exist continuous functions $x, y : (-\infty, +\infty) \to E$ such that

$$\begin{cases} x(t) - y(t) = H(x_t) - H(y_t) + g(t), & t \geqslant 0, \\ x_0 = y_0 = \varphi. \end{cases} \tag{27}$$

Then, for each $T > 0$, there exists a function $b \in L^{\infty}([0, T], \mathbb{R}^+)$ such that

$$\|x_t - y_t\|_{\mathcal{B}} \leqslant b(t) \sup_{0\leqslant s\leqslant t} |g(s)|, \quad t \in [0, T]. \tag{28}$$

Proof. Let $\varepsilon > 0$. By axiom (A)(iii) we obtain, for $t \in [0, \varepsilon]$,

$$\begin{aligned} \|x_t - y_t\|_{\mathcal{B}} &\leqslant K(t) \sup_{0\leqslant s\leqslant t} |x(s) - y(s)| \\ &\leqslant K_\varepsilon \mu_0 \sup_{0\leqslant s\leqslant t} \|x_s - y_s\|_{\mathcal{B}} + K_\varepsilon \sup_{0\leqslant s\leqslant t} \sup |g(s)|. \end{aligned}$$

We choose $\varepsilon > 0$ small enough such that $1 - K_\varepsilon \mu_0 > 0$. Then,

$$\sup_{0\leqslant s\leqslant t} \|x_s - y_s\|_{\mathcal{B}} \leqslant b_0(\varepsilon) \sup_{0\leqslant s\leqslant t} |g(s)|,$$

with $b_0(\varepsilon) = K_\varepsilon/(1 - K_\varepsilon\mu_0)$. Similarly, for $t \in [\varepsilon, 2\varepsilon]$,

$$\begin{aligned}\|x_t - y_t\|_{\mathcal{B}} &\leqslant K(t-\varepsilon) \sup_{\varepsilon\leqslant s\leqslant t} \left|x(s) - y(s)\right| + M(t-\varepsilon)\|x_\varepsilon - y_\varepsilon\|_{\mathcal{B}} \\ &\leqslant K_\varepsilon \mu_0 \sup_{0\leqslant s\leqslant t} \|x_s - y_s\|_{\mathcal{B}} + K_\varepsilon \sup_{0\leqslant s\leqslant t} \left|g(s)\right| \\ &\quad + M(t-\varepsilon) b_0(\varepsilon) \sup_{0\leqslant s\leqslant t} \left|g(s)\right| \\ &\leqslant b_1(\varepsilon) \sup_{0\leqslant s\leqslant t} \left|g(s)\right|,\end{aligned}$$

where

$$b_1(\varepsilon) = \frac{K_\varepsilon + M_\varepsilon b_0(\varepsilon)}{1 - K_\varepsilon \mu_0} = b_0(\varepsilon) + \frac{M_\varepsilon b_0^2(\varepsilon)}{K_\varepsilon}.$$

Using the same argument, we can see that for $t \in [2\varepsilon, 3\varepsilon]$,

$$\begin{aligned}b_2(\varepsilon) &= \frac{K_\varepsilon + M_\varepsilon b_1(\varepsilon)}{1 - K_\varepsilon \mu_0} = b_0(\varepsilon) + \frac{M_\varepsilon b_0(\varepsilon) b_1(\varepsilon)}{K_\varepsilon} \\ &= b_0(\varepsilon) + \frac{M_\varepsilon b_0^2(\varepsilon)}{K_\varepsilon} + \frac{M_\varepsilon^2 b_0^3(\varepsilon)}{K_\varepsilon^2}.\end{aligned}$$

Inductively, for $t \in [n\varepsilon, (n+1)\varepsilon]$ with n an integer such that $(n+1)\varepsilon \leqslant T$, we obtain

$$\|x_t - y_t\|_{\mathcal{B}} \leqslant b_n(\varepsilon) \sup_{0\leqslant s\leqslant t} \left|g(s)\right|, \quad \text{with } b_n(\varepsilon) = b_0(\varepsilon) \sum_{p=0}^{n} \frac{b_0^p(\varepsilon)}{K_\varepsilon^p} M_\varepsilon^p.$$

Then, the inequality (28) holds for any $T > 0$. This completes the proof of the lemma. □

Proof of Theorem 12. It suffices to show that for each $\varphi \in \mathcal{B}_{\mathcal{D}}$, $t > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\|U(t)(\varphi) - T(t)\varphi\right\|_{\mathcal{B}} \leqslant \varepsilon \|\varphi\|_{\mathcal{B}} \quad \text{for } \|\varphi\|_{\mathcal{B}} \leqslant \delta.$$

We have

$$\begin{aligned}&(\mathcal{D}+G)\big(U(t)(\varphi)\big) - \mathcal{D}\big(T(t)\varphi\big) \\ &\quad = \frac{d}{dt}\int_0^t S(t-s)\big[F\big(U(s)(\varphi)\big) - F\big(T(s)\varphi\big)\big]\,ds \\ &\qquad + S'(t)G(\varphi) + \frac{d}{dt}\int_0^t S(t-s)\big[F\big(T(s)\varphi\big) - L\big(T(s)\varphi\big)\big]\,ds.\end{aligned}$$

Then,

$$\begin{aligned}&(\mathcal{D}+G)\big(U(t)(\varphi)\big) - (\mathcal{D}+G)\big(T(t)\varphi\big) \\ &\quad = S'(t)G(\varphi) - G\big(T(t)\varphi\big) + \frac{d}{dt}\int_0^t S(t-s)\big[F\big(U(s)(\varphi)\big) - F\big(T(s)\varphi\big)\big]\,ds\end{aligned}$$

$$+\frac{d}{dt}\int_0^t S(t-s)\big[F\big(T(s)\varphi\big)-L\big(T(s)\varphi\big)\big]\,ds.$$

Let $x:(-\infty,+\infty)\to E$, $y:(-\infty,+\infty)\to E$ and $g:[0,+\infty)\to E$ be defined by

$$x(t)=\begin{cases}(U(t)(\varphi))(0) & \text{if } t\geqslant 0,\\ \varphi(t) & \text{if } t\in(-\infty,0],\end{cases}\qquad y(t)=\begin{cases}(T(t)\varphi)(0) & \text{if } t\geqslant 0,\\ \varphi(t) & \text{if } t\in(-\infty,0],\end{cases}$$

and

$$g(t)=S'(t)G(\varphi)-G\big(T(t)\varphi\big)+\frac{d}{dt}\int_0^t S(t-s)\big[F\big(U(s)(\varphi)\big)-F\big(T(s)\varphi\big)\big]\,ds$$
$$+\frac{d}{dt}\int_0^t S(t-s)\big[F\big(T(s)\varphi\big)-L\big(T(s)\varphi\big)\big]\,ds.$$

Then,

$$\begin{cases}(\mathcal{D}+G)(x_t)-(\mathcal{D}+G)(y_t)=g(t), & t\geqslant 0,\\ x_0=y_0=\varphi.\end{cases}$$

Using Lemma 13, we obtain $\|x_t-y_t\|_{\mathcal{B}}\leqslant b(t)\sup_{0\leqslant s\leqslant t}|g(s)|$. By virtue of the continuous differentiability of G and F at 0 and (vi) of Proposition 11, we deduce that for $\varepsilon>0$, there exists $\delta>0$ such that

$$\big|G\big(T(s)\varphi\big)-S'(t)G(\varphi)\big|\leqslant\varepsilon\|\varphi\|_{\mathcal{B}}$$

and

$$\int_0^t e^{-\omega s}\big|F\big(T(s)\varphi\big)-L\big(T(s)\varphi\big)\big|\,ds\leqslant\varepsilon\|\varphi\|_{\mathcal{B}}\quad\text{for } \|\varphi\|_{\mathcal{B}}\leqslant\delta.$$

Consequently,

$$\big|g(t)\big|\leqslant\bar{M}e^{\bar{\omega}t}\left(2\varepsilon\|\varphi\|_{\mathcal{B}}+\beta_0\int_0^t e^{-\bar{\omega}s}\big\|U(s)(\varphi)-T(s)\varphi\big\|_{\mathcal{B}}\,ds\right)$$

for $\bar{M}\geqslant 1$ and $\bar{\omega}\geqslant 0$ well chosen. It follows that

$$\big\|U(t)(\varphi)-T(t)\varphi\big\|_{\mathcal{B}}\leqslant b(t)\bar{M}e^{\bar{\omega}t}\left(2\varepsilon\|\varphi\|_{\mathcal{B}}+\beta_0\int_0^t e^{-\bar{\omega}s}\big\|U(s)(\varphi)-T(s)\varphi\big\|_{\mathcal{B}}\,ds\right).$$

By Gronwall's lemma, we obtain

$$\big\|U(t)(\varphi)-T(t)\varphi\big\|_{\mathcal{B}}\leqslant 2b(t)\bar{M}\varepsilon\|\varphi\|_{\mathcal{B}}\exp\big[\big(b(t)\bar{M}\beta_0+\bar{\omega}\big)t\big].$$

We conclude that $U(t)$ is differentiable at 0 and $D_\varphi U(t)(0)=T(t)$ for each $t\geqslant 0$. □

As in [23, Theorem 5.2, p. 281], for ordinary NFDE and in [10] for PNFDEs with finite delay, we have the following result.

Theorem 14. *Under the same assumptions as in Theorem* 12, *if the zero equilibrium of* $(T(t))_{t\geqslant 0}$ *is exponentially stable, then the zero equilibrium of* $(U(t))_{t\geqslant 0}$ *is locally exponentially stable in the sense that there exist* $\delta > 0, \mu > 0$ *and* $k \geqslant 1$ *such that*

$$\left\| U(t)(\varphi) \right\|_{\mathcal{B}} \leqslant k e^{-\mu t} \|\varphi\|_{\mathcal{B}} \quad \text{for } t \geqslant 0 \text{ and } \varphi \in \mathcal{B}_{\mathcal{D}} \text{ with } \|\varphi\|_{\mathcal{B}} \leqslant \delta.$$

The proof of this theorem is based on Proposition 11, Theorem 12 and the following result.

Theorem 15 (Desch and Schappacher [18]). *Let* $(V(t))_{t\geqslant 0}$ *be a nonlinear strongly continuous semigroup on a subset* Ω *of a Banach space* X. *Assume that* $x_0 \in \Omega$ *is an equilibrium of* $(V(t))_{t\geqslant 0}$ *such that* $V(t)$ *is Fréchet-differentiable at* x_0 *for each* $t \geqslant 0$, *with* $W(t)$ *the Fréchet-derivative at* x_0 *of* $V(t)$, $t \geqslant 0$. *Then,* $(W(t))_{t\geqslant 0}$ *is a strongly continuous semigroup of bounded linear operators on* E. *Moreover, if the zero equilibrium of* $(W(t))_{t\geqslant 0}$ *is exponentially stable, then* x_0 *is a locally exponentially stable equilibrium of* $(V(t))_{t\geqslant 0}$.

5. Application

To apply our previous results, we consider a special nonlinear case of the model (4),

$$\begin{cases} \frac{\partial}{\partial t}\left[v(t,\xi) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,\xi))\,d\theta\right] \\ \quad = \frac{\partial^2}{\partial \xi^2}\left[v(t,\xi) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,\xi))\,d\theta\right] + \int_{-\infty}^{0} K_2(\theta, v(t+\theta,\xi))\,d\theta, \\ \quad t \geqslant 0,\ 0 \leqslant \xi \leqslant 1, \\ v(t,0) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,0))\,d\theta = 0, \quad t \geqslant 0, \\ v(t,1) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,1))\,d\theta = 0, \quad t \geqslant 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \quad -\infty < \theta \leqslant 0,\ 0 \leqslant \xi \leqslant 1, \end{cases} \tag{29}$$

where $K_1, K_2 : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ and $v_0 : (-\infty, 0] \times [0,1] \to \mathbb{R}$ are continuous functions. We choose $E := C([0,1];\mathbb{R})$ endowed with the uniform norm topology and we consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \left\{ y \in C^2\left([0,1],\mathbb{R}\right) :\ y(0) = y(1) = 0 \right\} \quad \text{and} \quad Ay = y''.$$

It is well known (see [16]) that the operator A satisfies condition (H1) with $(0,+\infty) \subset \rho(A)$, $\|(\lambda I - A)^{-1}\| \leqslant 1/\lambda$ for $\lambda > 0$, and

$$\overline{D(A)} = \left\{ y \in E :\ y(0) = y(1) = 0 \right\} \neq E.$$

For the choice of a concrete phase space $\mathcal{B}$, we define for a positive constant γ the following standard space:

$$C_\gamma := \left\{ \phi : (-\infty, 0] \to E \text{ continuous such that } \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E \right\}.$$

Lemma 16 ([26] and [30]). *C_γ with the norm $\|\phi\|_\gamma = \sup_{\theta \leqslant 0}(e^{\gamma\theta}|\phi(\theta)|)$, $\phi \in C_\gamma$, satisfies the axioms* (A), (A1), (B), (C1) *and* (D) *with* $K(0) = 1$.

We define, for all $\xi \in [0,1]$ and $\phi \in C_\gamma$,

$$G(\phi)(\xi) = \int_{-\infty}^{0} K_1\big(\theta, \phi(\theta)(\xi)\big)\, d\theta \quad \text{and} \quad F(\phi)(\xi) = \int_{-\infty}^{0} K_2\big(\theta, \phi(\theta)(\xi)\big)\, d\theta.$$

If we put

$$\begin{cases} x(t)(\xi) = v(t,\xi), & t \in \mathbb{R},\ \xi \in [0,1], \\ \varphi(\theta)(\xi) = v_0(\theta,\xi), & \theta \leqslant 0,\ \xi \in [0,1], \end{cases}$$

Eq. (29) takes the following autonomous abstract form:

$$\begin{cases} \frac{d}{dt}[x(t) - G(x_t)] = A[x(t) - G(x_t)] + F(x_t), & t \geqslant 0, \\ x_0 = \varphi \in C_\gamma. \end{cases} \tag{30}$$

To study the existence of solutions of Eq. (30), we make the following assumptions.

(i) For each $i = 1,2$, $\theta \leqslant 0$ and $\zeta_1, \zeta_2 \in \mathbb{R}$, $|K_i(\theta,\zeta_1) - K_i(\theta,\zeta_2)| \leqslant k_i(\theta)|\zeta_1 - \zeta_2|$, where k_1, k_2 are measurable nonnegative functions on $(-\infty, 0]$ such that

$$\int_{-\infty}^{0} e^{-\gamma\theta} k_1(\theta)\, d\theta < 1 \quad \text{and} \quad \int_{-\infty}^{0} e^{-\gamma\theta} k_2(\theta)\, d\theta < \infty;$$

(ii) $\lim_{\theta \to -\infty} e^{\gamma\theta} v_0(\theta,\xi)$ exists uniformly for $\xi \in [0,1]$;
(iii) $v_0(0,.) - \int_{-\infty}^{0} K_1(\theta, v_0(\theta,.))\, d\theta \in \overline{D(A)}$.

Assumption (i) implies that, for $\phi_1, \phi_2 \in C_\gamma$,

$$\sup_{0 \leqslant \xi \leqslant 1} \big|G(\phi_1)(\xi) - G(\phi_2)(\xi)\big| \leqslant \left(\int_{-\infty}^{0} e^{-\gamma\theta} k_1(\theta)\, d\theta \right) \|\phi_1 - \phi_2\|_\gamma$$

and

$$\sup_{0 \leqslant \xi \leqslant 1} \big|F(\phi_1)(\xi) - F(\phi_2)(\xi)\big| \leqslant \left(\int_{-\infty}^{0} e^{-\gamma\theta} k_2(\theta)\, d\theta \right) \|\phi_1 - \phi_2\|_\gamma.$$

Consequently, (H2) and (H3) are true.

Assumption (iii) is true, for example, if $v_0(.,0) = v_0(.,1) = 0$ and $K_1(.,0) = 0$.

Also, (ii) and (iii) imply, respectively, that $\varphi \in C_\gamma$ and $\varphi(0) - G(\varphi) \in \overline{D(A)}$. In definitive, all conditions of Theorem 7 are satisfied. This proves the existence of a unique integral solution x of Eq. (30). To assert that x is a strict solution, we have to make more assumptions.

(iv) For each $i = 1, 2$, K_i is C^2-smooth and the second derivative of K_i with respect to the second variable satisfies the following estimate:

$$\left|\frac{\partial^2}{\partial \zeta^2} K_i(\theta, \zeta)\right| \leqslant \tilde{\beta}_i(\theta)|\zeta| \quad \text{for } \theta \leqslant 0 \text{ and } \zeta \in \mathbb{R},$$

where $\tilde{\beta}_i$ is a measurable nonnegative function on $(-\infty, 0]$ such that

$$\int_{-\infty}^{0} e^{-3\gamma\theta} \tilde{\beta}_i(\theta)\, d\theta < \infty.$$

By this assumption, F and G are continuously differentiable and satisfy, for $\phi, \psi \in C_\gamma$ and $\xi \in [0, 1]$,

$$\begin{cases} G'(\phi)(\psi)(\xi) = \int_{-\infty}^{0} \frac{\partial}{\partial \zeta} K_1(\theta, \phi(\theta)(\xi))(\psi)(\theta)(\xi)\, d\theta, \\ F'(\phi)(\psi)(\xi) = \int_{-\infty}^{0} \frac{\partial}{\partial \zeta} K_2(\theta, \phi(\theta)(\xi))(\psi)(\theta)(\xi)\, d\theta. \end{cases}$$

Moreover, as a consequence of assumption (iv), F' and G' are Lipschitz continuous in C_γ. In fact, this is a consequence of the following:

$$\int_{-\infty}^{0} \left|\frac{\partial}{\partial \zeta} K_i\big(\theta, \phi_1(\theta)(\xi)\big)(\psi)(\theta)(\xi)\, d\theta - \frac{\partial}{\partial \zeta} K_i\big(\theta, \phi_2(\theta)(\xi)\big)(\psi)(\theta)(\xi)\right| d\theta$$
$$\leqslant \int_{-\infty}^{0} e^{-2\gamma\theta} \tilde{\beta}_i(\theta)\big(e^{\gamma\theta}\big|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)\big|\big)\big(e^{\gamma\theta}\big|\psi(\theta)(\xi)\big|\big)\, d\theta,$$

which implies

$$\int_{-\infty}^{0} \left|\frac{\partial}{\partial \zeta} K_i\big(\theta, \phi_1(\theta)(\xi)\big)(\psi)(\theta)(\xi)\, d\theta - \frac{\partial}{\partial \zeta} K_i\big(\theta, \phi_2(\theta)(\xi)\big)(\psi)(\theta)(\xi)\right| d\theta$$
$$\leqslant \left(\int_{-\infty}^{0} e^{-2\gamma\theta} \tilde{\beta}_i(\theta)\, d\theta\right) \|\phi_1 - \phi_2\|_\gamma \|\psi\|_\gamma.$$

Then, (H4) is fulfilled. In addition, we assume that

(v) $(\partial/\partial\theta) v_0 \in C_\gamma$,

$$v_0(0, .) - \int_{-\infty}^{0} K_1\big(\theta, v_0(\theta, .)\big)\, d\theta \in D(A)$$

and

$$\frac{\partial}{\partial \theta} v_0(0, .) - \int_{-\infty}^{0} \frac{\partial}{\partial \zeta} K_1\big(\theta, v_0(\theta, .)\big) \frac{\partial}{\partial \theta} v_0(\theta, .)\, d\theta \in \overline{D(A)}.$$

The above condition is true if, for example, $v_0 \in C^2((-\infty, 0] \times [0, 1]; \mathbb{R})$,

$$\lim_{\theta \to -\infty} e^{\gamma\theta} \frac{\partial}{\partial\theta} v_0(\theta, \xi)$$

exists uniformly for $\xi \in [0, 1]$ and satisfies

$$\begin{cases} \frac{\partial}{\partial\theta} v_0(0, 0) - \int_{-\infty}^{0} \frac{\partial}{\partial\zeta} K_1(\theta, v_0(\theta, 0)) \frac{\partial}{\partial\theta} v_0(\theta, 0)\, d\theta = 0, \\ \frac{\partial}{\partial\theta} v_0(0, 1) - \int_{-\infty}^{0} \frac{\partial}{\partial\zeta} K_1(\theta, v_0(\theta, 1)) \frac{\partial}{\partial\theta} v_0(\theta, 1)\, d\theta = 0. \end{cases}$$

Moreover, by the dominated convergence theorem, we can see that the function

$$\frac{\partial}{\partial\theta} v_0(0, .) - \int_{-\infty}^{0} \frac{\partial}{\partial\zeta} K_1\big(\theta, v_0(\theta, .)\big) \frac{\partial}{\partial\theta} v_0(\theta, .)\, d\theta$$

is continuous on [0, 1].

Finally, condition (10) in Theorem 10 is formulated, for $\xi \in [0, 1]$, as

$$\frac{\partial}{\partial\theta} v_0(0, \xi) - \int_{-\infty}^{0} \frac{\partial}{\partial\zeta} K_1\big(\theta, v_0(\theta, \xi)\big) \frac{\partial}{\partial\theta} v_0(\theta, \xi)\, d\theta$$
$$= \frac{\partial^2}{\partial\xi^2} \left[v_0(0, \xi) - \int_{-\infty}^{0} K_1\big(\theta, v_0(\theta, \xi)\big)\, d\theta \right] + \int_{-\infty}^{0} K_2\big(\theta, v_0(\theta, \xi)\big)\, d\theta.$$

Hence, all the assumptions of Theorem 10 are satisfied. Then, we obtain that the integral solution x of Eq. (30) is strict. Consequently, the function v, defined by $v(t, \xi) = x(t)(\xi)$ for $t \geqslant 0$ and $\xi \in [0, 1]$, is a solution of Eq. (29).

Acknowledgment

The authors thank the anonymous referee for his/her remarks about the evaluation of the previous version of the manuscript. His/her valuable suggestions helped in improving the original version.

References

[1] M. Adimy, Abstract semilinear functional differential equations with nondense domain, Preprint URA 1204 Pau 95/18, 1995.

[2] M. Adimy, Integrated semigroups and delay differential equations, J. Math. Anal. Appl. 177 (1993) 125–134.

[3] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, Nonlinear Anal. 46 (2001) 91–112.

[4] M. Adimy, H. Bouzahir, K. Ezzinbi, Local existence and stability for some partial functional differential equations with infinite delay, Nonlinear Anal. 48 (2002) 323–348.

[5] M. Adimy, K. Ezzinbi, Local existence and linearized stability for partial functional differential equations, Dynam. Systems Appl. 7 (1998) 389–404.

[6] M. Adimy, K. Ezzinbi, Equations de type neutre et semi-groupes intégrés, C. R. Acad. Sci. Paris Sér. I 318 (1994) 529–534.
[7] M. Adimy, K. Ezzinbi, Semi-groupes inté grés et équations différentielles à retard en dimension infinie, C. R. Acad. Sci. Paris, Sér. I 323 (1996) 481–486.
[8] M. Adimy, K. Ezzinbi, A class of linear partial neutral functional differential equations with nondense domain, J. Differential Equations 147 (1998) 285–332.
[9] M. Adimy, K. Ezzinbi, Strict solutions of nonlinear hyperbolic neutral differential equations, Appl. Math. Lett. 12 (1999) 107–112.
[10] M. Adimy, K. Ezzinbi, Existence and linearized stability for partial neutral functional differential equations with nondense domains, Differential Equations Dynam. Systems 7 (1999) 371–417.
[11] M. Adimy, K. Ezzinbi, M. Laklach, Existence of solutions for a class of partial neutral differential equations, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 957–962.
[12] M. Adimy, K. Ezzinbi, M. Laklach, Spectral decomposition for partial neutral functional differential equations, Canad. Appl. Math. Quart. 9 (2001) 1–34.
[13] W. Arendt, Resolvent positive operators and integrated semigroups, Proc. London Math. Soc. (3) 54 (1987) 321–349.
[14] S. Busenberg, B. Wu, Convergence theorems for integrated semigroups, Differential Integral Equations 5 (1992) 509–520.
[15] B.D. Coleman, M.E. Gurtin, Equipresence and constitutive equations for rigid heat conductors, Z. Angew. Math. Phys. 18 (1967) 199–208.
[16] G. Da Prato, E. Sinestrari, Differential operators with nondense domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 14 (1987) 285–344.
[17] W. Desch, R. Grimmer, W. Schappacher, Wellposedness and wave propagation for a class of integrodifferential equations in Banach space, J. Differential Equations 74 (1988) 391–411.
[18] W. Desch, W. Schappacher, Linearized stability for nonlinear semigroups, in: A. Favini, E. Obrecht (Eds.), Differential Equations in Banach Spaces, in: Lecture Notes in Mathematics, vol. 1223, Springer-Verlag, 1986, pp. 61–73.
[19] M.E. Gurtin, A.C. Pipkin, A general theory of heat conduction with finite wave speeds, Arch. Rational Mech. Anal. 31 (1968) 113–126.
[20] J.K. Hale, Partial neutral functional differential equations, Rev. Roumaine Math. Pure Appl. 39 (1994) 339–344.
[21] J.K. Hale, Coupled oscillators on a circle, Resenhas 1 (1994) 441–457.
[22] J.K. Hale, J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978) 11–41.
[23] J.K. Hale, S. Verduyn-Lunel, Introduction to Functional Differential Equations, in: Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.
[24] E. Hernandez, H.R. Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, J. Math. Anal. Appl. 221 (1998) 452–475.
[25] E. Hernandez, H.R. Henriquez, Existence of periodic solutions for partial neutral functional differential equations with unbounded delay, J. Math. Anal. Appl. 221 (2) (1998) 499–522.
[26] Y. Hino, S. Murakami, T. Naito, Functional Differential Equations with Infinite Delay, in: Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.
[27] H. Kellermann, M. Hieber, Integrated semigroup, J. Funct. Anal. 15 (1989) 160–180.
[28] R.K. Miller, An integrodifferential equation for rigid heat conductions with memory, J. Math. Anal. Appl. 66 (1978) 313–332.
[29] T. Naito, J.S. Shin, S. Murakami, The generator of the solution semigroup for the general linear functional–differential equation, Bull. Univ. Electro-Comm. 11 (1998) 29–38.
[30] W.M. Ruess, W.H. Summers, Linearized stability for abstract differential equations with delay, J. Math. Anal. Appl. 198 (1996) 310–336.
[31] J. Wu, Theory and Applications of Partial Functional Differential Equations, in: Applied Mathematical Sciences, vol. 119, Springer-Verlag, New York, 1996.
[32] J. Wu, H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines, J. Differential Equations 124 (1996) 247–278.
[33] J. Wu, H. Xia, Rotating waves in neutral partial functional differential equations, J. Dynam. Differential Equations 1 (1999) 209–238.