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Existence for a class of partial functional differential equations with infinite delay

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1. Introduction

Let $(E, \|\,.\,\|)$ be a Banach space. Consider the following class of partial functional differential equations with infinite delay

$$\begin{aligned} &x'(t) = Ax(t) + F(t, x_t), \quad t \geq 0, \\ &x_0 = \varphi, \end{aligned} \tag{1}$$

where $A : D(A) \subseteq E \to E$ is a closed linear operator, φ is an element in a phase space $\mathscr{B}$ of functions mapping $(-\infty, 0]$ into E, which will be specified later, F is an appropriate mapping defined on $[0, +\infty) \times \mathscr{B}$ with values in E and, for each $x : (-\infty, b] \to E$ and $t \in [0, b]$, x_t represents the mapping defined from $(-\infty, 0]$ into E by

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in (-\infty, 0].$$

In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is unbounded, the selection of the state space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [10], Kappel

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and Schappacher [16], and Schumacher [22]. For a detailed discussion on this topic, we refer the reader to the book by Hino et al. [15].

In recent years, the theory of partial functional differential equations with delay has attracted widespread attention. The development was initiated for equations with finite delay and about existence and stability by Travis and Webb [25,26] and Webb [27,28]. For later development, we cite only a paper by Arino and Sanchez [5] and a recent book by Wu [29]. In the standard framework for Eq. (1), one assumes that the operator A is the infinitesimal generator of a C_0-semigroup on E. This in particular contains the density of the domain $D(A)$ in E by the Hille–Yosida theorem. More recently, it has been shown in [1,2,9] that the density condition is not necessary (in a certain sense) to deal with partial functional differential equations with finite delay. Non-density occurs in many situations due to restrictions on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or due to boundary conditions (for example, the space C^1 with null value on the boundary is non dense in the space of continuous functions) (see examples in [8,19,20]).

Concerning the case of infinite delay, an extensive theory is developed for Eq. (1) with $A=0$. We refer the reader to Hale and Kato [10], Corduneanu and Lakshmikantham [7], Hino et al. [15], Lakshmikantham et al. [18], and Shin [23]. The extension to the case when A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ was later studied by Henriquez in his three consecutive papers [11–13] and Ruan and Wu [21]. Following an axiomatic approach, he has developed several fundamental results on the existence of solutions, regularity, existence of periodic solutions and stability. Henriquez proved his results by using the following variation-of-constants formula:

$$x(t,\varphi)=\begin{cases} T(t)\varphi(0)+\displaystyle\int_0^t T(t-s)F(s,x_s)\,\mathrm{d}s & \text{for } t\geq 0,\\ \varphi(t) & \text{for } -\infty<t\leq 0.\end{cases}$$

The purpose of this paper is to state that even in the case of infinite delay, we can solve Eq. (1) without assuming necessarily that A is densely defined. We extend the results of Henriquez [11–13] to the situation where A satisfies the Hille–Yosida condition while the domain of A is not dense. Techniques employed in [1,2,9], have been generalized to the study of Eq. (1). That is, existence and uniqueness of solutions (called mild solutions) are first proved, using a fixed-point argument. Then, the mild solutions are shown to be classical solutions under more restrictive assumptions.

The main tool in the approach followed in this work is the theory of integrated semigroups. A brief reminder of this theory is provided in Section 2. We now propose a brief discussion about the advantage of using the integrated semigroups theory. In the case where the operator F of Eq. (1) is equal to zero, the problem can still be handled by using the classical semigroup theory because A generates a strongly continuous semigroup in the space $\overline{D(A)}$. But, when $F\neq 0$ it is necessary to impose additional restrictions, the simplest of which is that F takes values in $\overline{D(A)}$. It is the integrated semigroups theory that allows the range of the operator F to be a subset of E not necessarily contained in $\overline{D(A)}$.

2. Integrated semigroups and differential operators with non-dense domain

The purpose of this section is to collect some background materials required throughout this paper. These materials include integrated semigroups theory and differential operators with non-dense domain. We will only state results and for the details the reader may refer references.

The following definitions are due to Arendt [3,4].

Definition 1 (Arendt [3]). Let X be a Banach space. A family $(S(t))_{t\geq 0}\subset \mathscr{L}(X)$ is called an integrated semigroup if the following conditions are satisfied:

(i) $S(0)=0$;
(ii) for any $x\in X$, $S(t)x$ is a continuous function of $t\geq 0$ with values in X;
(iii) for any $t,s\geq 0$ $S(s)S(t)=\int_0^s (S(t+\tau)-S(\tau))\,\mathrm{d}\tau$.

Definition 2 (Arendt [3]). An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $M\geq 0$ and $\omega\in\mathbb{R}$ such that

$$\|S(t)\|\leq M\mathrm{e}^{\omega t}\quad \text{for } t\geq 0.$$

Moreover, $(S(t))_{t\geq 0}$ is called non-degenerate if $S(t)x=0$, for all $t\geq 0$, implies that $x=0$.

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda):=\lambda\int_0^{+\infty}\mathrm{e}^{-\lambda t}S(t)\,\mathrm{d}t$ exists for all λ with $\mathscr{Re}(\lambda)>\omega$. $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. $R(\lambda)$ satisfies the following expression

$$R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu)$$

and in the case when $(S(t))_{t\geq 0}$ is non-degenerate, there exists a unique operator A satisfying $(\omega,+\infty)\subset\rho(A)$ (the resolvent set of A) such that

$$R(\lambda)=R(\lambda,A):=(\lambda I-A)^{-1}\quad\text{for all } \mathscr{Re}(\lambda)>\omega.$$

This operator A is called the generator of $(S(t))_{t\geq 0}$.

We have the following general definition.

Definition 3 (Arendt [3]). An operator A is called a generator of an integrated semigroup, if there exists $\omega\in\mathbb{R}$ such that $(\omega,+\infty)\subset\rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0)=0$ and $(\lambda I-A)^{-1}=\lambda\int_0^{+\infty}\mathrm{e}^{-\lambda t}S(t)\,\mathrm{d}t$ for all $\lambda>\omega$.

Remark 1. If an operator A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$, then for all $\lambda\in\mathbb{R}$, $A-\lambda I$ is the generator of the integrated semigroup $(S_\lambda(t))_{t\geq 0}$ given by

$$S_\lambda(t)=\mathrm{e}^{-\lambda t}S(t)+\lambda\int_0^t \mathrm{e}^{-\lambda s}S(s)\,\mathrm{d}s.$$

Similar results as for semigroups can be obtained for integrated semigroups.

Proposition 4 (Arendt [3]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$,*

$$\int_0^t S(s)x\,\mathrm{d}s \in D(A) \quad \text{and} \quad S(t)x = A\left(\int_0^t S(s)x\,\mathrm{d}s\right) + tx.$$

Moreover, for all $x \in D(A)$, $t \geq 0$

$$S(t)x \in D(A), \quad AS(t)x = S(t)Ax$$

and

$$S(t)x = \int_0^t S(s)Ax\,\mathrm{d}s + tx.$$

Corollary 5 (Arendt [3]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$ one has $S(t)x \in \overline{D(A)}$. Moreover, let $x \in X$. Then $S(\cdot)x$ is right-sided differentiable in $t \geq 0$ if and only if $S(t)x \in D(A)$. In that case*

$$S'(t)x = AS(t)x + x.$$

Proposition 6 (Hieber [14]). *Let $A : D(A) \subseteq X \to X$ be a linear operator and $(S(t))_{t\geq 0} \subset \mathscr{L}(X)$ an exponentially bounded family. The following assertions are equivalent*

(i) $\int_0^t S(s)x\,\mathrm{d}s \in D(A)$ *and* $S(t)x = A(\int_0^t S(s)x\,\mathrm{d}s) + tx$ $(t \geq 0,\ x \in X)$,

(ii) $(S(t))_{t\geq 0}$ *is an integrated semigroup on X generated by A.*

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 7 (Kellermann and Hieber [17]). An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous, if for all $\tau > 0$ there exists a constant $k(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq k(\tau)|t-s| \quad \text{for all } t,s \in [0,\tau].$$

In this case, we know from [17] that $(S(t))_{t\geq 0}$ is exponentially bounded.

Definition 8 (Kellermann and Hieber [17]). We say that a linear operator A satisfies the Hille–Yosida condition (HY) if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda-\omega)^n \|R(\lambda,A)^n\|,\ n \in \mathbb{N},\ \lambda > \omega\} \leq M. \tag{HY}$$

The following theorem shows that the Hille–Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 9 (Kellermann and Hieber [17]). *The following assertions are equivalent.*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*

(ii) *A satisfies the condition* (HY).

In the sequel, we give some results for the existence of solutions of the following Cauchy problem:

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) + f(t), \quad t \geq 0, \\ &u(0) = x \in X, \end{aligned} \tag{2}$$

where A satisfies the condition (HY), without being densely defined.

By a solution of Eq. (2) on $[0, T]$ where $T > 0$, we understand a function $u \in C^1([0, T], X)$ satisfying $u(t) \in D(A)$ for $t \in [0, T]$, such that the two relations in (2) hold.

The following result is due to Da Prato and Sinestrari.

Theorem 10 (Da Prato and Sinestrari [8]). *Let* $A : D(A) \subseteq X \to X$ *be a linear operator,* $f : [0, T] \to X$, $x \in D(A)$ *such that*

(i) *A satisfies the condition* (HY),

(ii) $f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s$ *for some Bochner-integrable function* g,

(iii) $Ax + f(0) \in \overline{D(A)}$.

Then there exists a unique solution u of Eq. (2) *on the interval* $[0, T]$, *and for each* $t \in [0, T]$

$$|u(t)| \leq M\mathrm{e}^{\omega t}\left(|x| + \int_0^t \mathrm{e}^{-\omega s}|f(s)|\,\mathrm{d}s\right).$$

In the case where x is not sufficiently regular (that is, x is just in $\overline{D(A)}$) there may not exist a strong solution $u(t) \in X$ but, following the work of Da Prato and Sinestrari [8], Eq. (2) may still have an integral solution. This motives the following definition.

Definition 11 (Da Prato and Sinestrari [8]). Given $f \in L^1_{\mathrm{loc}}(0, +\infty; X)$ and $x \in X$, we say that $u : [0, +\infty) \to X$ is an integral solution of Eq. (2) if the following assertions are true

(i) $u \in C([0, +\infty); X)$,

(ii) $\int_0^t u(s)\,\mathrm{d}s \in D(A)$ for $t \geq 0$,

(iii) $u(t) = x + A(\int_0^t u(s)\,\mathrm{d}s) + \int_0^t f(s)\,\mathrm{d}s$ for $t \geq 0$.

From this definition, we deduce that for an integral solution u, we have $u(t) \in \overline{D(A)}$, for all $t > 0$, because $u(t) = \lim_{h \to 0} 1/h \int_t^{t+h} u(s)\,\mathrm{d}s$ and $\int_t^{t+h} u(s)\,\mathrm{d}s \in D(A)$. In particular, $x \in \overline{D(A)}$ is a necessary condition for the existence of an integral solution of Eq. (2).

Proposition 12 (Thieme [24]). *Let* $A : D(A) \subseteq X \to X$ *be a linear operator which satisfies the condition* (HY), $(S(t))_{t \geq 0}$ *be the integrated semigroup generated by* A *and* $G : [0, T] \to X$, $T > 0$, *be a Bochner-integrable function. Then, the function* $K : [0, T] \to X$ *defined by*

$$K(t) = \int_0^t S(t - s)G(s)\,\mathrm{d}s$$

is continuously differentiable on $[0,T]$ *and satisfies, for* $\lambda > \omega$ *and* $t \in [0,T]$,

$$R(\lambda,A)K'(t) = \int_0^t S'(t-s)R(\lambda,A)G(s)\,\mathrm{d}s.$$

This is suggestive to solve Eq. (2) by the variation-of-constants formula

$$u(t) = S'(t)x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t S(t-s)f(s)\,\mathrm{d}s\right) \quad \text{for } t \geq 0, \tag{3}$$

where $S(t)$ is the integrated semigroup generated by A.

Theorem 13 (Busenberg and Wu [6]). *Suppose that* A *satisfies the condition* (HY), $x \in \overline{D(A)}$ *and* $f : [0,+\infty) \to X$ *is a continuous function. Then the problem* (2) *has a unique integral solution which is given by* (3).

Furthermore, the function u *satisfies the inequality*

$$|u(t)| \leq M\mathrm{e}^{\omega t}\left(|x| + \int_0^t \mathrm{e}^{-\omega s}|f(s)|\,\mathrm{d}s\right) \quad \textit{for } t \geq 0.$$

3. Main results

Following the book [15], we will assume that the state space $\mathscr{B}$ is a linear space of functions mapping $(-\infty,0]$ into E, endowed with a seminorm $\|\,.\,\|_{\mathscr{B}}$ and satisfying the following fundamental axioms:

(A1) There exist a positive constant H and functions $K(\cdot), M(\cdot) : [0,+\infty) \to [0,+\infty)$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathscr{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for all t in $[\sigma, \sigma + a]$ the following conditions hold:
  (i) $x_t \in \mathscr{B}$,
  (ii) $\|x(t)\| \leq H\,\|x_t\|_{\mathscr{B}}$,
  (iii) $\|x_t\|_{\mathscr{B}} \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}\|x(s)\| + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$.
(A2) For the function $x(\cdot)$ in (A1), $t \mapsto x_t$ is a $\mathscr{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.
(B) The space $\mathscr{B}$ is complete.

Remark 2. (a) The condition (ii) is equivalent to
(ii)$'$ $\|\varphi(0)\| \leq H\,\|\varphi\|_{\mathscr{B}}$, for every $\varphi \in \mathscr{B}$.

(b) Since $\|\,.\,\|_{\mathscr{B}}$ is a seminorm, two elements ϕ, $\psi \in \mathscr{B}$ can verify $\|\phi - \psi\|_{\mathscr{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$. But, from (ii)$'$ we see that

$$\phi, \psi \in \mathscr{B} \quad \text{and} \quad \|\phi - \psi\|_{\mathscr{B}} \quad \text{implies that } \phi(0) = \psi(0).$$

Suppose that F is continuous from $[0,+\infty) \times \mathscr{B}$ with values in E. The initial value problem associated with Eq. (1) is the following: given $\varphi \in \mathscr{B}$, to find a continuous function $x : (-\infty, a] \to E$, $a > 0$, differentiable on $[0,a]$ such that $x(t) \in D(A)$, for

$t \in [0,a]$ and x satisfies Eq. (1), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + F(t,x_t), \quad t \in [0,a],$$
$$x(t) = \varphi(t), \quad -\infty < t \leq 0.$$

In [13], the following result has been shown:

Theorem 14 (Henriguez [13]). *Assuming that A is the generator of a strongly continuous semigroup $(U(t))_{t\geq 0}$ on E, and $F : [0,a] \times \mathscr{B} \to E$ is a continuous mapping in t and uniformly Lipschitz continuous on $\mathscr{B}$. Then, for given $\varphi \in \mathscr{B}$, the Cauchy problem* (1) *has exactly one mild solution which is given by*

$$x(t) = \begin{cases} U(t)\varphi(0) + \displaystyle\int_0^t U(t-s)F(s,x_s)\,\mathrm{d}s, & 0 \leq t \leq a, \\ \varphi(t), & -\infty < t \leq 0. \end{cases}$$

We start by introducing the following definitions:

Definition 15. Let $\varphi \in \mathscr{B}$. We say that a function $x : (-\infty,a] \to E$, $a > 0$, is an integral solution of Eq. (1) in $(-\infty,a]$ if the following conditions hold:

(i) x is continuous on $[0,a]$;

(ii) $\int_0^t x(s)\,\mathrm{d}s \in D(A)$, for $t \in [0,a]$;

(iii) $x(t) = \begin{cases} \varphi(0) + A\displaystyle\int_0^t x(s)\,\mathrm{d}s + \int_0^t F(s,x_s)\,\mathrm{d}s, & 0 \leq t \leq a, \\ \varphi(t), & -\infty < t \leq 0. \end{cases}$

Definition 16. Let $\varphi \in \mathscr{B}$. We say that a function $x : (-\infty,a] \to E$ is a strict solution of Eq. (1) in $(-\infty,a]$ if the following conditions hold:

(i) $x \in C^1([0,a];E) \cap C([0,a];D(A))$;

(ii) x satisfies Eq. (1) on $(-\infty,a]$.

Remark 3. From the closedness property of the operator A, we can see the following statements:

(i) If x is an integral solution of Eq. (1) in $(-\infty,a]$, then for all $t \in [0,a]$, $x(t) \in \overline{D(A)}$. In particular $\varphi(0) \in \overline{D(A)}$.

(ii) If x is an integral solution of Eq. (1) in $(-\infty,a]$, such that $x : [0,a] \to E$ belongs to $C^1([0,a];E)$ or $C([0,a];D(A))$, then x is also a strict solution of Eq. (1) in $(-\infty,a]$.

3.1. Local existence of integral solutions

Let Ω be a nonempty open subset of $\mathscr{B}$. We assume the following.

(H1) $A : D(A) \subseteq E \to E$ is a Hille–Yosida operator, i.e., there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda-\omega)^n \|R(\lambda,A)^n\|,\ n \in \mathbb{N},\ \lambda > \omega\} \leq \overline{M}.$$

Observe that Theorem 9 implies that, under this assumption, A is the generator of a locally Lipschitz-continuous integrated semigroup $(S(t))_{t\geq 0}$ on E. In addition, $S'(t) : \overline{D(A)} \to \overline{D(A)}$ is a C_0-semigroup satisfying

$$\|S'(t)y\| \leq \overline{M}\mathrm{e}^{\omega t}\|y\| \quad \text{for all } t \geq 0 \quad \text{and} \quad y \in \overline{D(A)}.$$

We need also the following compactness property of $(S'(t))_{t\geq 0}$.
(H2) The semigroup $(S'(t))_{t\geq 0}$ is compact on $(\overline{D(A)}, \|\cdot\|)$.

Theorem 17. *Let* $F : [0,a] \times \Omega \to E$ *be a continuous mapping and suppose that the conditions* (H1) *and* (H2) *hold. If* $\varphi \in \Omega$ *with* $\varphi(0) \in \overline{D(A)}$, *then Eq.* (1) *has an integral solution* $x : (-\infty, b] \to E$, *for some* $b \in (0,a]$. *Moreover*,

$$x(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t S(t-s)F(s,x_s)\,\mathrm{d}s, & 0 \leq t \leq b, \\ \varphi(t), & -\infty < t \leq 0. \end{cases}$$

Proof. Let $\varphi \in \Omega$ with $\varphi(0) \in \overline{D(A)}$. The main idea of the proof is to use the Schauder fixed-point theorem.

First, there exist constants $r_1 > 0$ and $\mu \geq 0$ such that $\overline{B_{r_1}(\varphi)} = \{\psi \in \mathscr{B}\colon \|\psi - \varphi\|_{\mathscr{B}} \leq r_1\} \subseteq \Omega$ and $\|F(s,\psi)\| \leq \mu$ for all $s \in [0, r_1]$ and $\psi \in \overline{B_{r_1}(\varphi)}$. Consider the function $y : (-\infty, +\infty) \to E$ defined by

$$y(t) = \begin{cases} S'(t)\varphi(0) & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \leq 0. \end{cases}$$

By virtue of Axioms (A1)(i) and (A2), $y_t \in \mathscr{B}$ and for $r_2 \in\,]0, r_1[$, there exists $b_1 \in\,]0, r_1]$ such that $\|y_t - \varphi\|_{\mathscr{B}} \leq r_2$ for all $t \in [0, b_1]$.

Set $K_a := \max_{0\leq t\leq a} K(t)$, $M_a := \sup_{0\leq t\leq a} M(t)$ and $\overline{M}_a := \sup_{0\leq s\leq a} \|S'(s)\|_{\overline{D(A)}}$ and, let b be a constant such that

$$0 < b \leq \min\left\{b_1, \frac{r_1 - r_2}{\overline{M}\,\overline{M}_a K_a \mu}\right\}.$$

We introduce the space

$$\mathbb{F}_b := \{u : (-\infty, b] \to E \text{ such that } u_0 \in \mathscr{B} \text{ and} \\ \text{the restriction } u : [0,b] \to E \text{ is continuous}\},$$

endowed with the seminorm $\|\cdot\|_{\mathbb{F}_b}$, defined by

$$\|u\|_{\mathbb{F}_b} := \|u_0\|_{\mathscr{B}} + \sup_{0\leq s\leq b} \|u(s)\|.$$

We set also, for $\varphi \in \mathscr{B}$, the following subset of $\mathbb{F}_b$

$$\mathbb{F}_b(\varphi) := \{u \in \mathbb{F}_b \text{ such that } \|u_0 - \varphi\|_{\mathscr{B}} = 0 \text{ and} \\ \text{for any } t \in [0,b],\ \|u_t - \varphi\|_{\mathscr{B}} \leq r_1\}.$$

It is clear that the restriction of y to $(-\infty, b]$ is an element of $\mathbb{F}_b(\varphi)$. Then, $\mathbb{F}_b(\varphi)$ is a nonempty.

For any $u \in \mathbb{F}_b(\varphi)$, we have

$$\begin{aligned}\|u\|_{\mathbb{F}_b} &= \|u_0\|_{\mathscr{B}} + \sup_{0\le s\le b} \|u(s)\|,\\ &\le \|u_0-\varphi\|_{\mathscr{B}} + \|\varphi\|_{\mathscr{B}} + \sup_{0\le s\le b} H\|u_s\|_{\mathscr{B}},\\ &\le \|\varphi\|_{\mathscr{B}} + H \sup_{0\le s\le b}\{\|(u_s-\varphi)+\varphi\|_{\mathscr{B}}\},\\ &\le \|\varphi\|_{\mathscr{B}} + H\left\{\sup_{0\le s\le b}\|(u_s-\varphi)\|_{\mathscr{B}} + \|\varphi\|_{\mathscr{B}}\right\},\\ &\le \|\varphi\|_{\mathscr{B}} + H(r_1+\|\varphi\|_{\mathscr{B}}).\end{aligned}$$

Then, $\mathbb{F}_b(\varphi)$ is bounded.

By using the triangular inequality in $\mathscr{B}$, it is clear that $\alpha y_1 + (1-\alpha)y_2 \in \mathbb{F}_b(\varphi)$, for any y_1, $y_2 \in \mathbb{F}_b(\varphi)$ and $\alpha \in]0,1[$. Then, $\mathbb{F}_b(\varphi)$ is convex.

Finally, $\mathbb{F}_b(\varphi)$ is closed in $\mathbb{F}_b$. To prove that, consider a convergent sequence $(u^n)_{n\ge 0}$ of $\mathbb{F}_b(\varphi)$ with $\lim_{n\to+\infty} u^n = u$ in $\mathbb{F}_b$. Then, for any n in $\mathbb{N}$, we have

$$\|u_0-\varphi\|_{\mathscr{B}} \le \|u_0-u_0^n\|_{\mathscr{B}} + \|u_0^n-\varphi\|_{\mathscr{B}},$$

letting n to $+\infty$, yields $\|u_0-\varphi\|_{\mathscr{B}} = 0$. In addition, Axiom (A1)(iii) implies that for any $t \in [0,b]$,

$$\begin{aligned}\|u_t^n-u_t\|_{\mathscr{B}} &\le K(t)\sup_{0\le s\le t}\|u^n(s)-u(s)\| + M(t)\|u_0^n-u_0\|_{\mathscr{B}}\\ &\le \max(K_a,M_a)\|u^n-u\|_{\mathbb{F}_b}.\end{aligned}$$

Then, for any $t\in[0,b]$, $u_t^n \to u_t$ in $\mathscr{B}$. From this together with the following inequality

$$\begin{aligned}\|u_t-\varphi\|_{\mathscr{B}} &\le \|u_t-u_t^n\|_{\mathscr{B}} + \|u_t^n-\varphi\|_{\mathscr{B}}\\ &\le \|u_t-u_t^n\|_{\mathscr{B}} + r_1\end{aligned} \qquad \text{for any } n\in\mathbb{N},$$

we deduce that $\|u_t-\varphi\|_{\mathscr{B}} \le r_1$. Consequently, $u \in \mathbb{F}_b(\varphi)$.

We have proved that $\mathbb{F}_b(\varphi)$ is a nonempty, bounded, convex and closed subset of $\mathbb{F}_b$.

Consider now the mapping $\mathscr{T}$ defined on $\mathbb{F}_b(\varphi)$ by

$$(\mathscr{T}u)(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t S(t-s)F(s,u_s)\,\mathrm{d}s, & t\in[0,b],\\ \varphi(t), & -\infty < t \le 0.\end{cases} \tag{4}$$

$\mathscr{T}$ maps elements of $\mathbb{F}_b(\varphi)$ into $\mathbb{F}_b(\varphi)$. In fact, by (H2) and Axiom (A2), for every $u \in \mathbb{F}_b(\varphi)$, the mapping $s \mapsto F(s,u_s)$ is continuous on $[0,b]$. So, for every $u\in\mathbb{F}_b(\varphi)$, the mapping $t\mapsto \int_0^t S(t-s)F(s,u_s)\,\mathrm{d}s$ is continuously differentiable on $[0,b]$. From this, the mapping $v := \mathscr{T}u$ is continuous on $[0,b]$. Then, $v\in\mathbb{F}_b$. To prove that $v\in\mathbb{F}_b(\varphi)$, we put $w = v - y$. Then, we get for any $t \in [0,b]$,

$$\begin{aligned}\|v_t - \varphi\|_{\mathscr{B}} &\le \|w_t\|_{\mathscr{B}} + \|y_t - \varphi\|_{\mathscr{B}}\\ &\le \|w_t\|_{\mathscr{B}} + r_2.\end{aligned}$$

By Axiom (A1)(iii), we have for any $t \in [0,b]$,

$$\|w_t\|_{\mathscr{B}} \le K_a \sup_{0\le s\le t} \|w(s)\|.$$

From Proposition 12, we obtain for $t \in [0,b]$

$$\begin{aligned}\|\lambda R(\lambda,A)w(t)\| &= \left\| \int_0^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}t \right\|\\ &\le \frac{\lambda}{\lambda-\omega}\overline{M}\,\overline{M}_a\mu b\\ &\le \frac{\lambda}{\lambda-\omega}\left(\frac{r_1-r_2}{K_a}\right).\end{aligned}$$

Letting $\lambda \to +\infty$, we obtain $\|w(t)\| \le (r_1 - r_2)/K_a$, and $\|v_t - \varphi\|_{\mathscr{B}} \le r_1$ for any $t \in [0,b]$, which shows that $v \in \mathbb{F}_b(\varphi)$.

We will prove now the continuity of $\mathscr{T}$. Axioms (A2) and (A1)(iii) imply that the mapping $G : [0,b] \times \mathbb{F}_b \to \mathscr{B}$, defined by $G(s,u) = u_s$ is continuous. On the other hand, if $(u^n)_{n\ge 1}$ is a convergent sequence in $\mathbb{F}_b(\varphi)$ with $\lim_{n\to+\infty} u^n = u^\infty$, then the set $\{u^\infty\} \cup \{u^n\colon n \ge 1\}$ is compact in $\mathbb{F}_b$. Hence, the set $W = \{(s,u_s^\infty)\colon 0 \le s \le b\} \cup \{(s,u_s^n)\colon n \ge 1,\ 0 \le s \le b\}$ is compact in $[0,b]\times\mathscr{B}$ and F is uniformly continuous in W. Since $(u^n)_{n\ge 1}$ converges to u^∞, (4) implies that $(\mathscr{T}u^n)_{n\ge 1}$ converges to $\mathscr{T}u^\infty$.

We will show now that the range of $\mathscr{T}$, *Range*$(\mathscr{T}) := \{\mathscr{T}u,\ u \in \mathbb{F}_b(\varphi)\}$, is relatively compact in $\mathbb{F}_b(\varphi)$. By the Arzela–Ascoli theorem, it suffices to prove that *Range*$(\mathscr{T})(t)$ is relatively compact in E for each $t \in [0,b]$, and *Range*$(\mathscr{T})$ is equicontinuous on $[0,b]$. To prove the first assertion, it is sufficient to show that the set $\{(\mathscr{T}u)(t) - S'(t)\varphi(0),\ u \in \mathbb{F}_b(\varphi)\}$ is relatively compact for each $t \in \,]0,b]$. Let $0 < \varepsilon < t$, we have for $\lambda > \omega$

$$\begin{aligned}&\lambda R(\lambda,A)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S'(t-s)F(s,u_s)\,\mathrm{d}s\\ &\quad= \int_0^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s,\\ &\quad= S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-s-\varepsilon)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\\ &\qquad+ \int_{t-\varepsilon}^t S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s.\end{aligned}$$

Since $S'(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$\left\{S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-s-\varepsilon)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\colon u\in \mathbb{F}_b(\varphi),\ \lambda>\omega\right\}\subseteq W_\varepsilon.$$

Furthermore,

$$\left\|\int_{t-\varepsilon}^{t} S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\right\| \le \frac{\lambda}{\lambda-\omega}\overline{M}M_a\mu\varepsilon \quad \text{for all } u\in\mathbb{F}_b(\varphi).$$

This implies, by letting λ to $+\infty$, that $\{(\mathscr{T}u)(t)-S'(t)\varphi(0),\ u\in\mathbb{F}_b(\varphi)\}$ is totally bounded, and $Range(\mathscr{T})(t)$ is relatively compact.

On the other hand, for every $0\le t_0<t\le b$ and $\lambda>\omega$, one has

$$\begin{aligned}\lambda R(\lambda,A)((\mathscr{T}u)(t)-(\mathscr{T}u)(t_0)) &= \lambda R(\lambda,A)(S'(t)-S'(t_0))\varphi(0)\\ &\quad+\int_{t_0}^{t} S'(t-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\\ &\quad+\int_0^{t_0}(S'(t-s)-S'(t_0-s))\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s.\end{aligned}$$

This leads to

$$\begin{aligned}&\|\lambda R(\lambda,A)((\mathscr{T}u)(t)-(\mathscr{T}u)(t_0))\|\\ &\quad\le \|\lambda R(\lambda,A)(S'(t)-S'(t_0))\varphi(0)\| + \frac{\lambda}{\lambda-\omega}\overline{M}M_a\mu(t-t_0)\\ &\qquad+\left\|(S'(t-t_0)-I)\int_0^{t_0} S'(t_0-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\right\|. \qquad (5)\end{aligned}$$

Besides, there exists a compact set W such that

$$\left\{\int_0^{t_0} S'(t_0-s)\lambda R(\lambda,A)F(s,u_s)\,\mathrm{d}s\colon u\in\mathbb{F}_b(\varphi),\ \lambda>\omega\right\}\subseteq W.$$

Letting λ to $+\infty$ and using the equicontinuity of $(S'(\cdot)x)_{x\in W}$, we obtain

$$\lim_{t\to t_0}\|(\mathscr{T}u)(t)-(\mathscr{T}u)(t_0)\|=0. \qquad (6)$$

Using similar argument for $0\le t<t_0\le b$, we can conclude that $Range(\mathscr{T})(t)$ is equicontinuous.

Finally, the Schauder fixed-point theorem implies that $\mathscr{T}$ has a fixed point x in $\mathbb{F}_b(\varphi)$. The fact that x is an integral solution of Eq. (1) is a consequence of the same property in Theorem 13. This ends the proof of Theorem 17. □

In order to define the solution to Eq. (1) in its maximal interval of existence, we will assume the following condition:

(H3) F is a continuous mapping from $[0,+\infty)\times\mathscr{B}$ into E and takes bounded sets of $[0,+\infty)\times\mathscr{B}$ into bounded sets of E.

Theorem 18. *Assume that the conditions* (H1)–(H3) *hold and let* $\varphi \in \mathscr{B}$ *such that* $\varphi(0) \in \overline{D(A)}$. *Then, Eq.* (1) *has a unique integral solution* $x(.,\varphi)$ *in a maximal interval* $(-\infty, b_\varphi)$ *and either* $b_\varphi = +\infty$ *or* $\limsup_{t\to b_\varphi} \|x(t,\varphi)\| = +\infty$.

Proof. Let $x(.,\varphi)$ be an integral solution of Eq. (1) in $(-\infty, b_1]$. We know that $x(t) \in \overline{D(A)}$, for all $t \in [0, b_1]$. Repeating the procedure used in the local existence result, yields existence of $b_2 > b_1$ and a function $x(.,x_{b_1}(.,\varphi)) : (-\infty, b_2] \to E$ which satisfies for $t \in [b_1, b_2]$

$$x(t, x_{b_1}(.,\varphi)) = S'(t-b_1)x(b_1,\varphi) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{b_1}^{t} S(t-s)F(s, x_s(.,x_{b_1}(.,\varphi)))\,\mathrm{d}s.$$

Proceeding inductively, we obtain the maximal interval of existence $(-\infty, b_\varphi)$ of the solution $x(.,\varphi)$. Assume that $b_\varphi < \infty$ and $\limsup_{t\to b_\varphi} \|x(t,\varphi)\|$ is bounded. Then, from (H3) there exists a constant $\mu > 0$ such that $\|F(t, x_t(.,\varphi))\| \le \mu$ for all $t \in [0, b_\varphi)$. As before, Let $\overline{M}_{b_\varphi} = \sup_{0\le s\le b_\varphi} \|S'(s)\|_{\overline{D(A)}}$ and $x : [t_0, b_\varphi) \to E$, $t_0 \in (0, b_\varphi)$, be the restriction of $x(.,\varphi)$ to $[t_0, b_\varphi)$. Consider $t_0 \le t \le t' < b_\varphi$ and $0 < \varepsilon < t_0$. Let $\eta_1 > 0$ be such that $\|(S'(t') - S'(t))\varphi(0)\| \le \varepsilon$ for $|t-t'| < \eta_1$. For $\lambda > \omega$, one has

$$\begin{aligned}
&\|\lambda R(\lambda,A)(x(t') - x(t))\| \\
&\quad \le \|\lambda R(\lambda,A)(S'(t') - S'(t))\varphi(0)\| \\
&\qquad + \int_{t-\varepsilon}^{t} \|(S'(t'-s) - S'(t-s))\lambda R(\lambda,A)F(s,x_s)\|\,\mathrm{d}s \\
&\qquad + \int_{t}^{t'} S'(t'-s)\lambda R(\lambda,A)F(s,x_s)\,\mathrm{d}s \\
&\qquad + \int_{0}^{t-\varepsilon} \|(S'(t'-\varepsilon-s) - S'(t-\varepsilon-s))S'(\varepsilon)\lambda R(\lambda,A)F(s,x_s)\|\,\mathrm{d}s.
\end{aligned}$$

Moreover, $S'(\varepsilon)\{\lambda R(\lambda,A)F(s,x_s)\colon \lambda > \omega\}$ is included in a compact set W. Since, the family $(S'(\cdot)x)_{x\in W}$ is uniformly equicontinuous in $[0, b_\varphi - \varepsilon]$, there exists $\eta_2 > 0$ such that

$$\|(S'(t') - S'(t))u\| < \varepsilon \quad \text{for } u \in W,$$

whenever $t, t' \in [0, b_\varphi - \varepsilon]$ and $|t - t'| < \eta_2$.

Consequently, if $|t' - t| < \inf(\eta_1, \eta_2, \varepsilon)$, then

$$\|\lambda R(\lambda,A)(x(t') - x(t))\| \le \frac{\lambda}{\lambda-\omega}\overline{M}(\varepsilon + 3\overline{M}_{b_\varphi}\varepsilon\mu) + (t-\varepsilon)\varepsilon.$$

By letting λ to $+\infty$, we obtain

$$\|x(t') - x(t)\| \le (\overline{M} + 3\overline{M}\overline{M}_{b_\varphi}\mu + b_\varphi)\varepsilon.$$

Using a similar argument for $t_0 \le t' \le t < b_\varphi$, we can conclude that

$$\lim_{|t-t'|\to 0} \|x(t') - x(t)\| = 0.$$

Therefore, $\lim_{t\to b_\varphi} x(t,\varphi)$ exists. If we define $x(b_\varphi,\varphi) := \lim_{t\to b_\varphi} x(t,\varphi)$, we can extend $x(.,\varphi)$ beyond b_φ and we contradict the maximality of $(-\infty, b_\varphi)$. This finishes the proof. □

3.2. Global existence and uniqueness of the integral solution

Our objective here is to give sufficient conditions for global existence and uniqueness of integral solution of Eq. (1). We keep the assumption (H1) and instead of the assumptions (H2) and (H3), we make the following condition:

(H4) $F : [0,+\infty)\times\mathscr{B}\to E$ is continuous and satisfies a Lipschitz condition: for each $a>0$ there exists a positive constant L such that $\|F(t,\varphi)-F(t,\psi)\|\le L\|\varphi-\psi\|_{\mathscr{B}}$, for every $t\in[0,a]$ and $\varphi,\psi\in\mathscr{B}$.

Theorem 19. *Assume that the conditions* (H1) *and* (H4) *are satisfied. Then, for* $\varphi\in\mathscr{B}$ *such that* $\varphi(0)\in\overline{D(A)}$, *Eq.* (1) *has a unique global integral solution* x *which is given by*

$$x(t)=\begin{cases} S'(t)\varphi(0)+\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t S(t-s)F(s,x_s)\,\mathrm{d}s, & t\ge 0,\\ \varphi(t), & -\infty<t\le 0. \end{cases} \tag{7}$$

Moreover, $x(t,\varphi)$ *is a continuous function of* φ, *in the sense that if* $\varphi\in\mathscr{B}$ *such that* $\varphi(0)\in\overline{D(A)}$ *and* $t\ge 0$, *then there exist positive constant* β *such that, for* $\psi\in\mathscr{B}$ *and* $\psi(0)\in\overline{D(A)}$, *we have*

$$\|x(t,\varphi)-x(t,\psi)\|\le\beta\|\varphi-\psi\|_{\mathscr{B}}.$$

Proof. Let $a>0$ be fixed and $\varphi\in\mathscr{B}$ such that $\varphi(0)\in\overline{D(A)}$. Consider the set $Z_a(\varphi):=\{y\in C([0,a];E)\colon y(0)=\varphi(0)\}$ and $z\in Z_a(\varphi)$.

Let $\tilde z:(-\infty,a]\to E$ be the mapping defined by

$$\tilde z(t)=\begin{cases} z(t), & t\in[0,a],\\ \varphi(t), & -\infty<t\le 0. \end{cases}$$

Set $K_a:=\max_{0\le t\le a}K(t)$ and $M_a:=\max_{0\le t\le a}M(t)$. By virtue of Condition (H4) and Axiom (A2) the mapping $s\mapsto F(s,\tilde z_s)$ is continuous on $[0,a]$. This implies that the mapping $t\mapsto\int_0^t S(t-s)F(s,\tilde z_s)\,\mathrm{d}s$ is continuously differentiable on $[0,a]$.

Consider the operator $P:Z_a(\varphi)\to Z_a(\varphi)$ defined by

$$(Pz)(t):=S'(t)\varphi(0)+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)F(s,\tilde z_s)\,\mathrm{d}s.$$

Owing to Remark 1, we can assume without loss of generality that $\omega>0$. By virtue of Proposition 12, we have for every $y,z\in Z_a(\varphi)$, $t\in[0,a]$ and $\lambda>\omega$,

$$\lambda R(\lambda,A)((Py)(t)-(Pz)(t))=\int_0^t S'(t-s)\lambda R(\lambda,A)(F(s,\tilde y_s)-F(s,\tilde z_s))\,\mathrm{d}s.$$

Then,

$$\|\lambda R(\lambda,A)((Py)(t)-(Pz)(t))\|\le\frac{\lambda}{\lambda-\omega}\overline{M}^2\mathrm{e}^{\omega a}\int_0^t\|F(s,\tilde y_s)-F(s,\tilde z_s)\|\,\mathrm{d}s.$$

Thus, from (H4) and by letting λ to $+\infty$, we obtain

$$\|(Py)(t)-(Pz)(t)\| \le L\overline{M}^2 \mathrm{e}^{\omega a}\int_0^t \|\tilde{y}_s-\tilde{z}_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

This yields

$$\begin{aligned}\|(Py)(t)-(Pz)(t)\| &\le L\overline{M}^2 \mathrm{e}^{\omega a}\int_0^t K(s)\sup_{0\le\xi\le s}\|y(\xi)-z(\xi)\|\,\mathrm{d}s\\ &\le L\overline{M}^2 K_a\mathrm{e}^{\omega a}\int_0^t \sup_{0\le\xi\le s}\|y(\xi)-z(\xi)\|\,\mathrm{d}s.\end{aligned}$$

Following the same reasoning, we can see that

$$\begin{aligned}\|(P^2y)(t)-(P^2z)(t)\| &\le L\overline{M}^2 K_a\mathrm{e}^{\omega a}\int_0^t \sup_{0\le\xi\le s}\|(Py)(\xi)-(Pz)(\xi)\|\,\mathrm{d}s\\ &\le (L\overline{M}^2 K_a\mathrm{e}^{\omega a})^2\int_0^t \sup_{0\le\xi\le s}\int_0^{\xi}\sup_{0\le\alpha\le p}\|y(\alpha)-z(\alpha)\|\,\mathrm{d}p\,\mathrm{d}s\\ &\le (L\overline{M}^2 K_a\mathrm{e}^{\omega a})^2\int_0^t\int_0^s \|y-z\|_{Z_a(\varphi)}\,\mathrm{d}p\,\mathrm{d}s\\ &\le \frac{(L\overline{M}^2 K_a\mathrm{e}^{\omega a})^2}{2}a^2\|y-z\|_{Z_a(\varphi)}.\end{aligned}$$

We can repeat the previous argument, and we obtain

$$\|(P^ny)(t)-(P^nz)(t)\| \le \frac{(L\overline{M}^2 K_a\mathrm{e}^{\omega a})^n}{n!}a^n\|y-z\|_{Z_a(\varphi)}.$$

Since there exists $m\in\mathbb{N}$ such that $((L\overline{M}^2 K_a\mathrm{e}^{\omega a})^m/m!)a^m<1$, it follows that P^m is a strict contraction on the closed subset $Z_a(\varphi)$ of the Banach space $C([0,a];E)$. Consequently, by the Banach fixed point theorem, we deduce the existence and uniqueness of $x:=x(.,\varphi)\in Z_a(\varphi)$ such that $Px=x$. Finally, for all $a>0$, Eq. (7) has a unique solution which is defined on the interval $(-\infty,a]$. Then, x is a global solution of Eq. (7).

On the other hand, if we consider two solutions $x:=x(.,\varphi)$ and $y:=y(.,\psi)$ for $\varphi\in\mathscr{B}$ with $\varphi(0)\in\overline{D(A)}$ and $\psi\in\mathscr{B}$ with $\psi(0)\in\overline{D(A)}$, then for every $t\in[0,a]$, with $a>0$ fixed

$$\begin{aligned}\|x(t)-y(t)\| &\le \|S'(t)\varphi(0)-S'(t)\psi(0)\| + L\overline{M}^2\mathrm{e}^{\omega a}\int_0^t \|\tilde{x}_s-\tilde{y}_s\|_{\mathscr{B}}\,\mathrm{d}s\\ &\le \overline{M}\mathrm{e}^{\omega a}\|\varphi(0)-\psi(0)\|\\ &\quad + L\overline{M}^2\mathrm{e}^{\omega a}\int_0^t\left(K(s)\max_{0\le\xi\le s}\|x(\xi)-y(\xi)\|+M(t)\|\varphi-\psi\|_{\mathscr{B}}\right)\mathrm{d}s\end{aligned}$$

$$\leq H\overline{M}\mathrm{e}^{\omega a}\|\varphi-\psi\|_{\mathscr{B}}$$

$$+L\overline{M}^2K_a\mathrm{e}^{\omega a}\int_0^t\max_{0\leq\xi\leq s}\|x(\xi)-y(\xi)\|\,\mathrm{d}s+aL\overline{M}^2M_a\mathrm{e}^{\omega a}\|\varphi-\psi\|_{\mathscr{B}}.$$

By the Bellman–Gronwall lemma, it follows that

$$\|x(t)-y(t)\|\leq\beta\|\varphi-\psi\|_{\mathscr{B}}\quad\text{for } t\in[0,a],$$

where

$$\beta=\overline{M}\mathrm{e}^{\omega a}(H+aL\overline{M}M_a)\exp(aL\overline{M}^2K_a\mathrm{e}^{\omega a}).$$

Hence the continuity of $x(t,\varphi)$ in terms of φ and the uniqueness of the solution of Eq. (7). This completes the proof. □

3.3. Existence of strict solution

Under more restrictive conditions, we obtain the strict solution of Eq. (1). We make the following assumption.

(H5) $F:(-\infty,0]\times\mathscr{B}\to E$ is continuously differentiable and the derivatives D_1F, D_2F satisfied the following locally Lipschitz condition:
for any compact set $Q\subset(-\infty,0]\times\mathscr{B}$, there exists a constant $L>0$ such that

$$\|D_1F(t,\varphi)-D_1F(t,\psi)\|\leq L\|\varphi-\psi\|_{\mathscr{B}},$$

$$\|D_2F(t,\varphi)-D_2F(t,\psi)\|\leq L\|\varphi-\psi\|_{\mathscr{B}}$$

for all (t,φ), $(t,\psi)\in Q$.

Let $a>0$. Note that, for every functions x and y verifying the conditions in Axiom (A), the sets $\{(s,x_s)\colon s\in[0,a]\}$ and $\{(s,y_s)\colon s\in[0,a]\}$ are in a compact set of $[0,a]\times\mathscr{B}$. Then, we deduce that

$$\|D_1F(t,x_s)-D_1F(t,y_s)\|\leq L\|x_s-y_s\|_{\mathscr{B}},$$

$$\|D_2F(t,x_s)-D_2F(t,y_s)\|\leq L\|x_s-y_s\|_{\mathscr{B}}$$

for all $s\in[0,a]$ and any functions x and y as in Axiom (A).

Theorem 20. *Assume that the conditions* (H1), (H4) *and* (H5) *are satisfied. Let* $\varphi\in\mathscr{B}$ *continuously differentiable such that* $\varphi'\in\mathscr{B}$, $\varphi(0)\in D(A),\varphi'(0)\in\overline{D(A)}$ *and* $\varphi'(0)=A\varphi(0)+F(0,\varphi)$. *Then, the integral solution of Eq.* (1) *is a strict solution.*

Proof. By Theorem 19, we know that Eq. (1) has a unique global integral solution x, which is given by

$$x(t)=S'(t)\varphi(0)+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^tS(t-s)F(s,x_s)\,\mathrm{d}s\quad\text{for } t\geq0. \tag{8}$$

The assumption $\varphi(0)\in D(A)$ implies that $S'(t)\varphi(0)=S(t)A\varphi(0)+\varphi(0)$, and x can be written as

$$x(t)=S(t)A\varphi(0)+\varphi(0)+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^tS(t-s)F(s,x_s)\,\mathrm{d}s. \tag{9}$$

Consider the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = Ay(t) + D_1F(t,x_t) + D_2F(t,x_t)y_t, \quad t \geq 0,$$
$$y(t) = \varphi'(t), \quad -\infty < t \leq 0. \tag{10}$$

Using Axiom (A2), we can see that the mapping $G : (-\infty, 0] \times \mathscr{B} \to E$ defined by $G(t,\psi) = D_1F(t,x_t) + D_2F(t,x_t)\psi$, satisfies (H4) By Theorem 19, Eq. (10) has a unique global integral solution y which is given by

$$\begin{aligned} y(t) &= S'(t)(A\varphi(0) + F(0,\varphi)) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)(D_1F(s,x_s) + D_2F(s,x_s)y_s)\,\mathrm{d}s, \quad t \geq 0, \\ y(t) &= \varphi'(t), \quad -\infty < t \leq 0. \end{aligned} \tag{11}$$

Consider the mapping $w : (-\infty, +\infty) \to E$, defined by

$$w(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)\,\mathrm{d}s, & t \geq 0, \\ \varphi(t), & -\infty < t \leq 0. \end{cases} \tag{12}$$

We will show that $x = w$. We have

$$\begin{aligned} w(t) &= S(t)(A\varphi(0) + F(0,\varphi)) + \varphi(0) \\ &\quad + \int_0^t S(t-s)(D_1F(s,x_s) + D_2F(s,x_s)y_s)\,\mathrm{d}s, \quad t \geq 0, \\ w(t) &= \varphi(t), \quad -\infty < t \leq 0. \end{aligned} \tag{13}$$

From (12), we can deduce that $w_t = \varphi + \int_0^t y_s\,\mathrm{d}s$. Then, the mappings $t \mapsto w_t$ and $t \mapsto \int_0^t S(t-s)F(s,w_s)\,\mathrm{d}s$ are continuously differentiable and satisfy

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)F(s,w_s)\,\mathrm{d}s \\ &\quad = S(t)F(0,\varphi) + \int_0^t S(t-s)(D_1F(s,w_s) + D_2F(s,w_s)y_s)\,\mathrm{d}s. \end{aligned} \tag{14}$$

From (9), (13) and (14), we obtain

$$\begin{aligned} x(t) - w(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)(F(s,x_s) - F(s,w_s))\,\mathrm{d}s \\ &\quad - \int_0^t S(t-s)(D_1F(s,x_s) - D_1F(s,w_s))\,\mathrm{d}s \\ &\quad - \int_0^t S(t-s)(D_2F(s,x_s) - D_2F(s,w_s))y_s\,\mathrm{d}s. \end{aligned}$$

Let

$$\begin{aligned}
I_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)(F(s,x_s)-F(s,w_s))\,\mathrm{d}s,\\
I_2 &= -\int_0^t S(t-s)(D_1F(s,x_s)-D_1F(s,w_s))\,\mathrm{d}s,\\
I_3 &= -\int_0^t S(t-s)(D_2F(s,x_s)-D_2F(s,w_s))y_s\,\mathrm{d}s,
\end{aligned}$$

and, for $a>0$ fixed

$$b_0 = \max\left\{\sup_{0\le s\le a}\|S(s)\|;\ \sup_{0\le s\le a}\|S'(s)\|;\ \sup_{0\le s\le a}(K(s)\|y(s)\|+M(s)\|\varphi'\|_{\mathscr{B}})\right\}.$$

Let $t\in[0,a]$. It is easy to see that

$$\|I_2\| \le b_0L\int_0^t \|x_s-w_s\|_{\mathscr{B}}\,\mathrm{d}s,$$

and by Axiom (A1)(iii),

$$\|I_3\| \le b_0^2L\int_0^t \|x_s-w_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

Proceeding as in the previous proofs, we get

$$\|I_1\| \le b_0\overline{M}L\int_0^t \|x_s-w_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

Let

$$b_1 = b_0L(1+b_0+\overline{M}).$$

Then, we obtain

$$\sup_{0\le s\le t}\|x(s)-w(s)\| \le b_1\int_0^t \|x_s-w_s\|_{\mathscr{B}}\,\mathrm{d}s$$

Since $x_0=w_0=\varphi$, Axiom (A1)(iii) implies that

$$\|x_t-w_t\|_{\mathscr{B}} \le K_a\sup_{0\le s\le t}\|x(s)-w(s)\|.$$

Then,

$$\|x_t-w_t\|_{\mathscr{B}} \le K_ab_1\int_0^t \|x_s-w_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

Using the Bellman–Gronwall lemma, we conclude that

$$\|x_t-w_t\|_{\mathscr{B}} = 0 \quad \text{for any } t\in[0,a].$$

Consequently, $x(t)=w(t)$ for all $t\in(-\infty,a]$. Hence, x is continuously differentiable on $(-\infty,a]$. This ends the proof. □

4. An application to partial integrodifferential equations with infinite delay

In this section, we make an attempt to apply the results obtained in the previous sections to the following partial integrodifferential equations:

$$\begin{aligned}
&\frac{\partial}{\partial t}w(t,\xi) = d\Delta w(t,\xi) + \int_{-\infty}^{t} k(t,s,w(s,\xi))\,\mathrm{d}s + f(t,w(t,\xi)), \quad t \geq 0,\ \ \xi \in \Omega,\\
&w(t,\xi) = 0, \quad t \geq 0,\ \ \xi \in \partial\Omega,\\
&w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ \ \xi \in \overline{\Omega},
\end{aligned} \tag{15}$$

where $d > 0$, Ω is a bounded open set in $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{n} \partial^2/\partial\zeta_i^2$ and $k : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $w_0 : (-\infty, 0] \times \overline{\Omega} \to \mathbb{R}^n$ are functions.

We choose $E = C(\overline{\Omega}, \mathbb{R}^n)$, and we consider the operator $A : D(A) \subset E \to E$ defined by

$$\begin{aligned}
&D(A) = \{y \in E\colon\ \Delta y \in E \text{ and } y_{/\partial\Omega} = 0\},\\
&Ay = d\Delta y.
\end{aligned}$$

We have $\overline{D(A)} = \{y \in E\colon\ y_{/\partial\Omega} = 0\} \neq E$ and

$$\begin{aligned}
&\rho(A) = (0, +\infty)\\
&\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.
\end{aligned}$$

This implies that A satisfies (H1) on E.

A simple example of the space of initial functions is the following:

$$\mathscr{B} = C_g^0 := \left\{\phi \in C((-\infty, 0]; E)\colon\ \lim_{\theta \to -\infty} \frac{\|\phi(\theta)\|}{g(\theta)} = 0\right\},$$

endowed with the following norm:

$$\|\phi\|_{\mathscr{B}} := \sup_{-\infty < \theta \leq 0} \frac{\|\phi(\theta)\|}{g(\theta)},$$

where $g : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfying

(g1) $g(0) = 1$.

(g2) The function $G : [0, +\infty) \to [0, +\infty)$ defined by

$$G(t) = \sup_{-\infty < \theta \leq -t} \frac{g(t+\theta)}{g(\theta)},$$

is locally bounded.

Theorems 3.2 and 3.6 in [15], implies that $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ satisfies Axioms (A1), (A2) and (B).

In addition, we set for $t \geq 0$ and $\xi \in \overline{\Omega}$

$$x(t)(\xi) = w(t,\xi),$$

$$\varphi(\theta)(\xi) = w_0(\theta,\xi),$$

$$F(t,\phi)(\xi) = f(t,\phi(0)(\xi)) + \int_{-\infty}^{0} k(t,t+\theta,\phi(\theta)(\xi))\,\mathrm{d}\theta \quad \text{for } \phi \in \mathscr{B}.$$

A case where the problem (15) can be handled by using the classical semigroup theory, is the case when

$$f(t,0) + \int_{-\infty}^{0} k(t,t+\theta,0)\,\mathrm{d}\theta = 0 \quad \text{for all } t \geq 0. \tag{16}$$

In that case, F takes its values in the space $\overline{D(A)}$ and the operator A generates a strongly continuous semigroup in $\overline{D(A)}$. On the other hand, the integrated semigroups theory allows the range of F to be in E, i.e., without the condition (16).

Now it will be easy to adapt our previous results to solving Eq. (15). The properties of the mapping F depend on k, f and on the choose of the space $\mathscr{B}$.

We assume that $k : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions and satisfy:

(a) $\|k(t,s,v(\cdot))\| \leq C(s-t)\|v\|$, $s \leq t$, $v \in E$;
(b) $\|k(t,s,u(\cdot)) - k(t,s,v(\cdot))\| \leq N(s-t)\|u-v\|$, $s \leq t$, $u,v \in E$;
(c) $\|f(t,u(\cdot)) - f(t,v(\cdot))\| \leq \eta\|u-v\|$, $t \geq 0$, $u,v \in E$;

where η is a fixed positive constant, $C,N : (-\infty,0] \to [0,+\infty)$ are two measurable functions such that, $C(\cdot)g(\cdot)$ and $N(\cdot)g(\cdot)$ are integrable on $(-\infty,0]$.

Under the above conditions, $F : [0,+\infty) \times \mathscr{B} \to E$ satisfies condition (H4). In fact, given $t \geq 0$, $\phi \in \mathscr{B}$ and a sequence $(t_n)_{n\geq 0}$ of $[0,+\infty)$ such that $t_n \to t$, we have

$$\begin{aligned}
&\|F(t_n,\phi) - F(t,\phi)\| \\
&\quad \leq \sup_{\xi\in\overline{\Omega}} |f(t_n,\phi(0)(\xi)) - f(t,\phi(0)(\xi))| \\
&\qquad + \sup_{\xi\in\overline{\Omega}} \left(\int_{-\infty}^{0} |k(t_n,t_n+\theta,\phi(\theta)(\xi)) - k(t,t+\theta,\phi(\theta)(\xi))|\,\mathrm{d}\theta \right),
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\|k(t_n,t_n+\theta,\phi(\theta)(\cdot))\| &\leq C(\theta)\|\phi(\theta)\|, \\
&\leq C(\theta)g(\theta) \sup_{-\infty<\theta\leq 0} \frac{\|\phi(\theta)\|}{g(\theta)}, \\
&= C(\theta)g(\theta)\|\phi\|_{\mathscr{B}}.
\end{aligned}$$

By continuity of k, we have

$$\lim_{n} k(t_n,t_n+\theta,\phi(\theta)(\xi)) = k(t,t+\theta,\phi(\theta)(\xi)) \quad \text{uniformly on } \xi \in \overline{\Omega}.$$

Then, the Lebesgue convergence theorem allows us to assert that

$$\sup_{\xi\in\overline{\Omega}}\left(\int_{-\infty}^{0}|k(t_n,t_n+\theta,\phi(\theta)(\xi))-k(t,t+\theta,\phi(\theta)(\xi))|\,\mathrm{d}\theta\right)\underset{n}{\to}0,$$

and by the continuity of f, we obtain

$$\sup_{\xi\in\overline{\Omega}}|f(t_n,\phi(0)(\xi))-f(t,\phi(0)(\xi))|\underset{n}{\to}0.$$

Then, F is continuous in $t\in[0,+\infty)$.

Furthermore, by Axiom (A1)(ii), we have for every $\phi,\psi\in\mathscr{B}$ and $t\geq 0$

$$\begin{aligned}\|f(t,\phi(0)(\cdot))-f(t,\psi(0)(\cdot))\| &\leq \eta\|\phi(0)-\psi(0)\|\\ &\leq \eta H\|\phi-\psi\|_{\mathscr{B}}\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{0}\|k(t,t+\theta,\phi(\theta)(\cdot))-k(t,t+\theta,\psi(\theta)(\cdot))\|\,\mathrm{d}\theta\\ &\quad\leq\|\phi-\psi\|_{\mathscr{B}}\int_{-\infty}^{0}N(\theta)g(\theta)\,\mathrm{d}\theta.\end{aligned}$$

This implies that

$$\|F(t,\phi)-F(t,\psi)\|\leq L\|\phi-\psi\|_{\mathscr{B}},$$

where

$$L:=\eta H+\int_{-\infty}^{0}N(\theta)g(\theta)\,\mathrm{d}\theta.$$

Hence, (H2) is satisfied.

Assume also that

(i) $w_0\in C((-\infty,0]\times\overline{\Omega};\mathbb{R}^n)$, with $\lim_{\theta\to-\infty}\|w_0(\theta,.)\|/g(\theta)=0$;
(ii) $w_0(0,\xi)=0$ for $\xi\in\partial\Omega$.

Then all conditions of Theorem 19 are satisfied, and Eq. (15) has a unique global integral solution on $(-\infty,+\infty)$.

Under more restrictive conditions, we obtain the strict solution.

(iii) $w_0\in C^2((-\infty,0]\times\overline{\Omega};\mathbb{R}^n)$, with

$$\lim_{\theta\to-\infty}\frac{1}{g(\theta)}\left\|\frac{\partial}{\partial\theta}w_0(\theta,.)\right\|=0;$$

(iv) $\Delta w_0(0,\xi)=0$ for $\xi\in\partial\Omega$;
(v) $w_0(\theta,\xi)=0$ for $-\infty<\theta\leq 0$ and $\xi\in\partial\Omega$;
(vi) $\frac{\partial}{\partial\theta}w_0(0,\xi)=\Delta w_0(0,\xi)+f(0,w_0(0,\xi))+\int_{-\infty}^{0}k(0,\theta,w_0(\theta,\xi))\,\mathrm{d}\theta$ for $\xi\in\overline{\Omega}$;
(vii) f and k are continuously differentiable, with the following conditions:
(a1) $\|D_1k(t,s,v(\cdot))\|\leq C_1(s-t)\|v\|$ for $s\leq t$, $v\in E$;
(b1) $\|D_1k(t,s,u(\cdot))-D_1k(t,s,v(\cdot))\|+\|D_2k(t,s,u(\cdot))-D_2k(t,s,v(\cdot))\|\leq N_1(s-t)\|u-v\|$ for $s\leq t$, $u,v\in E$;
(c1) $\|D_1f(t,u(\cdot))-D_1f(t,v(\cdot))\|\leq\eta_1\|u-v\|$ for $t\geq 0$, $u,v\in E$;

(a2) $\|D_3k(t,s,v(\cdot))\| \le C_2(s-t)\|v\|$ for $s \le t,\ v \in E$;
(b2) $\|D_3k(t,s,u(\cdot)) - D_3k(t,s,v(\cdot))\| \le N_2(s-t)\|u-v\|$ for $s \le t,\ u,v \in E$;
(c2) $\|D_2f(t,u(\cdot)) - D_2f(t,v(\cdot))\| \le \eta_2\|u-v\|$ for $t \ge 0,\ u,v \in E$;
where D_1, D_2 and D_3 are the derivatives, η_1 and η_2 are two fixed positive constants, $C_1,\ N_1,\ C_2,\ N_2 : (-\infty,0] \to [0,+\infty)$ are measurable functions such that $C_1(\cdot)g(\cdot)$, $N_1(\cdot)g(\cdot)$, $C_2(\cdot)g(\cdot)$ and $N_2(\cdot)g(\cdot)$ are integrable on $(-\infty,0]$. We can verify without difficulty that all conditions of Theorem 20 hold. Then the integral solution $w(t,\xi) = x(t)(\xi)$ is a strict solution.

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