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Local existence and stability for some partial functional differential equations with infinite delay☆

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1. Introduction

In this work, we are concerned with the local existence and stability for the following partial functional differential equations with infinite delay

$$\begin{aligned} &\frac{dx}{dt}(t) = Ax(t) + F(x_t), \quad t \geq 0, \\ &x_0 = \varphi \in \mathscr{B}, \end{aligned} \tag{1}$$

where A is a closed linear operator on a Banach space $(E, |.|)$, the phase space $\mathscr{B}$ is a linear space of functions mapping $(-\infty, 0]$ into E satisfying some axioms which are described in the sequel, F is an E-valued appropriate function, and for every $t \geq 0$, the function $x_t \in \mathscr{B}$ is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \in (-\infty, 0].$$

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In the literature devoted to equations with finite delay, the state space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the selection of the state space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [12], Kappel and Schappacher [18], and Schumacher [31]. For a detailed discussion on this topic, we refer the reader to the book by Hino et al. [17].

In recent years, the theory of partial functional differential equations with delay has attracted widespread attention. The development was initiated for equations with finite delay and about existence and stability by Travis and Webb [35,36], and Webb [37,38]. For later development, we cite only a paper by Arino and Sanchez [5] and a recent book by Wu [40]. In the standard framework for Eq. (1), one assumes that the operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators in E or, equivalently

(i) $\overline{D(A)} = E$,

(ii) there exist $\beta \geq 1$, $\omega \in \mathbb{R}$ such that if $\lambda > \omega$ $(\lambda I - A)^{-1} \in \mathscr{L}(E)$ and

$$\|(\lambda - \omega)^n (\lambda I - A)^{-n}\| \leq \beta \quad \text{for all } n \in \mathbb{N},$$

where $\mathscr{L}(E)$ is the space of bounded linear operators from E into E. In that case, the classical semigroups theory ensures the well posedness of Problem (1). More recently, it has been shown in [1,2] that the density condition is not necessary (in a certain sense) to deal with partial functional differential equations with finite or infinite delay. In the applications, it is sometimes convenient to take initial functions with more restrictions. There are many examples in concrete situations where evolution equations are not densely defined. Only hypothesis (ii) holds. One can refer for this to [8] for more details. Non-density occurs, in many situations, from restrictions made on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or from boundary conditions (e.g., the space $\mathscr{C}^1$ with null value on the boundary is non-dense in the space of continuous functions).

Concerning the case of infinite delay, an extensive theory is developed for Eq. (1) with $A = 0$. We refer the reader to Hale and Kato [12], Corduneanu and Lakshmikantham [7], Hale [10,11], Shin [32], Hino et al. [17], and Lakshmikantham et al. [20]. The extension to the case when A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ was later on studied by Henriquez in his three consecutive papers [13–15]. Following an axiomatic approach, he developed several fundamental results on the existence of solutions, regularity, existence of periodic solutions and stability. Henriquez proved his results by using the following variation-of-constants formula:

$$x(t, \varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s) F(s, x_s)\, \mathrm{d}s & \text{for } t \geq 0, \\ \varphi(t) & \text{for } -\infty < t \leq 0. \end{cases}$$

In [22], Milota obtained some results on existence and stability. In [29], Ruan and Wu developed a general theory of existence, comparison, invariance and monotonicity and provide some applications to reaction diffusion systems with general distributed delays. Recently, in [1], we treated Eq. (1) when A is non-densely defined and satisfies the

Hille–Yosida condition. We obtained some results about the global existence under the assumption that F is globally Lipschitz continuous.

As in [1], we study Eq. (1) without assuming necessarily that A is densely defined. We state the local existence and regularity of solutions under a locally Lipschitz condition on F. We extend the results obtained in [2] to the case of infinite delay. In the case of global existence, we give some properties of the solution map. Mainly, we discuss the stability of the trivial solutions. The method used here is based on the integrated semigroup theory.

Let us now briefly discuss about the advantage of using the integrated semigroups. In the case when the operator F of Eq. (1) is equal to zero, the problem can still be handled by using the classical semigroup theory because A generates a strongly continuous semigroup in the space $\overline{D(A)}$. But, when $F \neq 0$ it is necessary to impose additional restrictions, the simplest of which is that F takes values in $\overline{D(A)}$. It is the integrated semigroups theory that allows the range of the operator F to be a subset of E not necessarily contained in $\overline{D(A)}$.

The organization of this work is as follows: in Section 2, we collect some useful results on Hille–Yosida operators and integrated semigroups. In Section 3, we study the local existence and the regularity of solutions of Eq. (1). In Section 4, we state some properties of the solution operator associated to Eq. (1). We investigate the stability near an equilibrium and we prove that if the linearized semigroup around an equilibrium is exponentially stable, then the equilibrium of Eq. (1) is also exponentially stable. The last section is devoted to an application to a reaction diffusion equation with infinite delay.

2. Preliminaries

In this section, we collect some known definitions and results about integrated semigroups. For more details, we refer the reader to [3,4,6,16,24,33]

Definition 1 (Arendt [3]). A family $(S(t))_{t\geq 0} \subset \mathscr{L}(E)$ is called an integrated semigroup if the following conditions are satisfied:

(i) $S(0) = 0$;
(ii) for any $x \in E$, $S(t)x$ is a continuous function of $t \geq 0$ with values in E;
(iii) $S(s)S(t) = \int_0^s (S(t+\tau) - S(\tau))\,\mathrm{d}\tau$, for any $t, s \geq 0$.

Definition 2 (Arendt [3]). An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $\beta \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq \beta e^{\omega t} \quad \text{for } t \geq 0.$$

Definition 3 (Arendt [3]). An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of A), and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} \mathrm{e}^{-\lambda t} S(t)\,\mathrm{d}t$, for all $\lambda > \omega$.

Similar results as for semigroups can be obtained for integrated semigroups.

Proposition 1 (Arendt [3]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)x\,\mathrm{d}s \in D(A) \quad \textit{and} \quad S(t)x = A\left(\int_0^t S(s)x\,\mathrm{d}s\right) + tx.$$

Moreover, for all $x \in D(A)$ and $t \geq 0$

$$S(t)x \in D(A), \qquad AS(t)x = S(t)Ax$$

and

$$S(t)x = \int_0^t S(s)Ax\,\mathrm{d}s + tx.$$

Corollary 1 (Arendt [3]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x \in E$ and $t \geq 0$ one has $S(t)x \in \overline{D(A)}$.*

Moreover, for $x \in E$, $S(\cdot)x$ is right-sided differentiable in $t \geq 0$ if and only if $S(t)x \in D(A)$. In that case

$$S'(t)x = AS(t)x + x.$$

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 4 (Kellermann and Hieber [19]). An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous, if for all $\tau > 0$ there exists a constant $l(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq l(\tau)|t - s| \quad \text{for all } t, s \in [0, \tau].$$

In this case, we know from [19], that $(S(t))_{t\geq 0}$ is exponentially bounded.

Definition 5 (Kellermann and Hieber [19]). We say that a linear operator A is a Hille–Yosida operator if there exist $\beta \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n \|(\lambda I - A)^{-n}\| \colon n \in \mathbb{N},\ \lambda > \omega\} \leq \beta.$$

The following theorem shows that the Hille–Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroup.

Theorem 1 (Kellermann and Hieber [19]). *The following assertions are equivalent.*
(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*
(ii) *A is a Hille–Yosida operator.*

Remark 1. Let A be a Hille–Yosida operator and $(S(t))_{t\geq 0}$ be the locally Lipschitz continuous integrated semigroup generated by A. It is well known from [34], that the derivative $(S'(t))_{t\geq 0}$ on $\overline{D(A)}$ is a strongly continuous semigroup generated by the part A_0 of the operator A in $\overline{D(A)}$, which is defined by

$$D(A_0) = \{x \in D(A_0)\colon Ax \in \overline{D(A)}\},$$
$$A_0 x = Ax \quad \text{for } x \in D(A_0).$$

The following lemma is important in the sequel.

Lemma 1 (Kellermann and Hieber [19]). *Let $(S(t))_{t\geq 0}$ be a locally Lipschitz continuous integrated semigroup on E and $G : [0,a] \to E$, $a > 0$, be a Bochner-integrable function. Then, the function $B : [0,a] \to E$ defined by*

$$B(t) = \int_0^t S(t-s)G(s)\,\mathrm{d}s$$

is continuously differentiable on $[0,a]$ and satisfies,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right| \leq 2l \int_0^t |G(s)|\,\mathrm{d}s \quad \text{for } t \in [0,a],$$

where $l = l(a)$ is the Lipschitz constant of $S(\,.\,)$ on $[0,a]$.

3. Local existence and regularity of solutions

We first study the local existence and regularity of solutions for Eq. (1) in an integrated form. We use a variation-of-constants formula in the sense of integrated semigroups. In all this paper, we assume that the state space $(\mathscr{B}, \|.\|_{\mathscr{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following fundamental axioms introduced at first by Hale and Kato in [12]:

(A1) There exist a positive constant H and functions $K(\,.\,)$, $M(\,.\,)\colon \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathscr{B}$, and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:

(i) $x_t \in \mathscr{B}$,

(ii) $|x(t)| \leq H\,\|x_t\|_{\mathscr{B}}$, which is equivalent to

(ii)$'$ $|\varphi(0)| \leq H\|\varphi\|_{\mathscr{B}}$, for every $\varphi \in \mathscr{B}$,

(iii) $\|x_t\|_{\mathscr{B}} \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|x(s)| + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$.

(A2) For the function $x(.)$ in (A1), $t \mapsto x_t$ is a $\mathscr{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(B) The space $\mathscr{B}$ is complete.

Notice that Axiom (B) is equivalent to saying that the space of equivalence classes $\mathscr{B}/\|.\|_{\mathscr{B}}$ is a Banach space.

Let C_{00} be the set of continuous functions $\varphi : (-\infty, 0] \to E$ with compact support $supp(\varphi)$.

Remark 2 (Hino et al. [17]). Any $\varphi \in C_{00}$ belongs to $\mathscr{B}$. If *supp*(φ) is contained in $[-r,-s]$, $0 \le s \le r < \infty$, then there exists a constant $\delta(r,s)$ such that

$$\|\varphi\|_{\mathscr{B}} \le \delta(r,s) \sup_{\theta\in[-r,-s]} |\varphi(\theta)|.$$

Throughout this paper, we assume that
(H_1) $A : D(A) \subseteq E \to E$ is a Hille–Yosida operator.
We start by introducing the following definitions.

Definition 6. We say that a function $x : (-\infty,b] \to E$, $b > 0$, is an integral solution of Eq. (1) in $(-\infty,b]$ if the following conditions hold:
(i) x is continuous on $[0,b]$,
(ii) $\int_0^t x(s)\,\mathrm{d}s \in D(A)$, for $t \in [0,b]$,
(iii) $x(t) = \begin{cases} \varphi(0) + A\displaystyle\int_0^t x(s)\,\mathrm{d}s + \int_0^t F(x_s)\,\mathrm{d}s, & 0 \le t \le b, \\ \varphi(t), & -\infty < t \le 0. \end{cases}$

Definition 7. We say that a function $x : (-\infty,b] \to E$ is a strict solution of Eq. (1) in $(-\infty,b]$, if the following conditions hold:
(i) $x \in C^1([0,b];E) \cap C([0,b];D(A))$,
(ii) x satisfies Eq. (1) on $[0,b]$,
(iii) $x(t) = \varphi(t)$, for $-\infty < t \le 0$.

From the closedness property of the operator A, we can see the following result.

Lemma 2. (i) *If x is an integral solution of Eq.* (1) *in* $(-\infty,b]$, *then for all* $t \in [0,b]$, $x(t) \in \overline{D(A)}$. *In particular* $\varphi(0) \in \overline{D(A)}$.
(ii) *If x is an integral solution of Eq.* (1) *in* $(-\infty,b]$, *such that* $x : [0,b] \to E$ belongs to $C^1([0,b];E)$ *or* $C([0,b];D(A))$, *then x is a strict solution of Eq.* (1) *in* $(-\infty,b]$.

Recall from [1] that an integral solution $x : (-\infty,b] \to E$ of Eq. (1) is given by the following formula:

$$x(t,\varphi) = \begin{cases} S'(t)\varphi(0) + \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t S(t-s)F(x_s(.\,,\varphi))\,\mathrm{d}s & \text{for } t \in [0,b], \\ \varphi(t) & \text{for } t \in (-\infty,0], \end{cases} \tag{2}$$

where $(S(t))_{t\ge 0}$ is the locally Lipschitz continuous integrated semigroup generated by the operator A. Note that there exist $\beta \ge 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\|, \|S'(t)\| \le \beta e^{\omega t}$, for $t \ge 0$.

To prove the local existence of solutions of Eq. (1), we suppose a local Lipschitz condition on the nonlinear part F.
(H_2) F is Lipschitz continuous on the seminorm balls of $\mathscr{B}$, i.e., for each $r > 0$ there exists a constant $c_0(r) > 0$ such that if $\varphi_1, \varphi_2 \in \mathscr{B}$ and $\|\varphi_1\|_{\mathscr{B}}, \|\varphi_2\|_{\mathscr{B}} \le r$ then,

$$|F(\varphi_1) - F(\varphi_2)| \le c_0(r)\|\varphi_1 - \varphi_2\|_{\mathscr{B}}.$$

Theorem 2. *Assume that* (H_1) *and* (H_2) *hold. Let* $\varphi \in \mathscr{B}$ *such that* $\varphi(0) \in \overline{D(A)}$. *Then, there exists a maximal interval of existence* $(-\infty, b_\varphi)$, $b_\varphi > 0$, *and a unique integral solution* $x(.,\varphi)$ *of Eq.* (1), *defined on* $(-\infty, b_\varphi)$ *and either*

$$b_\varphi = +\infty \quad or \quad \limsup_{t \to b_\varphi^-} |x(t,\varphi)| = +\infty.$$

Moreover, $x(t,\varphi)$ *is a continuous function of* φ, *in the sense that if* $\varphi \in \mathscr{B}$, $\varphi(0) \in \overline{D(A)}$ *and* $t \in [0, b_\varphi)$, *then there exist positive constants* α *and* ε *such that, for* $\psi \in \mathscr{B}$, $\psi(0) \in \overline{D(A)}$ *and* $\|\varphi - \psi\|_{\mathscr{B}} < \varepsilon$, *we have*

$$|x(s,\varphi) - x(s,\psi)| \le \alpha \|\varphi - \psi\|_{\mathscr{B}} \quad \textit{for all } s \in [0,t].$$

Proof. Note that (H_2) implies that, for each $r > 0$ there exists $c_0(r) > 0$ such that for $\varphi \in \mathscr{B}$ and $\|\varphi\|_{\mathscr{B}} \le r$, we have

$$|F(\varphi)| \le c_0(r)\|\varphi\|_{\mathscr{B}} + |F(0)| \le c_0(r)r + |F(0)|.$$

Let $\varphi \in \mathscr{B}$, $\varphi(0) \in \overline{D(A)}$, $r = \|\varphi\|_{\mathscr{B}} + 1$ and $c_1 = c_0(r)r + |F(0)|$. Define the function

$$y(t) = \begin{cases} S'(t)\varphi(0) & \text{for } t \ge 0, \\ \varphi(t) & \text{for } t \le 0. \end{cases}$$

By virtue of the axioms (A1-i) and (A2), we deduce that $y_t \in \mathscr{B}$ and the map $t \to y_t$ is continuous. Then, for $b_1 \in (0,1)$, there exists $b_2 \in (0,1)$ such that

$$\|y_t - \varphi\|_{\mathscr{B}} \le b_1 \quad \text{for all } t \in [0, b_2].$$

Without loss of generality, we can assume that $\omega > 0$. Let $0 < b \le b_2$ such that

$$K_b \beta e^{\omega b} c_1 b < 1 - b_1,$$

where $K_b := \max_{0 \le t \le b} K(t)$.

Let us introduce the space

$$\mathbb{F}_b := \{u\colon (-\infty, b] \to E, \text{ such that } u_0 \in \mathscr{B} \text{ and } u : [0,b] \to E \text{ is continuous}\},$$

endowed with the seminorm $\|.\|_{\mathbb{F}_b}$, defined by

$$\|u\|_{\mathbb{F}_b} = \|u_0\|_{\mathscr{B}} + \sup_{0 \le s \le b} |u(s)|.$$

We can see from Axiom (B), that $(\mathbb{F}_b, \|.\|_{\mathbb{F}_b})$ is complete.

Consider the following set:

$$\mathbb{F}_b^\varphi := \left\{ u \in \mathbb{F}_b\colon \|u_0 - \varphi\|_{\mathscr{B}} = 0 \text{ and } \sup_{0 \le t \le b} \|u_t - \varphi\|_{\mathscr{B}} \le 1 \right\}.$$

$\mathbb{F}_b^\varphi$ is a nonempty subset of $\mathbb{F}_b$. In fact, it is clear that the restriction of the function y to $(-\infty, b]$ is an element of $\mathbb{F}_b^\varphi$.

$\mathbb{F}_b^\varphi$ is closed in $\mathbb{F}_b$. To prove that, consider a convergent sequence $(u^n)_{n\geq 0}$ of $\mathbb{F}_b^\varphi$ with $\lim_{n\to+\infty} u^n = u$ in $\mathbb{F}_b$. Then, for any $n \in \mathbb{N}$

$$\begin{aligned}\|u_0-\varphi\|_{\mathscr{B}} &\le \|u_0-u_0^n\|_{\mathscr{B}} + \|u_0^n-\varphi\|_{\mathscr{B}},\\ &= \|u_0-u_0^n\|_{\mathscr{B}},\\ &\le \|u-u^n\|_{\mathbb{F}_b}.\end{aligned}$$

Letting n to $+\infty$, yields $\|u_0-\varphi\|_{\mathscr{B}} = 0$. In addition, from Axiom (A1-iii) we deduce that, for any $t \in [0,b]$,

$$\begin{aligned}\|u_t^n-u_t\|_{\mathscr{B}} &\le K(t)\sup_{0\le s\le t}|u^n(s)-u(s)| + M(t)\|u_0^n-u_0\|_{\mathscr{B}},\\ &\le \sup(K_b,M_b)\|u^n-u\|_{\mathbb{F}_b},\end{aligned}$$

where $M_b := \sup_{0\le t\le b} M(t)$. Thus, for any $t\in[0,b]$, $u_t^n \to u_t$ in $\mathscr{B}$. From this together with the inequality

$$\|u_t-\varphi\|_{\mathscr{B}} \le \|u_t-u_t^n\|_{\mathscr{B}} + \|u_t^n-\varphi\|_{\mathscr{B}} \quad \text{for any } n\in\mathbb{N},$$

we deduce that $\|u_t-\varphi\|_{\mathscr{B}} \le 1$. Consequently, $u \in \mathbb{F}_b^\varphi$.

We have proved that $\mathbb{F}_b^\varphi$ is a nonempty closed subset of $\mathbb{F}_b$.

Consider now the mapping $\mathscr{T}$ defined on $\mathbb{F}_b^\varphi$ by

$$\begin{aligned}&(\mathscr{T}u)(t) = S'(t)\varphi(0) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)F(u_s)\,\mathrm{d}s \quad \text{for } t\in[0,b],\\ &(\mathscr{T}u)_0 = \varphi.\end{aligned} \tag{3}$$

$\mathscr{T}$ maps elements of $\mathbb{F}_b^\varphi$ into $\mathbb{F}_b^\varphi$. In fact, by (H_2) and Axiom (A1) for every $u\in\mathbb{F}_b^\varphi$ the function $s\mapsto F(u_s)$ is continuous on $[0,b]$. Then, Lemma 1 implies that $s\mapsto\int_0^t S(t-s)F(u_s)\,\mathrm{d}s$ is continuously differentiable on $[0,b]$. From this, $v := \mathscr{T}u$ is continuous on $[0,b]$ and $v\in\mathbb{F}_b$. To prove that $v\in\mathbb{F}_b^\varphi$, taking $w = v-y$. We get, for any $t\in[0,b]$,

$$\|v_t-\varphi\|_{\mathscr{B}} \le \|w_t\|_{\mathscr{B}} + \|y_t-\varphi\|_{\mathscr{B}} \le \|w_t\|_{\mathscr{B}} + b_1.$$

Since $\|u_s-\varphi\|_{\mathscr{B}} \le 1$, for $s\in[0,b]$ and $r = \|\varphi\|_{\mathscr{B}}+1$, we deduce that $\|u_s\|_{\mathscr{B}} \le r$, for $s\in[0,b]$. Then,

$$|F(u_s)| \le c_0(r)\|u_s\|_{\mathscr{B}} + |F(0)| \le c_1.$$

On the other hand, by Axiom (A1-iii) we see that for any $t\in[0,b]$

$$\|w_t\|_{\mathscr{B}} \le K_b \sup_{0\le s\le t}|w(s)|$$

and for suitable constants β and ω

$$\begin{aligned}|w(t)| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)F(u_s)\,\mathrm{d}s\right|,\\ &\le \beta e^{\omega t} c_1 \int_0^t e^{-\omega s}\,\mathrm{d}s,\end{aligned}$$

$$\le \beta e^{\omega t} c_1 b,$$
$$< \frac{1}{K_b}(1 - b_1).$$

Then, we deduce that

$$\|v_t - \varphi\|_{\mathscr{B}} < 1 \quad \text{for any } t \in [0, b].$$

Which implies that $v \in \mathbb{F}_b^{\varphi}$.

To prove that $\mathscr{T}$ is a strict contraction in $\mathbb{F}_b^{\varphi}$, we consider $u, v \in \mathbb{F}_b^{\varphi}$ and $t \in [0, b]$. Since

$$\|(\mathscr{T}u)_0 - (\mathscr{T}v)_0\|_{\mathscr{B}} = 0,$$

we obtain

$$\|\mathscr{T}u - \mathscr{T}v\|_{\mathbb{F}_b} = \sup_{t\in[0,b]} |(\mathscr{T}u)(t) - (\mathscr{T}v)(t)|$$

and

$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| = \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(t-s)(F(u_s) - F(v_s))\,\mathrm{d}s \right|,$$
$$\le \beta e^{\omega t} \int_0^t |F(u_s) - F(v_s)|\,\mathrm{d}s,$$
$$\le \beta e^{\omega t} c_0(r) \int_0^t \|u_s - v_s\|_{\mathscr{B}}\,\mathrm{d}s,$$
$$\le \beta e^{\omega b} c_0(r) K_b b \|u - v\|_{\mathbb{F}_b}.$$

Then,

$$\|\mathscr{T}u - \mathscr{T}v\|_{\mathbb{F}_b} \le \beta e^{\omega b} c_0(r) K_b b \|u - v\|_{\mathbb{F}_b}.$$

Note that $r \ge 1$, then

$$\beta e^{\omega b} c_0(r) K_b b \le \beta e^{\omega b} c_1 K_b b < 1.$$

This means that $\mathscr{T}$ is a strict contraction in $(\mathbb{F}_b^{\varphi}, \|.\|_{\mathbb{F}_b})$. So, $\mathscr{T}$ has a fixed point x in $\mathbb{F}_b^{\varphi}$. Let u be another fixed point of $\mathscr{T}$. Then,

$$\|u - x\|_{\mathbb{F}_b} = 0,$$

$$u_0 = x_0 = \varphi.$$

From the expression of the seminorm $\|.\|_{\mathbb{F}_b}$, we obtain

$$u(t) = x(t) \quad \text{for } t \in [0, b].$$

This means that $\mathscr{T}$ has one and only one fixed point x in $\mathbb{F}_b^{\varphi}$. We conclude that Eq. (1) has one and only one integral solution which is defined on the interval $(-\infty, b]$.

Let $[0, b_\varphi)$, $b_\varphi > 0$, be the maximal interval of existence of the integral solution $x(.,\varphi)$ of Eq. (1). Set $K_{b_\varphi} := \max_{0\le t\le b_\varphi} K(t)$ and $M_{b_\varphi} := \sup_{0\le t\le b_\varphi} M(t)$. Assume that $b_\varphi < +\infty$ and $\limsup_{t\to b_\varphi^-} |x(t,\varphi)| < +\infty$. Then, from Axiom (A1-iii), we can see that there exists a constant $r > 0$ such that $\|x_s(.,\varphi)\|_{\mathscr{B}} \le r$, for all $s \in [0, b_\varphi)$. Consider $t, t+h \in [0, b_\varphi)$ and $h > 0$,

$$
\begin{aligned}
|x(t+h) - x(t)| &\le |S'(t+h)\varphi(0) - S'(t)\varphi(0)| \\
&\quad + \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{t+h} S(t+h-s)F(x_s(.,\varphi))\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(t-s)F(x_s(.,\varphi))\,\mathrm{d}s \right|, \\
&\le |S'(t)(S'(h) - I)\varphi(0)| + \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_t^{t+h} S(t+h-s)F(x_s(.,\varphi))\,\mathrm{d}s \right| \\
&\quad + \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(s)(F(x_{t+h-s}(.,\varphi)) - F(x_{t-s}(.,\varphi)))\,\mathrm{d}s \right|, \\
&\le \beta \mathrm{e}^{\omega b_\varphi} |(S'(h) - I)\varphi(0)| + \beta \mathrm{e}^{\omega b_\varphi} c_1 h \\
&\quad + \beta \mathrm{e}^{\omega b_\varphi} c_0(r) \int_0^t \|x_{s+h}(.,\varphi) - x_s(.,\varphi)\|_{\mathscr{B}}\,\mathrm{d}s, \\
&\le \beta \mathrm{e}^{\omega b_\varphi} |S'(h)\varphi(0) - \varphi(0)| + \beta \mathrm{e}^{\omega b_\varphi} c_1 h + b_\varphi M_{b_\varphi} \|x_h(.,\varphi) - \varphi\|_{\mathscr{B}} \\
&\quad + \beta \mathrm{e}^{\omega b_\varphi} c_0(r) K_{b_\varphi} \int_0^t \sup_{0 \le \sigma \le s} |x(\sigma+h,\varphi) - x(\sigma,\varphi)|\,\mathrm{d}s.
\end{aligned}
$$

Using the same reasoning, one can show a similar result for $h < 0$. By applying Gronwall's lemma, we can deduce that x is uniformly continuous on $[0, b_\varphi)$. Consequently, $x(.,\varphi)$ can be extended to b_φ, which contradicts the maximality of $[0, b_\varphi)$.

We prove now that the solution depends continuously on the initial data. Let $\varphi \in \mathscr{B}$, $\varphi(0) \in \overline{D(A)}$ and $t \in [0, b_\varphi)$ be fixed. Set

$$
r = 1 + \sup_{0 \le s \le t} \|x_s(.,\varphi)\|_{\mathscr{B}}.
$$

Let $\varepsilon \in (0,1)$ and

$$
\psi \in \mathscr{B}, \quad \psi(0) \in \overline{D(A)} \quad \text{such that } \|\varphi - \psi\|_{\mathscr{B}} < \varepsilon.
$$

It is clear that

$$
\|\psi\|_{\mathscr{B}} \le \|\varphi\|_{\mathscr{B}} + \varepsilon < r.
$$

We put

$$
b_0 = \sup\{s > 0\colon \|x_\sigma(.,\psi)\|_{\mathscr{B}} \le r, \text{ for all } \sigma \in [0,s]\},
$$

$$
K_{b_0} := \max_{0 \le s \le b_0} K(s),
$$

$$
M_{b_0} := \sup_{0 \le s \le b_0} M(s),
$$

$$
c(t) = (M_{b_0} + HK_{b_0}\beta \mathrm{e}^{\omega t}) \exp(K_{b_0}\beta \mathrm{e}^{\omega t} c_0(r) t).
$$

We choose ε small enough such that

$$c(t)\varepsilon < 1.$$

Suppose that $b_0 < t$. We obtain for $s \in [0, b_0]$,

$$\begin{aligned}
\|x_s(.\,,\varphi) - x_s(.\,,\psi)\|_{\mathscr{B}} &\le M_{b_0}\|\varphi - \psi\|_{\mathscr{B}} + K_{b_0} \sup_{0\le\xi\le s} |x(\xi,\varphi) - x(\xi,\psi)|,\\
&\le M_{b_0}\|\varphi - \psi\|_{\mathscr{B}} + K_{b_0} \sup_{0\le\xi\le s} \{|S'(\xi)(\varphi(0) - \psi(0))|\\
&\quad + \left|\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} S(\xi-\sigma)(F(x_\sigma(.\,,\varphi)) - F(x_\sigma(.\,,\psi)))\,\mathrm{d}\sigma\right|\Bigg\},\\
&\le (HK_{b_0}\beta \mathrm{e}^{\omega t} + M_{b_0})\|\varphi - \psi\|_{\mathscr{B}}\\
&\quad + K_{b_0}\beta \mathrm{e}^{\omega t} c_0(r)\int_0^s \|x_\sigma(.\,,\varphi) - x_\sigma(.\,,\psi)\|_{\mathscr{B}}\,\mathrm{d}\sigma.
\end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|x_s(,\varphi) - x_s(,\psi)\|_{\mathscr{B}} \le c(t)\|\varphi - \psi\|_{\mathscr{B}}. \tag{4}$$

This implies that

$$\|x_s(.\,,\psi)\|_{\mathscr{B}} \le c(t)\varepsilon + r - 1 < r \quad \text{for all } s \in [0, b_0].$$

By continuity, there exists $\delta > 0$ such that

$$\|x_s(.\,,\psi)\|_{\mathscr{B}} \le c(t)\varepsilon + r - 1 < r \quad \text{for all } s \in [0, b_0 + \delta].$$

It follows that b_0 cannot be the largest number $s > 0$ such that $\|x_\sigma(.\,,\psi)\|_{\mathscr{B}} \le r$, for $\sigma \in [0, s]$. Thus, $b_0 \ge t$ and $t < b_\psi$. Furthermore, $\|x_s(.\,,\psi)\|_{\mathscr{B}} \le r$, for $s \in [0, t]$. Then, using inequality (4), we deduce the continuous dependence on the initial data. This completes the proof of the theorem. □

Our objective now is to give some sufficient conditions ensuring the regularity of the integral solution. Following the terminology of Nachbin [23] we introduce the notion of derivative with respect to $(\mathscr{B}, \|.\|_{\mathscr{B}})$.

Theorem 3. *Assume that* (H_1), *and* (H_2) *hold. In addition, assume that* $F : \mathscr{B} \to E$ *is continuously differentiable* (*in the sense of* [23]) *and* $F' : \mathscr{B} \to \mathscr{L}(\mathscr{B}, E)$ *is locally Lipschitz continuous. Let* $\varphi \in \mathscr{B}$ *be a continuously differentiable function with* $\varphi' \in \mathscr{B}$ *and*

$$\varphi(0) \in D(A), \quad \varphi'(0) \in \overline{D(A)} \quad \textit{and} \quad \varphi'(0) = A\varphi(0) + F(\varphi).$$

If $x(.\,,\varphi) : (-\infty, b_\varphi) \to E$ *is the integral solution of Eq.* (1) *given by Theorem* 2, *then* $x(.\,,\varphi)$ *is a strict solution of Eq.* (1).

Proof. Let $\varphi \in \mathscr{B}$ such that $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$. Let $x := x(.\,,\varphi)$ be the unique integral solution of Eq. (1) on $(-\infty, b_\varphi)$ and $b_1 \in (0, b_\varphi)$.

Consider the following equation:

$$\frac{dy}{dt}(t) = Ay(t) + F'(x_t)y_t, \quad t \geq 0,$$

$$y_0 = \varphi' \in \mathscr{B}.$$

Using a similar argument as in the prove of Theorem 2, we can show that the above equation has a unique integral solution $y : [0, b_1] \to E$, which is given by

$$y(t) = \begin{cases} S'(t)\varphi'(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F'(x_s)y_s\,ds & \text{for } t \in [0, b_1], \\ \varphi'(t) & \text{for } t \in (-\infty, 0]. \end{cases}$$

Define the function z by

$$z(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)\,ds & \text{for } t \in [0, b_1], \\ \varphi(t) & \text{for } t \in (-\infty, 0]. \end{cases} \tag{5}$$

We will prove that $x = z$. Using the expression of y, we obtain

$$z(t) = \varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)F'(x_s)y_s\,ds \quad \text{for } t \in [0, b_1].$$

From $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$, we deduce

$$S(t)\varphi'(0) = S(t)A\varphi(0) + S(t)F(\varphi).$$

Using Corollary 1, we get

$$S(t)\varphi'(0) = S'(t)\varphi(0) - \varphi(0) + S(t)F(\varphi).$$

On the other hand, using Axiom (A1-(ii)$'$), we can infer from (5) and the continuity of φ' that

$$z_t = \varphi + \int_0^t y_s\,ds \quad \text{for } t \in [0, b_1].$$

Consequently, the maps $t \mapsto z_t$ and $t \mapsto \int_0^t S(t-s)F(z_s)\,ds$ are continuously differentiable and we have

$$\frac{d}{dt}\int_0^t S(t-s)F(z_s)\,ds = S(t)F(\varphi) + \int_0^t S(t-s)F'(z_s)y_s\,ds.$$

So, we deduce that

$$S(t)F(\varphi) = \frac{d}{dt}\int_0^t S(t-s)F(z_s)\,ds - \int_0^t S(t-s)F'(z_s)y_s\,ds.$$

Consequently, z satisfies for $t \in [0, b_1]$

$$z(t) = S'(t)\varphi(0) + S(t)F(\varphi) + \int_0^t S(t-s)F'(z_s)y_s\,ds.$$

This implies that

$$z(t) = S'(t)\varphi(0) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)F(z_s)\,\mathrm{d}s - \int_0^t S(t-s)F'(z_s)y_s\,\mathrm{d}s + \int_0^t S(t-s)F'(x_s)y_s\,\mathrm{d}s.$$

Therefore,

$$x(t) - z(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-s)(F(x_s) - F(z_s))\,\mathrm{d}s - \int_0^t S(t-s)(F'(x_s) - F'(z_s))y_s\,\mathrm{d}s.$$

Then, we deduce

$$|x(t) - z(t)| \le \beta \mathrm{e}^{\omega b_1}\left(\int_0^t |F(x_s) - F(z_s)|\,\mathrm{d}s + \int_0^t \|F'(x_s) - F'(z_s)\|_{\mathscr{L}(\mathscr{B},E)}\|y_s\|_{\mathscr{B}}\,\mathrm{d}s\right).$$

Let

$$r = \max\left(\sup_{0\le s\le b_1}\|x_s\|_{\mathscr{B}},\ \sup_{0\le s\le b_1}\|y_s\|_{\mathscr{B}},\ \sup_{0\le s\le b_1}\|z_s\|_{\mathscr{B}}\right).$$

There exist $c_0(r)$, $v_0(r) \ge 0$ such that if $\varphi_1, \varphi_2 \in \mathscr{B}$ and $\|\varphi_1\|_{\mathscr{B}}$, $\|\varphi_2\|_{\mathscr{B}} \le r$, then

$$|F(\varphi_1) - F(\varphi_2)| \le c_0(r)\|\varphi_1 - \varphi_2\|_{\mathscr{B}},$$

$$\|F'(\varphi_1) - F'(\varphi_2)\|_{\mathscr{L}(\mathscr{B},E)} \le v_0(r)\|\varphi_1 - \varphi_2\|_{\mathscr{B}}.$$

This implies that

$$\sup_{0\le s\le t}|x(s) - z(s)| \le \beta \mathrm{e}^{\omega b_1}(c_0(r) + r v_0(r))\int_0^t \|x_s - z_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

Applying Axiom (A1-iii), we obtain

$$\|x_t - z_t\|_{\mathscr{B}} \le K_{b_1}\sup_{0\le s\le t}|x(s) - z(s)|,$$

where $K_{b_1} := \max_{0\le t\le b_1} K(t)$. Thus, we obtain

$$\|x_t - z_t\|_{\mathscr{B}} \le K_{b_1}\beta \mathrm{e}^{\omega b_1}(c_0(r) + r\, v_0(r))\int_0^t \|x_s - z_s\|_{\mathscr{B}}\,\mathrm{d}s.$$

By Gronwall's lemma, $\|x_t - z_t\|_{\mathscr{B}} = 0$ for any $t \in [0, b_1]$. Using Axiom (A1-ii), we conclude that $x(t) = z(t)$, for all $t \in (-\infty, b_1]$. Consequently, the integral solution x is continuously differentiable. Then, from Lemma 2 we conclude that x is a strict solution of Eq. (1). This completes the proof of the theorem. □

4. The solution semigroup and stability

In this section, we give some properties of the solution map associated to Eq. (1). We keep the assumption (H_1) and instead of (H_2), we make the following condition. (H_3) F is globally Lipschitz continuous on $\mathscr{B}$:

$$|F(\varphi_1)-F(\varphi_2)| \le L\|\varphi_1-\varphi_2\|_{\mathscr{B}} \quad \text{for } \varphi_1,\varphi_2 \in \mathscr{B}.$$

Using this assumption and a fixed point theorem, we can show that for all $\varphi \in \mathscr{B}$ such that $\varphi(0) \in \overline{D(A)}$, Eq. (1) has a unique integral solution which is defined on $\mathbb{R}$

Let

$$X = \{\varphi \in \mathscr{B} \colon \varphi(0) \in \overline{D(A)}\}.$$

Define the following operator $U(t)$ on X for $t \ge 0$ by

$$U(t)\varphi = x_t(.,\varphi),$$

where $x(.,\varphi)$ is the unique integral solution of Eq. (1). We can prove the following result.

Proposition 2. *Assume that* (H_1) *and* (H_3) *hold. Then, the family* $(U(t))_{t\ge 0}$ *is a strongly continuous semigroup on X, that is*

(i) $U(0)=I$,

(ii) $U(t+s)=U(t)U(s)$, *for all* $t,s \ge 0$,

(iii) *for all* $\varphi \in X$, $U(t)\varphi$ *is a continuous function of* $t \ge 0$ *with values in* X. *Moreover*,

(iv) $(U(t))_{t\ge 0}$ *satisfies, for* $t \ge 0$ *and* $\theta \in (-\infty,0]$, *the following translation property*:

$$(U(t)\varphi)(\theta) = \begin{cases} (U(t+\theta)\varphi)(0) & \text{if } t+\theta \ge 0,\\ \varphi(t+\theta) & \text{if } t+\theta \le 0,\end{cases}$$

(v) *there exist two positive locally bounded functions* $m(.)$, $n(.) : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for all* $\varphi_1,\varphi_2 \in X$ *and* $t \ge 0$,

$$\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathscr{B}} \le m(t)\mathrm{e}^{n(t)t}\|\varphi_1-\varphi_2\|_{\mathscr{B}}.$$

Proof. (i) and (ii) are consequence of the uniqueness of the integral solution of Eq. (1). (iii) and (iv) can be proved by using some properties of the integral Eq. (2) and Axiom (A2). To prove (v), we can see that

$$\begin{aligned}
\|U(t)\phi - U(t)\psi\|_{\mathscr{B}} &= \|x_t(.,\phi)-x_t(.,\psi)\|_{\mathscr{B}},\\
&\le K(t)\sup_{0\le s\le t}|x(s,\phi)-x(s,\psi)| + M(t)\|\phi-\psi\|_{\mathscr{B}},\\
&\le K(t)\sup_{0\le s\le t}|S'(s)(\phi(0)-\psi(0))| + M(t)\|\phi-\psi\|_{\mathscr{B}},\\
&\quad + K(t)\sup_{0\le s\le t}\left|\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s S(s-\sigma)(F(x_\sigma(.,\phi))-F(x_\sigma(.,\psi)))\,\mathrm{d}\sigma\right|.
\end{aligned}$$

From Lemma 1, we deduce that

$$\|U(t)\phi - U(t)\psi\|_{\mathscr{B}} \leq K(t)\beta e^{\omega t} H\|\phi - \psi\|_{\mathscr{B}} + M(t)\|\phi - \psi\|_{\mathscr{B}}$$
$$+ 2K(t)l(t)L\int_0^t \|x_\sigma(.,\phi) - x_\sigma(.,\psi)\|_{\mathscr{B}}\,d\sigma.$$

This completes the proof by using the Gronwall's lemma.

We will give now another property of $(U(t))_{t\geq 0}$. To do this, we need to suppose that the phase space $\mathscr{B}$ satisfies a further axiom (see [17] for discussion on this axiom):

(C) If $\{\phi_n\}_{n\geq 0}$ is a Cauchy sequence in $\mathscr{B}$ with respect to the seminorm and if ϕ_n converges compactly to ϕ on $(-\infty, 0]$, then ϕ is in $\mathscr{B}$ and $\|\phi_n - \phi\|_{\mathscr{B}} \to 0$.

We also make a compactness assumption on $(S'(t))_{t\geq 0}$:

(H_4) The semigroup $(S'(t))_{t\geq 0}$ is compact on $(\overline{D(A)}, |.|)$.

Proposition 3. *Assume that* (H_1), (H_3), (H_4) *and* (C) *hold. Then, the semigroup* $(U(t))_{t\geq 0}$ *is decomposed on* X *as follows*:

$$U(t) = U_1(t) + U_2(t), \quad t \geq 0,$$

where $(U_2(t))_{t\geq 0}$ *is the semigroup on* X *given, for* $\varphi \in X$, $t \geq 0$ *and* $\theta \in (-\infty, 0]$, *by*

$$(U_2(t)\varphi)(\theta) = \begin{cases} S'(t+\theta)\varphi(0), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0 \end{cases}$$

and $U_1(t)$ *is compact on* X *for* $t > 0$.

Proof. Let $(\psi_n)_{n\geq 0}$ be a bounded sequence in X. First, we prove that the sequence $((U_1(t)\psi_n)(\theta))_{n\geq 0}$, $t > 0$, $\theta \in (-\infty, 0]$, is totally bounded in E and the sequence $(U_1(t)\psi_n)_{n\geq 0}$, $t > 0$, is equicontinuous in $(-\infty, 0]$. Let $t > 0$ and $\theta \in (-\infty, 0]$ be fixed. We have

$$(U_1(t)\psi_n)(\theta) = \begin{cases} (U_1(t+\theta)\psi_n)(0) & t+\theta \geq 0, \\ 0 & t+\theta \leq 0. \end{cases}$$

Let $0 < \varepsilon < t + \theta$. Then, for $\lambda > \omega$, we have

$$\lambda R(\lambda, A)(U_1(t)\psi_n)(\theta) = \int_0^{t+\theta} S'(t+\theta-s)\lambda R(\lambda, A)F(x_s(.,\psi_n))\,ds,$$
$$= S'(\varepsilon)\int_0^{t+\theta-\varepsilon} S'(t+\theta-\varepsilon-s)\lambda R(\lambda, A)F(x_s(.,\psi_n))\,ds,$$
$$+ \int_{t+\theta-\varepsilon}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda, A)F(x_s(.,\psi_n))\,ds,$$

where $x(.,\psi_n)$ is the unique integral solution of Eq. (1) for the initial condition ψ_n. Since $S'(\varepsilon)$ is compact, there exists a compact set W_ε such that

$$S'(\varepsilon)\left\{\int_0^{t+\theta-\varepsilon} S'(t+\theta-\varepsilon-s)\lambda R(\lambda, A)F(x_s(.,\psi_n))\,ds\colon n \geq 0,\ \lambda > \omega\right\} \subset W_\varepsilon.$$

Furthermore, for all $n \geq 0$

$$\left| \int_{t+\theta-\varepsilon}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda, A)F(x_s(.\,,\psi_n))\,\mathrm{d}s \right| \leq \frac{\lambda\beta}{\lambda-\omega}\beta_t\alpha_t\varepsilon,$$

where

$$\alpha_t = \sup_{s\in[0,t]} \{|F(x_s(.\,,\psi_n)|\colon n \geq 0\} \quad \text{and} \quad \beta_t = \sup_{s\in[0,t]} \|S'(s)\|_{\overline{D(A)}}.$$

This shows, by letting λ to $+\infty$, that $((U_1(t)\psi_n)(\theta))_{n\geq 0}$, $t > 0$, $\theta \in (-\infty, 0]$, is totally bounded. To establish the equicontinuity, let $t > 0$ and $\theta_0 \in (-\infty, 0]$ be fixed. For $\theta \in (-\infty, 0]$ such that $|\theta - \theta_0|$ small enough and $\theta_0 < \theta$, we obtain

$$(U_1(t)\psi_n)(\theta) - (U_1(t)\psi_n)(\theta_0) = \begin{cases} (U_1(t+\theta)\psi_n)(0) - (U_1(t+\theta_0)\psi_n)(0), & t+\theta_0 > 0, \\ (U_1(\theta-\theta_0)\psi_n)(0), & t+\theta_0 = 0, \\ 0, & t+\theta_0 < 0. \end{cases}$$

Therefore, for $-t < \theta_0 < \theta \leq 0$ and $\lambda > \omega$, we obtain

$$\begin{aligned} &\lambda R(\lambda, A)((U_1(t)\psi_n)(\theta) - (U_1(t)\psi_n)(\theta_0)) \\ &\quad = \int_0^{t+\theta_0} (S'(t+\theta-s) - S'(t+\theta_0-s))\lambda R(\lambda, A)F(x_s(.\,,\psi_n))\,\mathrm{d}s \\ &\qquad + \int_{t+\theta_0}^{t+\theta} S'(t+\theta-s)\lambda R(\lambda, A)F(x_s(.\,,\psi_n))\,\mathrm{d}s. \end{aligned}$$

Which leads to

$$\begin{aligned} &|\lambda R(\lambda, A)((U_1(t)\psi_n)(\theta) - (U_1(t)\psi_n)(\theta_0))| \\ &\quad \leq \left| (S'(\theta-\theta_0) - I) \int_0^{t+\theta_0} S'(t+\theta_0-s)\lambda R(\lambda, A)F(x_s(.\,,\psi_n))\,\mathrm{d}s \right| \\ &\qquad + \frac{\lambda\beta}{\lambda-\omega}\beta_t\alpha_t(\theta-\theta_0). \end{aligned}$$

Besides, there exists a compact set W such that

$$\left\{ \int_0^{t+\theta_0} |S'(t+\theta_0-s)\lambda R(\lambda, A)F(x_s(.\,,\psi_n))|\,\mathrm{d}s\colon n \geq 0,\ \lambda > \omega \right\} \subseteq W.$$

Letting λ to $+\infty$ and using the fact that $(S'(.)x)_{x\in W}$ is equicontinuous at the right in 0, gives

$$\lim_{\substack{\theta\to\theta_0 \\ \theta>\theta_0}} |(U_1(t)\psi_n)(\theta) - (U_1(t)\psi_n)(\theta_0)| = 0.$$

By a similar argument we deduce, for $-\infty < \theta < \theta_0 \leq 0$, the claimed equicontinuity. We will conclude by using Axiom (C). Proceeding by Arzela–Ascoli's theorem, we can see that there is a continuous function $\phi:(-\infty,0]\to E$ and a subsequence ϕ_n of $(U_1(t)\psi_n)_{n\geq 0}$, which converges compactly to ϕ in $(-\infty,0]$. By Remark 2, $(\phi_n)_{n\geq 0}$ is also a Cauchy sequence in $\mathscr{B}$ with respect to the seminorm. Then from Axiom (C), ϕ is in $\mathscr{B}$ and $\|\phi_n-\phi\|_{\mathscr{B}}\to 0$ as $n\to\infty$. We have proved that the image of any bounded sequence contains a converging subsequence in $\mathscr{B}$ with respect to the seminorm. This ends the proof of the proposition.

We focus now our attention on the stability near an equilibrium of the nonlinear semigroup $(U(t))_{t\geq 0}$ on X. We keep the assumptions (H_1) and (H_3), and we add the following:
(H_5) F is continuously differentiable at 0 with respect to $(\mathscr{B},\|.\|_{\mathscr{B}})$ and $F(0)=0$.

Consider the linearized equation of (1) corresponding to the derivative $F'(0)$ at 0:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t}(t)=Ax(t)+L(x_t),\quad t\geq 0,\\&x_0=\varphi\in\mathscr{B},\end{aligned}\tag{6}$$

where $L:=F'(0)$ and $(T(t))_{t\geq 0}$ is the solution semigroup on X associated to Eq. (6).

Proposition 4. *Assume that* (H_1), (H_3) *and* (H_5) *hold. Then, the derivative at zero of the nonlinear semigroup* $(U(t))_{t\geq 0}$, *associated to Eq.* (1) *is the linear semigroup* $(T(t))_{t\geq 0}$ *associated to Eq.* (6).

Proof. It suffices to show that for each $t\geq 0$ and $\varepsilon>0$, there exists $\delta>0$ such that

$$\|U(t)\varphi-T(t)\varphi\|_{\mathscr{B}}\leq\varepsilon\|\varphi\|_{\mathscr{B}}\quad\text{for }\|\varphi\|_{\mathscr{B}}\leq\delta.$$

Note that

$$\|U(t)\varphi-T(t)\varphi\|_{\mathscr{B}}=\|x_t(.,\varphi)-y_t(.,\varphi)\|_{\mathscr{B}},$$

where $x(.,\varphi)$ (resp. $y(.,\varphi)$) is the unique integral solution of Eq. (1) (resp. Eq. (6)).

From Axiom (A-iii), we see that for all $t\geq 0$

$$\begin{aligned}\|U(t)\varphi-T(t)\varphi\|_{\mathscr{B}}&\leq K(t)\sup_{0\leq\sigma\leq t}|x(\sigma,\varphi)-y(\sigma,\varphi)|,\\&\leq K(t)\sup_{0\leq\sigma\leq t}\left|\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_0^\sigma S(\sigma-s)(F(U(s)\varphi)-L(T(s)\varphi))\,\mathrm{d}s\right|,\\&\leq K(t)\beta\mathrm{e}^{\omega t}\int_0^t\mathrm{e}^{-\omega s}|F(U(s)\varphi)-L(T(s)\varphi)|\,\mathrm{d}s,\\&\leq K(t)\beta\mathrm{e}^{\omega t}\left(\int_0^t\mathrm{e}^{-\omega s}|F(U(s)\varphi)-L(U(s)\varphi)|\,\mathrm{d}s\right.\\&\quad\left.+\int_0^t\mathrm{e}^{-\omega s}|L(U(s)\varphi)-L(T(s)\varphi)|\,\mathrm{d}s\right).\end{aligned}$$

By virtue of the continuous differentiability of F at 0 and from (v) of Proposition 2, we deduce that for $\varepsilon>0$ there exists $\delta>0$ such that

$$\int_0^t\mathrm{e}^{-\omega s}|F(U(s)\varphi)-L(U(s)\varphi)|\,\mathrm{d}s\leq\varepsilon\|\varphi\|_{\mathscr{B}}\quad\text{for }\|\varphi\|_{\mathscr{B}}\leq\delta.$$

On the other hand, we find that

$$\int_0^t \mathrm{e}^{-\omega s}|L(U(s)\varphi) - L(T(s)\varphi)|\,\mathrm{d}s \le \|L\|_{\mathscr{L}(\mathscr{B},E)} \int_0^t \mathrm{e}^{-\omega s}\|U(s)\varphi - T(s)\varphi\|_{\mathscr{B}}\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned}&\|U(t)\varphi - T(t)\varphi\|_{\mathscr{B}}\\ &\quad\le K(t)\beta\mathrm{e}^{\omega t}\left(\varepsilon\|\varphi\|_{\mathscr{B}} + \|L\|_{\mathscr{L}(\mathscr{B},E)}\int_0^t \mathrm{e}^{-\omega s}\|U(s)\varphi - T(s)\varphi\|_{\mathscr{B}}\,\mathrm{d}s\right).\end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|U(t)\varphi - T(t)\varphi\|_{\mathscr{B}} \le K_t\beta\varepsilon\|\varphi\|_{\mathscr{B}}\exp((\|L\|_{\mathscr{L}(\mathscr{B},E)}K_t\beta + \omega)t), \quad t \ge 0,$$

where $K_t = \max_{0\le s\le t} K(s)$. Hence, we conclude that $U(t)$ is differentiable at 0 and $(D_\varphi U(t))(0) = T(t)$, for each $t \ge 0$.

Definition 8 (*Desch and Schappachev* [9]). Let $(V(t))_{t\ge 0}$ be a strongly continuous semigroup on a Banach space $(Y, |.|_Y)$. A point $x_0 \in Y$ is called an equilibrium of $(V(t))_{t\ge 0}$ *if* $V(t)x_0 = x_0$ for all $t \ge 0$. An equilibrium $x_0 \in Y$ is called exponentially stable if there exist $\delta > 0$, $\mu > 0$ and $k \ge 1$ such that

$$|V(t)x - x_0|_Y \le k\mathrm{e}^{-\mu t}|x - x_0|_Y \quad \text{for } t \ge 0 \quad \text{and} \quad x \in Y \quad \text{with } |x - x_0|_Y \le \delta.$$

Theorem 4 (Desch and Schappachev [9]). *Let $(V(t))_{t\ge 0}$ be a nonlinear strongly continuous semigroup in a Banach space Y. Assume that $x_0 \in Y$ is an equilibrium of $(V(t))_{t\ge 0}$ such that $V(t)$ is Fréchet-differentiable at x_0 for each $t \ge 0$, with $W(t)$ the Fréchet-derivative at x_0 of $V(t)$. Then, $(W(t))_{t\ge 0}$ is a strongly continuous semigroup of bounded linear operators on Y. Moreover, if $(W(t))_{t\ge 0}$ is exponentially stable, then x_0 is an exponentially stable equilibrium of $(V(t))_{t\ge 0}$.*

By using the above theorem in the Banach space of equivalence classes $\mathscr{B}/\|.\|_{\mathscr{B}}$, we get our main result concerning the stability of an equilibrium of $(U(t))_{t\ge 0}$. (See [2,26] for an earlier use of Theorem 4).

Theorem 5. *Assume that* (H$_1$), (H$_3$) *and* (H$_5$) *hold. If $(T(t))_{t\ge 0}$ is exponentially stable on X, i.e., there exist constants $\bar\beta \ge 1$ and $\bar\omega > 0$ such that $\|T(t)\varphi\|_{\mathscr{B}} \le \bar\beta\mathrm{e}^{-\bar\omega t}\|\varphi\|_{\mathscr{B}}$, for $t \ge 0$ and $\varphi \in X$, then zero is an exponentially stable equilibrium of $(U(t))_{t\ge 0}$ on X, i.e., there exist $\delta > 0$, $\mu > 0$ and $k \ge 1$ such that*

$$\|U(t)\psi - \varphi\|_{\mathscr{B}} \le k\mathrm{e}^{-\mu t}\|\psi - \varphi\|_{\mathscr{B}} \quad \textit{for } t \ge 0 \textit{ and } \psi \in X \quad \textit{with } \|\psi - \varphi\|_{\mathscr{B}} \le \delta.$$

Next, we characterize the infinitesimal generator of the semigroup $(T(t))_{t\ge 0}$ on X. We assume that $\mathscr{B}$ satisfies the following axiom which was introduced by Ruess and Summers [30]:

(D) For a sequence $(\varphi_n)_{n\ge 0}$ in $\mathscr{B}$, if $\|\varphi_n\|_{\mathscr{B}} \to 0$ then $|\varphi_n(s)| \to 0$ for each $s \in (-\infty, 0]$.

Let A_T: $D(A_T) \subseteq X \to X$ be defined by

$$D(A_T) = \{\varphi \in X\colon \varphi' \in X, \varphi(0) \in D(A) \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi)\},$$

$$A_T\varphi = \varphi'.$$

Proposition 5. *Assume that $\mathscr{B}$ is a subspace of $C((-\infty,0];E)$ (the space of continuous functions from $(-\infty,0]$ into E) satisfying Axioms* (A1), (A2), (B) *and* (D). *Then, A_T is the infinitesimal generator of $(T(t))_{t\geq 0}$.*

Proof. Let $\varphi \in X$ be continuously differentiable such that

$$\varphi' \in X, \quad \varphi(0) \in D(A) \quad \text{and} \quad \varphi'(0) = A\varphi(0) + L(\varphi).$$

Let $x(.,\varphi)$ be the integral solution of Eq. (1). In the proof of Theorem 3, we showed that there exists a continuous mapping $y : \mathbb{R} \to E$ with $y_0 = \varphi'$ such that

$$x_t(.,\varphi) = \varphi + \int_0^t y_s\,\mathrm{d}s \quad \text{for } t \geq 0.$$

Which implies that $\lim_{t\to 0^+} 1/t(x_t(.,\varphi)-\varphi)$ exists and is equal to φ'. Then, $\varphi \in D(A_T)$.

Conversely, let $\varphi \in X$ such that

$$\lim_{t\to 0^+} \frac{1}{t}(x_t(.,\varphi) - \varphi) = \psi = A_T\varphi \quad \text{exists in } X.$$

Axiom (D) implies that

$$\lim_{t\to 0^+} \frac{1}{t}(x_t(\theta,\varphi) - \varphi(\theta)) \quad \text{exists for all } \theta \leq 0 \quad \text{and is equal to } \psi(\theta).$$

Then, for $\theta \in (-\infty,0)$, we have

$$\psi(\theta) = \lim_{t\to 0^+} \frac{1}{t}(\varphi(t+\theta) - \varphi(\theta)) = D^+\varphi(\theta),$$

where $D^+\varphi$ is the right derivative of φ, and $D^+\varphi=\psi$ in $(-\infty,0)$. Since ψ is continuous, $D^+\varphi$ is also continuous in $(-\infty,0)$. Let us introduce the following result:

Lemma 3 (Pazy [27]). *Let φ be continuous and differentiable on the right on $[a,b)$. If $D^+\varphi$ is continuous on $[a,b)$, then φ is continuously differentiable on $[a,b)$.*

From the above lemma, we deduce that the function φ is continuously differentiable in $(-\infty,0)$ and $\varphi' = \psi$. On the other hand, for $\theta = 0$, one has $\lim_{\theta\to 0^-} \varphi'(\theta)$ exists and equals to $\psi(0)$. From this we infer that the function φ is continuously differentiable in $(-\infty,0]$ and $\varphi' = \psi \in X$. Since $t \to x_t(.,\varphi)$ is continuously differentiable in $[0,+\infty)$ and using Axiom (D), we can state that the map $t \to x(t,\varphi)$ is continuously differentiable in $[0,+\infty)$. Then, from Lemma 2, x is a strict solution of Eq. (6), and for $\theta = 0$, one has

$$\varphi'(0) = \psi(0) = \lim_{t\to 0} \frac{1}{t}(x(t) - \varphi(0)) = x'(0) \quad \text{and} \quad x'(0) = A\varphi(0) + L(\varphi).$$

From this we conclude that

$$\varphi'(0) = A\varphi(0) + L(\varphi).$$

This completes the proof of proposition. □

Let γ be a positive real constant and

$$C_\gamma = \left\{\phi\colon (-\infty, 0] \to E \text{ continuous such that } \lim_{\theta\to-\infty} \mathrm{e}^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\}.$$

Then, it is shown in Theorem 1.3.7 of the book [17] that C_γ together with the norm $\|\phi\|_\gamma = \sup_{\theta\le 0} \mathrm{e}^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$, is a standard example of phase space $\mathscr{B}$ that satisfies axioms (A1), (A2), (B), (C) and (D).

In the sequel, we give a necessary and sufficient condition to ensure the stability of the linear semigroup $(T(t))_{t\ge 0}$ on X. We need the following results. Let $\mathscr{H} : Z \to Z$ be a closed linear operator with a dense domain $D(\mathscr{H})$ in a Banach space Z. We denote by $\sigma(\mathscr{H})$ the spectrum of $\mathscr{H}$, and by $\sigma_p(\mathscr{H})$ the point spectrum of $\mathscr{H}$.

We call essential spectrum of $\mathscr{H}$ (denotation by $\sigma_{\mathrm{e}}(\mathscr{H})$) the set of all λ in $\sigma(\mathscr{H})$ for which at least one of the following holds:

(i) $\mathrm{Im}(\lambda I - \mathscr{H}) = \{(\lambda I - \mathscr{H})z\colon z \in Z\}$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathscr{H}) = \bigcup_{k\ge 1} \ker(\lambda I - \mathscr{H})^k$ of λ is infinite dimensional,
(iii) λ is a limit point of $\sigma(\mathscr{H})$.

If we suppose that $\mathscr{H}$ is bounded then, we define the Kuratowski measure of noncompactness of $\mathscr{H}$ by

$$\alpha(\mathscr{H}) = \inf\{k \in \mathbb{R}^+\colon \alpha(\mathscr{H}(B)) \le k\alpha(B) \text{ for every bounded subset } B \text{ of } Z\}.$$

The radius of $\sigma_{\mathrm{e}}(\mathscr{H})$ is given by

$$r_{\mathrm{e}}(\mathscr{H}) = \sup\{|\lambda|\colon \lambda \in \sigma_{\mathrm{e}}(\mathscr{H})\}.$$

Nussbaum proved in [25] that

$$r_{\mathrm{e}}(\mathscr{H}) = \lim_{n\to+\infty} [\alpha(\mathscr{H}^n)]^{1/n}. \tag{7}$$

Since $\alpha(\mathscr{H}) \le \|\mathscr{H}\|$, by Definition 7 one has

$$r_{\mathrm{e}}(\mathscr{H}) \le \|\mathscr{H}\|. \tag{8}$$

Let $\lambda \in \sigma(\mathscr{H}) - \sigma_{\mathrm{e}}(\mathscr{H})$, it follows from [39] that λ is an isolate point, $\dim M_\lambda(\mathscr{H}) < \infty$ and $\lambda \in \sigma_p(\mathscr{H})$.

Define the growth bound of the semigroup $(T(t))_{t\ge 0}$ by

$$\omega((T(t))_{t\ge 0}) := \inf\left\{\omega \in \mathbb{R}\colon \sup_{t\ge 0} \mathrm{e}^{-\omega t}\|T(t)\| < \infty\right\}.$$

Set

$$s_1(A_T) = \sup\{Re\,\lambda\colon \lambda \in \sigma(A_T) - \sigma_{\mathrm{e}}(A_T)\}.$$

It is well known in general that

$$-\infty \le s_1(A_T) \le \omega((T(t))_{t\ge 0}) < \infty.$$

Define the essential growth bound of $(T(t))_{t\geq 0}$ by

$$\omega_{\mathrm{e}}((T(t))_{t\geq 0}) := \lim_{t\to\infty} \frac{1}{t}\log(\alpha(T(t))) = \inf\left\{\frac{1}{t}\log(\alpha(T(t))):\ t>0\right\}.$$

According to [39], we have

$$\begin{aligned} &\omega((T(t))_{t\geq 0}) = \max\{\omega_{\mathrm{e}}((T(t))_{t\geq 0}),\ s_1(A_T)\} \\ &r_{\mathrm{e}}(T(t)) = \exp(t\omega_{\mathrm{e}}((T(t))_{t\geq 0})), \quad t>0. \end{aligned} \tag{9}$$

We make the following assumption.

(H_6) There exist two positive constants μ and δ such that

$$\|S'(t)\|_{\overline{D(A)}} \leq \mu \mathrm{e}^{-\delta t}, \quad t\geq 0.$$

Proposition 6. *Assume that* (H_1), (H_3), (H_4), (H_5) *and* (H_6) *hold and* $\mathscr{B} = C_\gamma$. *If* $s_1(A_T) < 0$, *then zero is an exponentially stable equilibrium of* $(U(t))_{t\geq 0}$.

Proof. Since from Proposition 3, the semigroup $(T(t))_{t\geq 0}$ is decomposed as follows:

$$T(t) = T_1(t) + T_2(t), \quad t\geq 0,$$

where $(T_2(t))_{t\geq 0}$ is the semigroup on X given, for $\varphi\in X$, $t\geq 0$ and $\theta\in(-\infty,0]$, by

$$(T_2(t)\varphi)(\theta) = \begin{cases} S'(t+\theta)\varphi(0), & t+\theta\geq 0, \\ \varphi(t+\theta), & t+\theta\leq 0 \end{cases}$$

and $T_1(t): X\to X$ is compact for $t>0$. Then we have

$$r_{\mathrm{e}}(T(t)) = r_{\mathrm{e}}(T_2(t)).$$

For every $\phi\in X$, we obtain

$$\begin{aligned} \|T_2(t)\phi\|_\gamma &= \sup_{s\leq 0} \mathrm{e}^{\gamma s}|(T_2(t)\phi)(s)|, \\ &= \max\left\{\sup_{-t\leq s\leq 0}(\mathrm{e}^{\gamma s}|S'(t+s)\phi(0)|),\ \sup_{s\leq -t}\mathrm{e}^{\gamma s}|\phi(t+s)|\right\}, \\ &\leq \max\left\{\sup_{-t\leq s\leq 0}(\mu\mathrm{e}^{\gamma s}\mathrm{e}^{-\delta(t+s)}|\phi(0)|),\ \sup_{s\leq -t}\mathrm{e}^{-\gamma t}\mathrm{e}^{\gamma(t+s)}|\phi(t+s)|\right\}, \\ &\leq \mu\max\left\{\mathrm{e}^{-\delta t}\sup_{-t\leq s\leq 0}(\mathrm{e}^{-(\delta-\gamma)s}), \mathrm{e}^{-\gamma t}\right\}\|\phi\|_\gamma, \end{aligned}$$

which implies that $\|T_2(t)\phi\|_\gamma \leq \mu\mathrm{e}^{-\min(\delta,\gamma)t}\|\phi\|_\gamma$. Using (8), we deduce that

$$r_{\mathrm{e}}(T(t)) \leq \mu\mathrm{e}^{-\min(\delta,\gamma)t}, \quad t>0.$$

Which implies that for t large enough, $r_{\mathrm{e}}(T(t)) < 1$, and by (9) we remark that $\omega_{\mathrm{e}}((T(t))_{t\geq 0}) < 0$. So, we conclude that $\omega((T(t))_{t\geq 0}) < 0$ and the semigroup $(T(t))_{t\geq 0}$ is exponentially asymptotically stable. Consequently, from Theorem 5, we deduce that zero is an exponentially stable equilibrium of $(U(t))_{t\geq 0}$. This completes the proof of the proposition.

5. Application

In this section, we apply our abstract results to the following partial functional differential equations with infinite delay

$$\frac{\partial}{\partial t}w(t,\xi) = a\frac{\partial^2}{\partial \xi^2}w(t,\xi) + bw(t,\xi) + c\int_{-\infty}^{0} G(\theta)w(t+\theta,\xi)\,\mathrm{d}\theta + f(w(t-\tau,\xi)), \quad t \geq 0,\ 0 \leq \xi \leq \pi,$$

$$w(t,0) = w(t,\pi) = 0, \quad t \geq 0,$$

$$w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ 0 \leq \xi \leq \pi. \tag{10}$$

where a, b, c and τ are positive constants, $f:\mathbb{R} \to \mathbb{R}$ is a continuous function, G is a positive integrable function on $(-\infty,0]$ and $w_0 : (-\infty,0]\times[0,\pi] \to \mathbb{R}$ is an appropriate continuous function. (Compare with [28]).

System (10) can be written as the abstract Eq. (1). We choose $E = C([0,\pi];\mathbb{R})$ and we consider the operator $A : D(A) \subseteq E \to E$ defined by

$$\begin{cases} D(A) = \{y \in C^2([0,\pi];\mathbb{R}):\ y(0) = y(\pi) = 0\}, \\ Ay = ay''. \end{cases}$$

It is well known, see [8], that

$$(0,+\infty) \subset \rho(A),$$

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

This implies that Assumption (H_1) is satisfied. On the other hand, we can see that

$$\overline{D(A)} = \{y \in E:\ y(0) = y(\pi) = 0\} \neq E.$$

Let, for $\gamma > 0$,

$$\mathscr{B} = C_\gamma = \left\{\varphi \in C((-\infty,0];E):\ \lim_{\theta\to-\infty} \mathrm{e}^{\gamma\theta}\varphi(\theta) \text{ exists in } E\right\}$$

with the norm $\|\phi\|_\gamma = \sup_{\theta\leq 0} \mathrm{e}^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$. This space satisfies axioms (A1), (A2), (B), (C) and (D). Set

$$x(t)(\xi) = w(t,\xi), \quad t \geq 0,\ \xi \in [0,\pi],$$

$$\varphi(\theta)(\xi) = w_0(\theta,\xi), \quad \theta \leq 0,\ \xi \in [0,\pi],$$

$$F(\phi)(\xi) = b\phi(0)(\xi) + f(\phi(-\tau)(\xi)) + c\int_{-\infty}^{0} G(\theta)\phi(\theta)(\xi)\,\mathrm{d}\theta, \quad \xi \in [0,\pi],\ \phi \in C_\gamma.$$

Then, Eq. (10) can be transformed as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + F(x_t), \quad t \geq 0,$$
$$x_0 = \varphi \in C_\gamma.$$

We suppose that:

(i) f is locally Lipschitz continuous,

(ii) $G(.)\mathrm{e}^{-\gamma\cdot}$ is integrable on $(-\infty,0]$,

(iii) $w_0 \in C((-\infty,0]\times[0,\pi];\mathbb{R})$, with $\lim_{\theta\to-\infty}(\mathrm{e}^{\gamma\theta}\sup_{0\le\xi\le\pi}|w_0(\theta,\xi)|)$ exists, and $w_0(0,0)=w_0(0,\pi)=0$.

We have, for every $\phi_1,\phi_2\in C_\gamma$,

$$\begin{aligned}
&\sup_{0\le\xi\le\pi}\int_{-\infty}^{0} G(\theta)|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|\,\mathrm{d}\theta\\
&\quad=\sup_{0\le\xi\le\pi}\int_{-\infty}^{0}\mathrm{e}^{-\gamma\theta}G(\theta)(\mathrm{e}^{\gamma\theta}|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|)\,\mathrm{d}\theta,\\
&\quad\le\left(\int_{-\infty}^{0}\mathrm{e}^{-\gamma\theta}G(\theta)\,\mathrm{d}\theta\right)\sup_{\substack{-\infty<\theta\le0\\0\le\xi\le\pi}}\mathrm{e}^{\gamma\theta}|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|.
\end{aligned}$$

Let $r>0$ and $\phi_1,\phi_2\in C_\gamma$ such that $\|\phi_1\|_\gamma,\|\phi_2\|_\gamma\le r$.

Then, $|\phi_1(-\tau)(\xi)|,|\phi_2(-\tau)(\xi)|\le r\mathrm{e}^{\gamma\tau}$, for every $\xi\in[0,\pi]$. Thus, from Assumption (i), there exists a positive constant $b(r)$ which depends only on r such that

$$\begin{aligned}
\sup_{0\le\xi\le\pi}|f(\phi_1(-\tau)(\xi))-f(\phi_2(-\tau)(\xi))|&\le b(r)\sup_{0\le\xi\le\pi}|\phi_1(-\tau)(\xi)-\phi_2(-\tau)(\xi)|,\\
&\le b(r)\mathrm{e}^{\gamma\tau}\|\phi_1-\phi_2\|_\gamma.
\end{aligned}$$

We conclude that the assumptions (i), (ii) and (iii) imply that F is locally Lipschitz continuous and $\varphi\in C_\gamma$ with $\varphi(0)\in\overline{D(A)}$. Consequently, Theorem 2 ensures the existence of a maximal interval of existence $(-\infty,b_{w_0})$ and a unique integral solution $w(t,\xi)$ on $(-\infty,b_{w_0})\times[0,\pi]$.

To investigate that the integral solution w of Eq. (10) is a strict one, we add the following assumptions

(iv) f is continuously differentiable and f' is locally Lipschitz continuous,

(v) $w_0\in C^2((-\infty,0]\times[0,\pi];E)$, with $\lim_{\theta\to-\infty}(\mathrm{e}^{\gamma\theta}\sup_{0\le\xi\le\pi}((\partial/\partial\theta)w_0(\theta,\xi)))$ exists, $(\partial/\partial\theta)w_0(0,0)=(\partial/\partial\theta)w_0(0,\pi)=0$ and

$$\begin{aligned}
\frac{\partial}{\partial\theta}w_0(0,\xi)&=a\frac{\partial^2}{\partial\xi^2}w_0(0,\xi)+bw_0(0,\xi))\\
&\quad+c\int_{-\infty}^{0}G(\theta)w_0(\theta,\xi))\,\mathrm{d}\theta+f(w_0(-\tau,\xi))\quad\text{for }\xi\in[0,\pi].
\end{aligned}$$

Then, F is continuously differentiable on C_γ and for $\phi,\psi\in C_\gamma$, $\xi\in[0,\pi]$, we have

$$F'(\phi)(\psi)(\xi)=b\psi(0)(\xi)+c\int_{-\infty}^{0}G(\theta)\psi(\theta)(\xi)\,\mathrm{d}\theta+f'(\phi(-\tau)(\xi))\psi(-\tau)(\xi).$$

F' is also locally Lipschitz continuous on C_γ. Consequently, all the conditions in Theorem 3 are satisfied. Hence, w is a strict solution of Eq. (10).

Now, if instead of Assumption (iv), we assume that
(vi) f is continuously differentiable at 0, $f(0)=0$ $f'(0)=0$ and f is globally Lipschitz.

Then, F is continuously differentiable at 0 with $F(0)=0$ and F is globally Lipschitz on C_γ.

Consider now the linearized equation of (10) corresponding to the derivative $F'(0)$ at 0:

$$\begin{aligned}
&\frac{\partial}{\partial t}w(t,\xi) = a\frac{\partial^2}{\partial \xi^2}w(t,\xi) + bw(t,\xi) + c\int_{-\infty}^{0} G(\theta)w(t+\theta,\xi)\,\mathrm{d}\theta,\\
&\qquad\qquad t \geq 0,\ \ 0 \leq \xi \leq \pi,\\
&w(t,0) = w(t,\pi) = 0, \qquad t \geq 0,\\
&w(\theta,\xi) = w_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ \ 0 \leq \xi \leq \pi.
\end{aligned} \tag{11}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$ given by

$$D(A_0) = \{y \in C^2([0,\pi];\mathbb{R})\colon\ y(0) = y''(0) = y(\pi) = y''(\pi) = 0\},$$

$$A_0 y = ay''.$$

It is well known that A_0 generates a strongly continuous compact exponentially stable semigroup in $\overline{D(A)}$ (in the sense of (H_4) and (H_6)).

Let $\sigma_p(A_T)$ denotes the point spectrum of A_T. Then, $\lambda \in \sigma_p(A_T)$ if and only if there exists $\phi \in D(A_T)$, $\phi \neq 0$ such that $A_T\phi = \lambda\phi$. It follows that $\phi(\theta) = \mathrm{e}^{\lambda\theta}u$ with $u \neq 0$, $u \in D(A)$ and $\lambda u = Au + F'(0)(\mathrm{e}^{\lambda\cdot}u)$. On the other hand, $\phi \in C_\gamma$. This is true only if $Re\,\lambda \geq -\gamma$. Then, $\lambda \in \sigma_p(A_T)$ if and only if there exists $u \in D(A)$ and $u \neq 0$ such that

$$\begin{aligned}
&Re\,\lambda \geq -\gamma,\\
&\lambda u = Au + bu + c\int_{-\infty}^{0} G(\theta)\mathrm{e}^{\lambda\theta}u\,\mathrm{d}\theta.
\end{aligned}$$

This means that

$$\begin{aligned}
&Re\,\lambda \geq -\gamma,\\
&\lambda - b - c\int_{-\infty}^{0} G(\theta)\mathrm{e}^{\lambda\theta}\,\mathrm{d}\theta \in \sigma_p(A) = \sigma_p(A_0).
\end{aligned}$$

We know that the point spectrum $\sigma_p(A_0)$ of A_0 is given by

$$\sigma_p(A_0) = \{-an^2\colon\ n \in \mathbb{N}^*\}.$$

It follows that the exponential stability of solutions of Eq. (11) is determined by the following characteristic equation:

$$\begin{aligned}
&Re\,\lambda \geq -\gamma,\\
&\lambda - b - c\int_{-\infty}^{0} G(\theta)\mathrm{e}^{\lambda\theta}\,\mathrm{d}\theta = -an^2, \quad n \geq 1.
\end{aligned} \tag{12}$$

In [21], Lenhart and Travis established the following result.

Lemma 4 (Lenhart and Travis [21]). *All roots of System* (12) *have negative real part if*

$$\int_{-\infty}^{0} G(\theta)\,\mathrm{d}\theta = 1, \quad c < a - b. \tag{13}$$

From this lemma, we conclude that if Condition 13 is satisfied, then the solution semigroup associated to Eq. (11) is exponentially stable. In that case, by Proposition 6 we conclude that zero is an exponentially stable equilibrium of the solution semigroup associated to Eq. (10).

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