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Variation of constants formula and almost periodic solutions for some partial functional differential equations with infinite delay [☆]

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Abstract

In this work, we give a variation of constants formula for partial functional differential equations with infinite delay. We assume that the linear part is not necessarily densely defined and its resolvent operator satisfies the Hille–Yosida condition. We establish a reduction of the problem to a finite-dimensional space which allows us to prove the existence of almost periodic solutions.

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1. Introduction

The aim of this work is to establish a variation of constants formula and to study the existence of almost periodic solutions for the following partial functional differential equation with infinite delay

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t), & t \geq \sigma, \\ x_\sigma = \phi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \rightarrow X$ is a nondensely defined linear operator on a Banach space X , \mathcal{B} is a normed linear space of functions mapping $(-\infty, 0]$ to X and satisfying some fundamental axioms which are introduced in Section 2. For every $t \geq \sigma$, the function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in (-\infty, 0],$$

L is a bounded linear operator from \mathcal{B} to X , and f is a continuous X -valued function on \mathbb{R} . We assume that A satisfies the Hille–Yosida condition, which means that A satisfies the following spectral condition:

(H₀) there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$, and

$$|(\lambda I - A)^{-n}| \leq \frac{M_0}{(\lambda - \omega_0)^n}, \quad n \in \mathbb{N}, \lambda > \omega_0,$$

where $\rho(A)$ is the resolvent set of A .

The well posedness of Eq. (1.1) has been studied in [1,2]. In particular, the authors proved several results concerning the existence, regularity and stability of solutions.

The problem of periodic and almost periodic solutions for differential equations has been studied extensively in the literature, for the reader, we refer to [4,8,11,15–18,20], and the references therein. We recall a classical theorem for the existence of almost periodic solutions for differential equations in finite-dimensional space. Let consider the following ordinary differential equation

$$\frac{d}{dt}x(t) = Qx(t) + e(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where Q is a $n \times n$ -matrix and e is an almost periodic function from \mathbb{R} to \mathbb{R}^n . Bohr and Neugebauer proved that the existence of a bounded solution on \mathbb{R}^+ of Eq. (1.2) implies the existence of an almost periodic solution, see [11]. The result of Bohr and Neugebauer still holds for finite delay differential equations in finite-dimensional space, namely the existence of a bounded solution on \mathbb{R}^+ implies the existence of a periodic or almost periodic solution. The proof of this result is based on the fact that the solution operator is compact for t greater than the delay. When the delay is infinite, the compactness property becomes false. However, if \mathcal{B} is a uniform fading memory space, then spectral decomposition of the phase space could be done under additional assumptions and Bohr and Neugebauer result could be also true.

Variation of constants formula for partial functional differential equations plays an important role to study the properties of solutions, for more details about this topic, we refer

to [7,13–15,17,22,23]. In [14,17], the authors established a variation of constants formula for Eq. (1.1), in the case where A generates a C_0 -semigroup on X . This means by the Hille–Yosida theorem, see [19], that A satisfies Condition (H_0) and $\overline{D(A)} = X$. They proved that the mild solution $x(\cdot, \varphi)$ of Eq. (1.1) is represented by

$$x_t(\cdot, \varphi) = W(t - \sigma)\varphi + \lim_{n \rightarrow \infty} \int_{\sigma}^t W(t - s)\Gamma^n f(s) ds, \quad \text{for } t \geq \sigma, \tag{1.3}$$

where $\Gamma^n f(s)$ is defined by

$$(\Gamma^n f(s))(\theta) = \begin{cases} (n\theta + 1)f(s) & \text{if } -\frac{1}{n} \leq \theta \leq 0, \\ 0 & \text{if } \theta \leq -\frac{1}{n}, \end{cases}$$

and $(W(t))_{t \geq 0}$ is the solution semigroup of Eq. (1.1) in the case $f = 0$. They used the formula (1.3) to establish the existence of an almost periodic solution. However, formula (1.3) cannot be used in the case where A is nondensely defined.

Let us now briefly discuss about the use of an operator with a nondense domain. There are many examples where A is nondensely defined. In particular, nondensity occurs in many situations due to restrictions on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or due to boundary conditions (for example, the space C^1 with null value on the boundary is nondense in the space of continuous functions).

Recently, in [5] the authors established a new variation of constants formula for partial neutral functional differential equations. We use the same method as in [5] to establish a variation of constants formula for Eq. (1.1) when A is nondensely defined. In the case where \mathcal{B} is a uniform fading memory space, we establish a spectral decomposition of the solutions of Eq. (1.1). This allows us to study the existence of almost periodic solutions. This work extends the corresponding results in [14,17]. More precisely, we prove that the existence of a bounded solution for Eq. (1.1) on \mathbb{R}^+ implies the existence of an almost periodic solution.

This work is organized as follows, in Section 2, we give some fundamental axioms that should be satisfied by \mathcal{B} in order to study Eq. (1.1). In Section 3, we establish a variation of constants formula. In Section 4, we study the spectral decomposition of the solutions and we reduce Eq. (1.1) in a finite-dimensional space. As a consequence of this reduction, we establish in Section 5, the existence of an almost periodic solution. To illustrate our studies, we propose in the last section, an application to the Lotka–Volterra model with diffusion.

2. Integral solutions of Eq. (1.1)

In this work, we employ an axiomatic definition of the phase space \mathcal{B} which has been introduced at first by Hale and Kato [12]. We assume that $(\mathcal{B}, \|\cdot\|)$ is a normed space of functions mapping $(-\infty, 0]$ into a Banach space $(X, |\cdot|)$ and satisfying the following fundamental axioms:

- (A) There exist a positive constant N , a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \rightarrow X$ is continuous on $[\sigma, a]$ with $x \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,
 - (i) $x_t \in \mathcal{B}$;
 - (ii) $t \mapsto x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$;
 - (iii) $N|x(t)| \leq \|x_t\| \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma)\|x_\sigma\|$.
- (B) \mathcal{B} is a Banach space.

We assume furthermore that

- (D₁) if $(\phi_n)_{n \geq 0}$ is a sequence in \mathcal{B} such that $\phi_n \rightarrow 0$ in \mathcal{B} as $n \rightarrow +\infty$, then for all $\theta \leq 0$, $\phi_n(\theta) \rightarrow 0$ in X as $n \rightarrow +\infty$;
- (D₂) $\mathcal{B} \subset C((-\infty, 0]; X)$, where $C((-\infty, 0]; X)$ is the space of continuous functions from $(-\infty, 0]$ into X ;
- (D₃) there exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > \lambda_0$ and $x \in X$, we have $e^{\lambda \cdot} x \in \mathcal{B}$ and

$$K_0 := \sup_{\substack{\text{Re } \lambda > \lambda_0, x \in X \\ x \neq 0}} \frac{\|e^{\lambda \cdot} x\|}{|x|} < \infty,$$

where $(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x$, for $\theta \in (-\infty, 0]$ and $x \in X$.

In the whole of this work, we assume that the operator A satisfies the Hille–Yosida condition (H₀) on X .

The following lemma is well known.

Lemma 2.1. [6, p. 140] *Let A_0 be the part of the operator A in $D(A)$, which is defined by*

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

Then A_0 generates a C_0 -semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.

We start with an existence result for Eq. (1.1) in an integrated form. The following definition and result are taken from [2].

Definition 2.2. [2] Let $\phi \in \mathcal{B}$. A function $u : \mathbb{R} \rightarrow X$ is called an integral solution of Eq. (1.1) on \mathbb{R} if the following conditions hold

- (i) u is continuous on $[\sigma, \infty)$,
- (ii) $u_\sigma = \phi$,
- (iii) $\int_\sigma^t u(s) ds \in D(A)$, for $t \geq \sigma$,
- (iv) $u(t) = \phi(0) + A \int_\sigma^t u(s) ds + \int_\sigma^t L(u_s) ds + \int_\sigma^t f(s) ds$, for $t \geq \sigma$.

If $\overline{D(A)} = X$, then the integral solutions coincide with the known mild solutions given in the literature.

Theorem 2.3. [2, p. 336] *Assume that \mathcal{B} satisfies (A) and (B). Then, for all $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A)}$, Eq. (1.1) has a unique integral solution $u = u(\cdot, \phi, L, f)$ on \mathbb{R} which is given by*

$$u(t) = \begin{cases} T_0(t - \sigma)\phi(0) + \lim_{\lambda \rightarrow +\infty} \int_{\sigma}^t T_0(t - s)\lambda R(\lambda, A)[L(u_s) + f(s)] ds, & t \geq \sigma, \\ \phi(t), & t \leq \sigma. \end{cases}$$

Let $\mathcal{B}_A := \{\phi \in \mathcal{B}: \phi(0) \in \overline{D(A)}\}$ be the phase space corresponding to Eq. (1.1). Define $U(t)$, for $t \geq 0$ and $\phi \in \mathcal{B}_A$, by

$$U(t)\phi = u_t(\cdot, \phi, L, 0),$$

where $u(\cdot, \phi, L, 0)$ is the integral solution of Eq. (1.1) in the case $f = 0$ and $\sigma = 0$.

Proposition 2.4. [2, Proposition 2] *Assume that \mathcal{B} satisfies (A) and (B). Then, $(U(t))_{t \geq 0}$ is a C_0 -semigroup on \mathcal{B}_A . That is*

- (i) $U(0) = \text{Id}$,
- (ii) $U(t + s) = U(t)U(s)$, for $t, s \geq 0$,
- (iii) for all $\phi \in \mathcal{B}_A$, $t \mapsto U(t)\phi$ is continuous from $[0, \infty)$ to \mathcal{B}_A .

Moreover, $(U(t))_{t \geq 0}$ satisfies, for $t \geq 0, \phi \in \mathcal{B}_A$, the translation property

$$(U(t)\phi)(\theta) = \begin{cases} (U(t + \theta)\phi)(0), & t + \theta \geq 0, \\ \phi(t + \theta), & t + \theta \leq 0. \end{cases}$$

In order to establish a variation of constants formula adapted to Eq. (1.1), we follow the same approach used in [5]. To achieve this goal, we recall the following result.

Lemma 2.5. [2, Proposition 5] *Let \mathcal{B} satisfy Axioms (A), (B), (D₁) and (D₂). Then, the infinitesimal generator A_U of $(U(t))_{t \geq 0}$ is given by*

$$\left\{ \begin{array}{l} D(A_U) = \left\{ \begin{array}{l} \phi \in C^1((-\infty, 0]; X) \cap \mathcal{B}_A: \phi' \in \mathcal{B}_A, \phi(0) \in D(A) \text{ and} \\ \phi'(0) = A\phi(0) + L(\phi) \end{array} \right\}, \\ A_U\phi = \phi'. \end{array} \right.$$

3. Variation of constants formula

To construct a variation of constants formula associated to Eq. (1.1), we need to extend A_U to the larger space $\mathfrak{X} := \mathcal{B}_A \oplus \langle X_0 \rangle$, where

$$\langle X_0 \rangle = \{X_0x: x \in X\}$$

and X_0x is the discontinuous function defined by

$$(X_0x)(\theta) = \begin{cases} 0 & \text{if } \theta \in (-\infty, 0), \\ x & \text{if } \theta = 0. \end{cases}$$

The space \mathfrak{X} endowed with the norm $\|\phi + X_0x\| = \|\phi\| + |x|$ is a Banach space.

In the beginning, by Axiom (D₃), we define for each complex number λ such that $\text{Re}(\lambda) > \lambda_0$, the linear operator $\Delta(\lambda) : D(A) \rightarrow X$, by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I).$$

Our first fundamental result of this work, is given in the next theorem.

Theorem 3.1. *Assume that \mathcal{B} satisfies Axioms (A), (B), (D₁), (D₂) and (D₃). Then the continuous extension \widetilde{A}_U of the operator A_U defined on \mathfrak{X} by*

$$\begin{cases} D(\widetilde{A}_U) = \{\phi \in \mathcal{B}_A : \phi' \in \mathcal{B}_A \text{ and } \phi(0) \in D(A)\}, \\ \widetilde{A}_U\phi = \phi' + X_0(A\phi(0) + L\phi - \phi'(0)), \end{cases}$$

satisfies the Hille–Yosida condition on \mathfrak{X} .

The proof will be done through the following fundamental lemma.

Lemma 3.2. *Assume that \mathcal{B} satisfies Axioms (A), (B), (D₁), (D₂) and (D₃). Then there exists $\bar{\omega} > \lambda_0$ such that, for $\lambda > \bar{\omega}$,*

- (i) $\Delta(\lambda)$ is invertible with $|\Delta(\lambda)^{-1}| \leq \frac{M_0}{\lambda - \bar{\omega}}$;
- (ii) $D(\widetilde{A}_U) = D(A_U) \oplus \langle e^{\lambda \cdot} \rangle_A$, where $\langle e^{\lambda \cdot} \rangle_A = \{e^{\lambda \cdot} x : x \in D(A)\}$;
- (iii) $(\bar{\omega}, \infty) \subset \rho(\widetilde{A}_U)$ and for $\lambda > \bar{\omega}$, $n \in \mathbb{N}^*$ and $(\phi, x) \in \mathcal{B}_A \times X$, we have $R(\lambda, \widetilde{A}_U)^n(\phi + X_0x) = R(\lambda, A_U)^n\phi + R(\lambda, A_U)^{n-1}(e^{\lambda \cdot} \Delta(\lambda)^{-1}x)$.

Proof. (i) Let $\lambda > \omega_1 := \max(0, \omega_0, \lambda_0)$, where ω_0 is given in Assumption (H₀) and λ_0 is given in Axiom (D₃). Then

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I) = (\lambda I - A)(I - R(\lambda, A)L(e^{\lambda \cdot} I)),$$

and

$$|R(\lambda, A)L(e^{\lambda \cdot} x)| \leq \frac{M_0|L|}{\lambda - \omega_0} \|e^{\lambda \cdot} x\| \leq \frac{M_0K_0|L|}{\lambda - \omega_0} |x|, \quad x \in X,$$

where M_0 is given in Assumption (H₀) and K_0 is given in Axiom (D₃). Consequently

$$|R(\lambda, A)L(e^{\lambda \cdot} I)| \leq \frac{\bar{M}}{\lambda - \omega_0} < 1, \quad \text{for all } \lambda > \omega_2 := \omega_1 + \bar{M},$$

where $\bar{M} := M_0K_0|L|$. We conclude that the operator $(I - R(\lambda, A)L(e^{\lambda \cdot} I))$ is invertible, and

$$|(I - R(\lambda, A)L(e^{\lambda \cdot} I))^{-1}| \leq \frac{1}{1 - |R(\lambda, A)L(e^{\lambda \cdot} I)|} \leq \frac{\lambda - \omega_0}{\lambda - \omega_0 - \bar{M}},$$

for all $\lambda > \omega_2$.

Consequently, $\Delta(\lambda)$ is invertible for $\lambda > \omega_2$ and satisfies

$$|\Delta(\lambda)^{-1}| \leq \frac{M_0}{\lambda - \omega_2}.$$

(ii) Let $\lambda > \omega_2$ and $e^{\lambda \cdot} x \in D(A_U) \cap \langle e^{\lambda \cdot} \rangle_A$. Then $\lambda x = Ax + L(e^{\lambda \cdot} x)$, with $x \in D(A)$. That is

$$\Delta(\lambda)x = 0.$$

Since $\Delta(\lambda)$ is invertible for $\lambda > \omega_2$, we conclude that $D(A_U) \cap \langle e^{\lambda \cdot} \rangle_A = \{0\}$. On the other hand, let $\tilde{\psi} \in D(\tilde{A}_U)$ and ψ given by

$$\psi = \tilde{\psi} + e^{\lambda \cdot} \Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)).$$

Then,

$$\begin{aligned} A\psi(0) + L\psi &= A\tilde{\psi}(0) + L\tilde{\psi} + A\Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)) \\ &\quad + L(e^{\lambda \cdot} \Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0))) \\ &= A\tilde{\psi}(0) + L\tilde{\psi} - \Delta(\lambda)\Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)) \\ &\quad + \lambda\Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)) \\ &= \tilde{\psi}'(0) + \lambda\Delta(\lambda)^{-1} (A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)) \\ &= \psi'(0). \end{aligned}$$

Hence $\psi \in D(A_U)$, which implies that $D(\tilde{A}_U) = D(A_U) \oplus \langle e^{\lambda \cdot} \rangle_A$.

(iii) Let $\lambda > \omega_2$ and $\tilde{\psi} \in \mathfrak{X}$. Then $\tilde{\psi} = \psi + X_0x$, for some $\psi \in \mathcal{B}_A$ and $x \in X$. We seek for $\tilde{\phi} = \phi + e^{\lambda \cdot} a \in D(\tilde{A}_U)$ such that $(\lambda I - \tilde{A}_U)\tilde{\phi} = \tilde{\psi}$, where $\phi \in D(A_U)$ and $a \in D(A)$. Remark that

$$\begin{cases} (\lambda I - \tilde{A}_U)\phi = (\lambda I - A_U)\phi, \\ (\lambda I - \tilde{A}_U)(e^{\lambda \cdot} a) = X_0\Delta(\lambda)a. \end{cases}$$

Then, the equation $(\lambda I - \tilde{A}_U)(\phi + e^{\lambda \cdot} a) = \psi + X_0x$ is equivalent to find $(a, \phi) \in D(A) \times D(A_U)$ such that

$$\begin{cases} (\lambda I - A_U)\phi = \psi, \\ \Delta(\lambda)a = x. \end{cases}$$

Since A_U is the generator of the semigroup $(U(t))_{t \geq 0}$ on \mathcal{B}_A , by the Hille–Yosida theorem [19], there exist positive constants M_1 and ω_3 such that $(\omega_3, \infty) \subset \rho(A_U)$ and

$$|R(\lambda, A_U)^n| \leq \frac{M_1}{(\lambda - \omega_3)^n}, \quad n \in \mathbb{N}, \lambda > \omega_3.$$

It follows that $(\lambda I - \tilde{A}_U)^{-1}$ exists for $\lambda > \bar{\omega} := \max(\omega_2, \omega_3)$, and

$$(\lambda I - \tilde{A}_U)^{-1}(\psi + X_0x) = (\lambda I - A_U)^{-1}\psi + e^{\lambda \cdot} \Delta(\lambda)^{-1}x.$$

Finally, for $n \in \mathbb{N}^*$ and $\lambda > \bar{\omega}$, we have

$$R(\lambda, \tilde{A}_U)^n(\psi + X_0x) = R(\lambda, A_U)^n\psi + R(\lambda, A_U)^{n-1}(e^{\lambda \cdot} \Delta(\lambda)^{-1}x). \quad \square$$

Proof of Theorem 3.1. By Lemma 3.2, there exist $\bar{\omega}$ and $\tilde{M} := M_1(1 + M_0K_0)$ such that

$$|R(\lambda, \tilde{A}_U)^n| \leq \frac{\tilde{M}}{(\lambda - \bar{\omega})^n}, \quad n \in \mathbb{N}, \lambda > \bar{\omega}.$$

Then, \tilde{A}_U satisfies the Hille–Yosida condition on \mathfrak{X} . \square

Corollary 3.3. *The part of the operator \tilde{A}_U in \mathcal{B}_A is the operator A_U .*

Proof. From Lemma 2.5, the operator A_U generates a C_0 -semigroup on \mathcal{B}_A . Then, by the Hille–Yosida theorem [19], $\overline{D(A_U)} = \mathcal{B}_A$. Since $D(A_U) \subset D(\tilde{A}_U) \subset \mathcal{B}_A$, we get

$$\overline{D(A_U)} = \overline{D(\tilde{A}_U)} = \mathcal{B}_A.$$

Let C be the part of the operator \tilde{A}_U in \mathcal{B}_A . This means that

$$\begin{cases} D(C) = \{\phi \in D(\tilde{A}_U) : \tilde{A}_U\phi \in \mathcal{B}_A\}, \\ C\phi = \tilde{A}_U\phi. \end{cases}$$

Then $D(A_U) \subseteq D(C)$ and $A_U\phi = C\phi$, for all $\phi \in D(A_U)$. Conversely, let $\phi \in D(C)$. We have

$$\begin{cases} \phi \in C^1((-\infty, 0]; X) \cap \mathcal{B}_A, \quad \phi' \in \mathcal{B}_A, \quad \phi(0) \in D(A), \quad \text{and} \\ \phi' + X_0(A\phi(0) + L\phi - \phi'(0)) \in \mathcal{B}_A. \end{cases}$$

By Assumption (D₂), it follows that

$$\phi \in D(\tilde{A}_U), \quad \phi'(0) = A\phi(0) + L\phi \quad \text{and} \quad C\phi = \phi'.$$

We conclude that $C = A_U$. \square

Now, consider the following abstract nonhomogeneous Cauchy problem

$$\begin{cases} \frac{d}{dt}\xi(t) = \tilde{A}_U\xi(t) + X_0f(t), \quad \text{for } t \geq 0, \\ \xi(0) = \tilde{\phi} \in \mathfrak{X}. \end{cases} \tag{3.1}$$

Definition 3.4. A continuous function $\xi : [0, +\infty) \rightarrow \mathfrak{X}$ is called an integral solution of Eq. (3.1) if

- (i) $\int_0^t \xi(s) ds \in D(\tilde{A}_U)$, for $t \geq 0$,
- (ii) $\xi(t) = \tilde{\phi} + \tilde{A}_U \int_0^t \xi(s) ds + X_0 \int_0^t f(s) ds$, for $t \geq 0$.

Theorem 3.5. *Assume that (D₁), (D₂) and (D₃) hold. Let $\phi \in \mathcal{B}_A$. If u is an integral solution of Eq. (1.1) with $\sigma = 0$, then the function given by $\xi(t) = u_t, t \geq 0$, is an integral solution of Eq. (3.1) for $\tilde{\phi} = \phi$. Conversely, if ξ is an integral solution of Eq. (3.1) with $\tilde{\phi} = \phi$, then the function u defined by*

$$u(t) = \begin{cases} \xi(t)(0) & \text{if } t \geq 0, \\ \phi(t) & \text{if } t \leq 0, \end{cases}$$

is an integral solution of Eq. (1.1).

Proof. Let $\phi \in \mathcal{B}_A$ and u be the integral solution of Eq. (1.1). Define the function $\xi : [0, \infty) \rightarrow \mathcal{B}_A$ by

$$\xi(t) = u_t, \quad t \geq 0.$$

We need to compute the integral in \mathcal{B} in terms of the integral in X . We use the following lemma.

Lemma 3.6. [3] Assume that (D_1) holds and $\chi : [a, b] \rightarrow \mathcal{B}$ is a continuous function. Then

$$\left(\int_a^b \chi(s) ds \right) (\theta) = \int_a^b \chi(s)(\theta) ds, \quad \text{for } \theta \leq 0.$$

By Lemma 3.6, we have, for $t \geq 0$ and $\theta \leq 0$,

$$\begin{aligned} \frac{d}{d\theta} \left(\int_0^t u_s ds \right) (\theta) &= \frac{d}{d\theta} \left(\int_0^t u(s + \theta) ds \right) \\ &= \frac{d}{d\theta} \left(\int_\theta^{t+\theta} u(s) ds \right) \\ &= u_t(\theta) - \phi(\theta). \end{aligned}$$

Then

$$\widetilde{A}_U \int_0^t \xi(s) ds = u_t - \phi + X_0 \left(A \int_0^t u(s) ds + L \left(\int_0^t u_s ds \right) - u(t) + \phi(0) \right).$$

Since u is an integral solution of Eq. (1.1), it follows that

$$u(t) = \phi(0) + A \int_0^t u(s) ds + L \left(\int_0^t u_s ds \right) + \int_0^t f(s) ds, \quad \text{for } t \geq 0.$$

This implies that

$$\xi(t) = \phi + \widetilde{A}_U \int_0^t \xi(s) ds + X_0 \int_0^t f(s) ds, \quad \text{for } t \geq 0.$$

Consequently, ξ is an integral solution of Eq. (3.1). Conversely, let ξ be an integral solution of Eq. (3.1) for $\tilde{\phi} = \phi$. Then, ξ satisfies the following translation property

$$\xi(t)(\theta) = \begin{cases} \xi(t + \theta)(0) & \text{if } t + \theta \geq 0, \\ \phi(t + \theta) & \text{if } t + \theta \leq 0. \end{cases}$$

In fact, we have, for $t + \theta \geq 0$,

$$\xi(t)(\theta) = (U(t)\phi)(\theta) + \lim_{\lambda \rightarrow +\infty} \int_0^t (U(t-s)\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(\theta) ds.$$

Then

$$\begin{aligned} \xi(t)(\theta) &= (U(t+\theta)\phi)(0) + \lim_{\lambda \rightarrow +\infty} \int_0^{t+\theta} (U(t+\theta-s)\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(0) ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t (U(t-s)\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(\theta) ds. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t (U(t-s)\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(\theta) ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t (\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(t-s+\theta) ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t e^{\lambda(t-s+\theta)} \lambda \Delta(\lambda)^{-1} f(s) ds \\ &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} \xi(t)(\theta) &= (U(t+\theta)\phi)(0) + \lim_{\lambda \rightarrow +\infty} \int_0^{t+\theta} (U(t+\theta-s)\lambda R(\lambda, \tilde{A}_U)X_0 f(s))(0) ds \\ &= \xi(t+\theta)(0). \end{aligned}$$

Consider the function $u : \mathbb{R} \rightarrow X$ defined by

$$u(t) = \begin{cases} \xi(t)(0) & \text{if } t > 0, \\ \phi(t) & \text{if } t \leq 0. \end{cases}$$

Then for all $t \geq 0$, $\xi(t) = u_t$ and

$$u_t = \phi + \tilde{A}_U \left(\int_0^t u_s ds \right) + X_0 \int_0^t f(s) ds.$$

This implies that u is an integral solution of Eq. (1.1), for $t \geq 0$. \square

Now, we can state a variation of constants formula representing the integral solutions of Eq. (1.1).

Theorem 3.7. *Assume that (D₁), (D₂), (D₃) hold and $\phi \in \mathcal{B}_A$. Then the integral solution u of Eq. (1.1) is given by the following variation of constants formula*

$$u_t = U(t - \sigma)\phi + \lim_{n \rightarrow +\infty} \int_{\sigma}^t U(t - s)\widetilde{B}_n(X_0 f(s)) ds, \quad t \geq \sigma, \tag{3.2}$$

where $\widetilde{B}_n = n(nI - \widetilde{A}_U)^{-1}$, for n large enough.

Proof. First, we can observe that if u is an integral solution of Eq. (1.1), then the function v defined by

$$v(t) = \begin{cases} u(t + \sigma), & t \geq 0, \\ \phi(t + \sigma), & t \leq 0, \end{cases} \tag{3.3}$$

is an integral solution of the equation

$$\begin{cases} \frac{d}{dt} \zeta(t) = \widetilde{A}_U \zeta(t) + X_0 f(t + \sigma), & \text{for } t \geq 0, \\ \zeta(0) = u_{\sigma}. \end{cases}$$

Then Theorem 3.7 is an immediate consequence of Theorem 3.5 and the following lemma. \square

Lemma 3.8. [21] *Let Y be a Banach space and $C : D(C) \subset Y \rightarrow Y$ be a linear operator on Y and $\alpha : \mathbb{R}^+ \rightarrow Y$ be a continuous function. Consider the following problem*

$$\begin{cases} \frac{d}{dt} x(t) = Cx(t) + \alpha(t), & t \geq 0, \\ x(0) = x_0 \in Y. \end{cases}$$

Assume that C satisfies the Hille–Yosida condition on Y and $x_0 \in \overline{D(C)}$. Then, there exists a unique continuous function $x : [0, \infty) \rightarrow \overline{D(C)}$ such that

- (i) $\int_0^t x(s) ds \in D(C)$, $t \geq 0$;
- (ii) $x(t) = x_0 + C \int_0^t x(s) ds + \int_0^t \alpha(s) ds$, $t \geq 0$.

Moreover, x is given by the following variation of constants formula

$$x(t) = S_0(t)x_0 + \lim_{n \rightarrow +\infty} \int_0^t S_0(t - s)C_n \alpha(s) ds, \quad t \geq 0,$$

where $C_n := n(nI - C)^{-1}$, for n large enough and $(S_0(t))_{t \geq 0}$ is the C_0 -semigroup generated by the part of C in $\overline{D(C)}$.

4. Spectral decomposition and reduction principle

Let C_{00} be the space of X -valued continuous functions on $(-\infty, 0]$ with compact support. We add the following axiom.

- (C) If a uniformly bounded sequence $(\varphi_n)_{n \in \mathbb{N}}$ in C_{00} converges to a function φ compactly on $(-\infty, 0]$, then φ is in \mathcal{B} and $\|\varphi_n - \varphi\| \rightarrow 0$ as $n \rightarrow \infty$.

Let $(S_0(t))_{t \geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 := \{\phi \in \mathcal{B}: \phi(0) = 0\}$$

by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t + \theta) & \text{if } t + \theta \leq 0, \\ 0 & \text{if } t + \theta \geq 0. \end{cases}$$

Definition 4.1. Assume that the space \mathcal{B} satisfies Axioms (A), (B) and (C). \mathcal{B} is said to be a fading memory space if for all $\phi \in \mathcal{B}_0$,

$$S_0(t)\phi \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in } \mathcal{B}.$$

Moreover, \mathcal{B} is said to be a uniform fading memory space, if

$$S_0(t) \xrightarrow[t \rightarrow \infty]{} 0, \quad \text{in the uniform operator topology.}$$

The following results give some properties of fading and uniform fading memory spaces.

Lemma 4.2. [13, p. 190] *The following statements hold:*

- (i) *If \mathcal{B} is a fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ in Axiom (A) can be chosen to be constants.*
- (ii) *If \mathcal{B} is a uniform fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ can be chosen such that $K(\cdot)$ is constant and $M(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proposition 4.3. [13] *If \mathcal{B} is a fading memory space, then the space $\mathcal{BC}((-\infty, 0]; X)$ of bounded continuous X -valued functions on $(-\infty, 0]$, endowed with the uniform norm topology, is continuously embedding in \mathcal{B} .*

One can observe that if \mathcal{B} is a fading memory space then \mathcal{B} satisfies (D_3) , for $\lambda_0 > 0$. Now, we assume that

- (H₁) $T_0(t)$ is compact on $\overline{D(A)}$, for each $t > 0$.
- (H₂) \mathcal{B} is a uniform fading memory space.

Let V be a bounded subset of a Banach space Y . The Kuratowski measure of noncompactness $\alpha(V)$ of V is defined by

$$\alpha(V) = \inf \left\{ d > 0 \text{ such that there exists a finite number of sets } V_1, \dots, V_n \text{ with } \begin{array}{l} \text{diam}(V_i) \leq d \text{ such that } V \subseteq \bigcup_{i=1}^n V_i \end{array} \right\}.$$

Moreover, for a bounded linear operator P on Y , we define $|P|_\alpha$ by

$$|P|_\alpha = \inf \{ k > 0: \alpha(P(V)) \leq k\alpha(V), \text{ for all bounded set } V \text{ of } Y \}.$$

For a C_0 -semigroup $(S(t))_{t \geq 0}$ on Y , we define the essential growth bound $\omega_{\text{ess}}(S)$ by

$$\omega_{\text{ess}}(S) = \lim_{t \rightarrow \infty} \frac{1}{t} \log |S(t)|_\alpha.$$

We give the following fundamental result.

Theorem 4.4. [8] *Assume that \mathcal{B} satisfies Axioms (A), (B), (D₁) and that Assumptions (H₁), (H₂) hold. Then*

$$\omega_{\text{ess}}(U) < 0.$$

From [10, Corollary IV.2.11], it follows that

$$\{ \lambda \in \sigma(A_U): \text{Re}(\lambda) \geq 0 \}$$

is a finite subset and \mathcal{B}_A is decomposed as follows:

$$\mathcal{B}_A = \mathcal{S} \oplus \mathcal{V},$$

where \mathcal{S}, \mathcal{V} are two closed subspaces of \mathcal{B}_A which are invariant under the semigroup $(U(t))_{t \geq 0}$. Let $U^{\mathcal{S}}(t)$ be the restriction of $U(t)$ on \mathcal{S} . Then, there exist positive constants N and μ such that

$$\|U^{\mathcal{S}}(t)\phi\| \leq Ne^{-\mu t} \|\phi\|, \text{ for all } \phi \in \mathcal{S}.$$

On the other hand, \mathcal{V} is a finite-dimensional space. Then the restriction $U^{\mathcal{V}}(t)$ of $U(t)$ on \mathcal{V} becomes a group. Let $\Pi^{\mathcal{S}}$ and $\Pi^{\mathcal{V}}$ denote the projections on \mathcal{S} and \mathcal{V} respectively and $d = \dim \mathcal{V}$. Take a basis $\{\phi_1, \dots, \phi_d\}$ in \mathcal{V} . Then there exists a d -elements $\{\psi_1, \dots, \psi_d\}$ in the dual space \mathcal{B}_A^* of \mathcal{B}_A such that $\langle \psi_i, \phi_j \rangle = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\langle \psi_i, \phi \rangle = 0$, for $\phi \in \mathcal{S}$ and $i = 1, \dots, d$, with $\langle \cdot, \cdot \rangle$ is the canonical pairing between the dual space and the original space. Denote by $\Phi := (\phi_1, \dots, \phi_d)$ and by Ψ the transpose of (ψ_1, \dots, ψ_d) . One has

$$\langle \Psi, \Phi \rangle = I_{\mathbb{R}^d},$$

where $I_{\mathbb{R}^d}$ is the identity $d \times d$ -matrix. For each $\phi \in \mathcal{B}_A$, $\Pi^{\mathcal{V}}\phi$ is computed as

$$\Pi^{\mathcal{V}}\phi = \Phi \langle \Psi, \phi \rangle = \sum_{i=1}^d \langle \psi_i, \phi \rangle \phi_i.$$

Let u be the integral solution of Eq. (1.1) and $\zeta(t) := (\zeta_1(t), \dots, \zeta_d(t))$ be the components of $\Pi^\mathcal{V} u_t$ in the basis vector Φ . Then

$$\Pi^\mathcal{V} u_t = \Phi \zeta(t) \quad \text{and} \quad \zeta(t) = \langle \Psi, u_t \rangle.$$

Since $(U^\mathcal{V}(t))_{t \geq 0}$ is a group on the finite-dimensional space \mathcal{V} , then there exists a $d \times d$ -matrix G such that

$$U^\mathcal{V}(t)\phi = \Phi e^{Gt} \langle \Psi, \phi \rangle, \quad \text{for all } t \in \mathbb{R} \text{ and } \phi \in \mathcal{V}.$$

This means that

$$U^\mathcal{V}(t)\Phi = \Phi e^{Gt}, \quad \text{for all } t \in \mathbb{R}.$$

Let $n_0 \in \mathbb{N}$ such that $n_0 > \bar{\omega}$. We define, for $n \in \mathbb{N}$ such that $n \geq n_0$ and $i \in \{1, \dots, d\}$, the functional x_n^{*i} by

$$\langle x_n^{*i}, x \rangle = \langle \psi_i, \widetilde{B}_n(X_0 x) \rangle, \quad \text{for all } x \in X.$$

By Lemma 3.2, we have $\widetilde{B}_n(X_0 x) = n e^{n\tau} \Delta^{-1}(n)x$, for $n \geq n_0$. Then, we can choose n_0 large enough such that

$$|\widetilde{B}_n(X_0 x)| \leq M_0 |x|, \quad \text{for all } x \in X \text{ and } n \geq n_0.$$

Which implies that x_n^{*i} is a bounded linear operator on X with $|x_n^{*i}| \leq M_0 |\psi_i|$. Define the d -columns vector x_n^* as an element of $\mathcal{L}(X, \mathbb{R}^d)$ given by the transpose of $(x_n^{*1}, \dots, x_n^{*d})$. Then, for all $n \geq n_0$ and $x \in X$, we have

$$\langle x_n^*, x \rangle = \langle \Psi, \widetilde{B}_n(X_0 x) \rangle \quad \text{and} \quad \sup_{n \geq n_0} |x_n^*| \leq M_0 \sup_{i=1, \dots, d} |\psi_i| < \infty.$$

Theorem 4.5. *The sequence $(x_n^*)_{n \geq n_0}$ converges weakly in $\mathcal{L}(X, \mathbb{R}^d)$, in the sense that*

$$\langle x_n^*, x \rangle \xrightarrow{n \rightarrow \infty} \langle x^*, x \rangle, \quad \text{for all } x \in X.$$

Proof. The proof will be done through two fundamental lemmas. \square

Lemma 4.6. (Banach–Alaoglu–Bourbaki [24]) *Let Y be a separable Banach space and $(\mathcal{K}_n)_{n \geq 0}$ be a bounded sequence in Y^* . Then there exist $\mathcal{K} \in Y^*$ and a subsequence $(\mathcal{K}_{n_k})_{k \geq 0}$ such that*

$$\langle \mathcal{K}_{n_k}, x \rangle \xrightarrow{k \rightarrow \infty} \langle \mathcal{K}, x \rangle, \quad \text{for all } x \in Y.$$

Let Y_0 be any separable closed subspace of X . By Lemma 4.6, the restriction $(x_n^{Y_0*})_{n \geq n_0}$ of $(x_n)_{n \geq n_0}$ in Y_0 has a subsequence $(x_{n_k}^{Y_0*})_{k \geq 0}$ such that

$$\lim_{k \rightarrow \infty} \langle x_{n_k}^{Y_0*}, y \rangle = \langle x^{Y_0*}, y \rangle, \quad \text{for all } y \in Y_0,$$

where $x^{Y_0^*} \in Y_0^*$. We claim that the whole sequence $(x_n^{Y_0^*})_{n \geq n_0}$ converges weakly in Y_0^* to $x^{Y_0^*}$. We proceed by contradiction and we assume that there exists a subsequence $(x_{m_k}^{Y_0^*})_{k \geq 0}$ of $(x_n^{Y_0^*})_{n \geq n_0}$ such that

$$x_{m_k}^{Y_0^*} \xrightarrow[k \rightarrow \infty]{} \bar{x}^{Y_0^*} \quad \text{weakly in } Y_0,$$

with $x^{Y_0^*} \neq \bar{x}^{Y_0^*}$. To get a contradiction, we use the following lemma.

Lemma 4.7. *For any continuous function $h : \mathbb{R} \rightarrow X$, we have*

$$\lim_{n \rightarrow \infty} \int_0^t U^{\mathcal{V}}(t-s) \Pi^{\mathcal{V}}(\widetilde{B}_n(X_0 h(s))) ds = \Phi \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)G} \langle x_n^*, h(s) \rangle ds.$$

Proof. It is not difficult to prove that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^t U^{\mathcal{V}}(t-s) \Pi^{\mathcal{V}}(\widetilde{B}_n(X_0 h(s))) ds \\ &= \lim_{n \rightarrow \infty} \int_0^t (U^{\mathcal{V}}(t-s) \Phi) \langle \Psi, \widetilde{B}_n(X_0 h(s)) \rangle ds \\ &= \lim_{n \rightarrow \infty} \int_0^t \Phi e^{(t-s)G} \langle x_n^*, h(s) \rangle ds \\ &= \Phi \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)G} \langle x_n^*, h(s) \rangle ds. \end{aligned}$$

Let $y \in Y_0$. Thanks to Lemma 4.7 with $h(\cdot) = y$, we obtain that

$$\int_0^t e^{(t-s)G} \langle x^{Y_0^*}, y \rangle ds = \int_0^t e^{(t-s)G} \langle \bar{x}^{Y_0^*}, y \rangle ds, \quad \text{for all } t \geq 0. \tag{4.1}$$

The formula (4.1) is valid for any $y \in Y_0$. This is true if and only if

$$\langle x^{Y_0^*}, y \rangle = \langle \bar{x}^{Y_0^*}, y \rangle, \quad \text{for all } y \in Y_0.$$

This yields a contradiction. Consequently, the whole sequence $(x_n^{Y_0^*})_{n \geq n_0}$ converges weakly in $\mathcal{L}(Y_0, \mathbb{R}^d)$ to $x^{Y_0^*}$.

Let Y_1 be another separable closed space of X . Then the restriction $(x_n^{Y_1^*})_{n \geq n_0}$ of $(x_n^*)_{n \geq n_0}$ in Y_1 converges weakly to some $x^{Y_1^*} \in Y_1^*$. We get that $x^{Y_0^*} = x^{Y_1^*}$ in $Y_0 \cap Y_1$. Since $(x_n^*)_{n \geq n_0}$ converges weakly in $Y_0 \cap Y_1$ and by the uniqueness of the limit, we obtain that $x^{Y_0^*} = x^{Y_1^*}$ in $Y_0 \cap Y_1$. Let x^* be the operator defined on X by

$$\langle x^*, x \rangle = \langle x^{Y^*}, x \rangle,$$

for any separable closed space Y of X such that $x \in Y$. Then x^* is well defined and belongs to $\mathcal{L}(X, \mathbb{R}^d)$. Moreover,

$$\langle x_n^*, x \rangle \xrightarrow{n \rightarrow \infty} \langle x^*, x \rangle, \quad \text{for all } x \in X. \quad \square$$

As a consequence, we get the following corollary.

Corollary 4.8. *For any continuous function $h : [\sigma, T] \rightarrow X$, we have*

$$\lim_{n \rightarrow \infty} \int_{\sigma}^t U^{\mathcal{V}}(t-s) \Pi^{\mathcal{V}}(\widetilde{B}_n(X_0 h(s))) ds = \Phi \int_{\sigma}^t e^{(t-s)G} \langle x^*, h(s) \rangle ds,$$

for all $t \in [\sigma, T]$.

In the next theorem, we state a finite-dimensional reduction of Eq. (1.1).

Theorem 4.9. *Assume that (A), (B), (D₁), (D₂) and (H₁) hold. Let u be an integral solution of Eq. (1.1) on \mathbb{R} . Then $\zeta(t) = \langle \Psi, u_t \rangle$, $t \in \mathbb{R}$, is a solution of the following ordinary differential equation*

$$\dot{\zeta}(t) = G\zeta(t) + \langle x^*, f(t) \rangle, \quad t \in \mathbb{R}. \tag{4.2}$$

Conversely, if f is bounded and ζ is a solution of (4.2), then the function

$$\left(\Phi \zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}}(\widetilde{B}_n(X_0 f(s))) ds \right) (0) \tag{4.3}$$

is an integral solution of Eq. (1.1) on \mathbb{R} .

Proof. Using the variation of constants formula (3.2), we obtain for $t \geq \sigma$,

$$\begin{aligned} \langle \Psi, u_t \rangle &= \langle \Psi, U(t-\sigma)u_{\sigma} \rangle + \left\langle \Psi, \lim_{n \rightarrow +\infty} \int_{\sigma}^t U(t-s) (\widetilde{B}_n(X_0 f(s))) ds \right\rangle \\ &= e^{(t-\sigma)G} \langle \Psi, u_{\sigma} \rangle + \lim_{n \rightarrow \infty} \int_{\sigma}^t e^{(t-s)G} \langle \Psi, (\widetilde{B}_n(X_0 f(s))) \rangle ds \\ &= e^{(t-\sigma)G} \langle \Psi, u_{\sigma} \rangle + \lim_{n \rightarrow \infty} \int_{\sigma}^t e^{(t-s)G} \langle x_n^*, f(s) \rangle ds \\ &= e^{(t-\sigma)G} \langle \Psi, u_{\sigma} \rangle + \int_{\sigma}^t e^{(t-s)G} \langle x^*, f(s) \rangle ds. \end{aligned}$$

This implies that the function $\zeta(t) = \langle \Psi, u_t \rangle$, is defined for $t \in \mathbb{R}$ and it is a solution of the ordinary differential equation (4.2). Conversely, assume that f is bounded on \mathbb{R} . Since

the restriction of the semigroup on \mathcal{S} is exponentially stable, then formula (4.3) is well defined. Let y be defined by

$$y(t) := \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}}(\widetilde{B}_n(X_0 f(s))) ds, \quad \text{for } t \in \mathbb{R}.$$

Then, for $t \geq \sigma$,

$$\begin{aligned} & U^{\mathcal{S}}(t-\sigma)y(\sigma) + \lim_{n \rightarrow +\infty} \int_{\sigma}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}}(\widetilde{B}_n(X_0 f(s))) ds \\ &= \lim_{n \rightarrow +\infty} \left(\int_{-\infty}^{\sigma} U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}}(\widetilde{B}_n(X_0 f(s))) ds \right. \\ & \quad \left. + \int_{\sigma}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}}(\widetilde{B}_n(X_0 f(s))) ds \right) \\ &= y(t). \end{aligned} \tag{4.4}$$

Moreover, the solution ζ of Eq. (4.2) is given, for $t \geq \sigma$, by

$$\zeta(t) = e^{(t-\sigma)G} \zeta(\sigma) + \int_{\sigma}^t e^{(t-s)G} \{x^*, f(s)\} ds.$$

Corollary 4.8 gives for $t \geq \sigma$, that

$$\Phi \zeta(t) = \Phi e^{(t-\sigma)G} \zeta(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t U^{\mathcal{V}}(t-s) \Pi^{\mathcal{V}}(\widetilde{B}_n(X_0 f(s))) ds.$$

This means that

$$\Phi \zeta(t) = U^{\mathcal{V}}(t-\sigma) \Phi \zeta(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t U^{\mathcal{V}}(t-s) \Pi^{\mathcal{V}}(\widetilde{B}_n(X_0 f(s))) ds. \tag{4.5}$$

Set $\xi(t) = \Phi \zeta(t) + y(t)$ on \mathbb{R} . By (4.4) and (4.5), we obtain for $t \geq \sigma$, that

$$\begin{aligned} \xi(t) &= U(t-\sigma)(\Phi \zeta(\sigma) + y(\sigma)) \\ & \quad + \lim_{n \rightarrow \infty} \int_{\sigma}^t U(t-s) [\Pi^{\mathcal{V}} + \Pi^{\mathcal{S}}](\widetilde{B}_n(X_0 f(s))) ds \\ &= U(t-\sigma)\xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t U(t-s) (\widetilde{B}_n(X_0 f(s))) ds. \end{aligned}$$

From Theorem 3.5, we conclude that the function

$$\left(\Phi \zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^S(t-s) \Pi^S(\widetilde{B}_n(X_0 f(s))) ds \right)(0)$$

is an integral solution of Eq. (1.1). \square

5. Almost periodic solutions of Eq. (1.1)

First of all, we recall some properties about almost periodic functions. Let $\mathcal{BC}(\mathbb{R}, X)$ be the space of all bounded continuous functions from \mathbb{R} to X , provided with the uniform norm topology. For $g \in \mathcal{BC}(\mathbb{R}, X)$ and for every $\tau \in \mathbb{R}$, we define the function g_τ by

$$g_\tau(s) = g(\tau + s), \quad \text{for all } s \in \mathbb{R}.$$

Definition 5.1. [11] The function g is said to be almost periodic if the set

$$\{g_\tau : \tau \in \mathbb{R}\}$$

is relatively compact in $\mathcal{BC}(\mathbb{R}, X)$.

In this section, we assume that

(H₃) f is almost periodic.

Theorem 5.2. Assume that (A), (B), (D₁), (D₂), (H₁), (H₂) and (H₃) hold. If Eq. (1.1) has a bounded integral solution on \mathbb{R}^+ , then it has an almost periodic integral solution on \mathbb{R} .

Proof. Since f is almost periodic, then the function

$$t \rightarrow \langle x^*, f(t) \rangle,$$

is also almost periodic, where x^* is obtained by Theorem 4.5. Consequently, to prove Theorem 5.2, we use Theorem 4.9 and the following lemma.

Lemma 5.3. [11, p. 86] Let $e : \mathbb{R} \rightarrow \mathbb{R}^n$ be an almost periodic function. Then, the ordinary differential equation (1.2) has an almost periodic solution on \mathbb{R} if and only if it has a bounded solution on \mathbb{R}^+ .

Let u be a bounded integral solution of Eq. (1.1) on \mathbb{R}^+ . Then, the function

$$t \rightarrow \langle \Psi, u_t \rangle$$

is a bounded solution on \mathbb{R}^+ of the reduced ordinary differential equation (4.2). By Lemma 5.3, we deduce that Eq. (4.2) has an almost periodic solution ζ . By Theorem 4.9, we know that the function v given by

$$v(t) = \left(\Phi \zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^S(t-s) \Pi^S(\widetilde{B}_n(X_0 f(s))) ds \right)(0), \quad t \in \mathbb{R}, \quad (5.1)$$

is a bounded integral solution of Eq. (1.1) on \mathbb{R} . We claim that v is almost periodic. To achieve this goal, we prove that the function

$$t \rightarrow \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^S(t-s) \Pi^S(\widetilde{B}_n(X_0 f(s))) ds$$

is almost periodic. In fact, let $(\tau_m)_{m \geq 0}$ be any real sequence. Then, there exists a subsequence $(\tau_{m_k})_{k \geq 0}$ such that $f(t + \tau_{m_k})$ converges uniformly on \mathbb{R} to some function h . By the dominated convergence theorem, we conclude that the sequence of the functions

$$t \rightarrow \lim_{n \rightarrow +\infty} \int_{-\infty}^{t+m_k} U^S(t+m_k-s) \Pi^S(\widetilde{B}_n(X_0 f(s))) ds$$

converges uniformly for $t \in \mathbb{R}$ to the following function

$$t \rightarrow \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^S(t-s) \Pi^S(\widetilde{B}_n(X_0 h(s))) ds.$$

This property holds for any sequence $(\tau_m)_{m \geq 0}$. This implies that the function v given by (5.1) is almost periodic. \square

6. Application to the Lotka–Volterra model

In order to illustrate the previous results, we consider the following Lotka–Volterra model with diffusion, which has been studied in periodic case in [8]

$$\begin{cases} \frac{\partial}{\partial t} v(t, x) = \frac{\partial^2}{\partial x^2} v(t, x) + a \int_{-\infty}^0 \eta(\theta) v(t + \theta, x) d\theta + F(t, x), \\ \qquad \qquad \qquad t \geq \sigma \text{ and } 0 \leq x \leq \pi, \\ v(t, 0) = v(t, \pi) = 0, \quad t \geq \sigma, \\ v(\sigma + \theta, x) = v_0(\theta, x), \quad -\infty < \theta \leq 0, \text{ and } 0 \leq x \leq \pi, \end{cases} \tag{6.1}$$

where a is a positive constant, η is a positive function on $(-\infty, 0]$, $v_0 : (-\infty, 0] \times [0, \pi] \rightarrow \mathbb{R}$ is a continuous function and $F : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R}$ is a continuous function and almost periodic in t uniformly with respect to $x \in [0, \pi]$. This means that for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property that

$$\sup_{t \in \mathbb{R}, x \in [0, \pi]} |F(t + \tau, x) - F(t, x)| < \varepsilon.$$

Let $X := C([0, \pi]; \mathbb{R})$ be the space of continuous functions from $[0, \pi]$ to \mathbb{R} endowed with the uniform norm topology. Consider the operator $A : D(A) \subset X \rightarrow X$ defined by

$$\begin{cases} D(A) = \{z \in C^2([0, \pi]; \mathbb{R}) : z(0) = z(\pi) = 0\}, \\ Az = z''. \end{cases}$$

Lemma 6.1. [9] *A satisfies the Hille–Yosida condition on X.*

This lemma implies that (H_0) is satisfied. Moreover,

$$\overline{D(A)} = \{ \psi \in C([0, \pi]; \mathbb{R}) : \psi(0) = \psi(\pi) = 0 \}.$$

In [8], the authors have chosen the following space $\mathcal{B} = C_\gamma$, $\gamma > 0$, where

$$C_\gamma = \left\{ \phi \in C((-\infty, 0]; X) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } X \right\}$$

with the following norm

$$\|\phi\|_\gamma = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta} |\phi(\theta)|, \quad \text{for } \phi \in C_\gamma.$$

Lemma 6.2. [13] *The space C_γ , for $\gamma > 0$, is a uniform fading memory space satisfying (D_1) and (D_2) .*

As in [8], we make the following assumptions.

(E₁) $\eta(\cdot)e^{-\gamma\cdot}$ is integrable on $(-\infty, 0]$;

(E₂) $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} v_0(\theta, x)$ exists uniformly for $x \in [0, \pi]$ and $v_0(0, 0) = v_0(0, \pi) = 0$.

Define

$$\begin{cases} (L\phi)(x) = a \int_{-\infty}^0 \eta(\theta)\phi(\theta)(x) d\theta, & x \in [0, \pi], \phi \in \mathcal{B}, \\ f(t)(x) = F(t, x), & t \in \mathbb{R}, x \in [0, \pi]. \end{cases}$$

Then L is a bounded linear operator from \mathcal{B} into X and $f : \mathbb{R} \rightarrow X$ is a continuous function.

If we put

$$\begin{cases} u(t)(x) = v(t, x), & t \geq \sigma, x \in [0, \pi], \\ \phi(\theta)(x) = v_0(\theta, x), & \theta \leq 0, x \in [0, \pi], \end{cases}$$

then, Eq. (6.1) takes the following abstract form

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + Lu_t + f(t), & t \geq \sigma, \\ u_\sigma = \phi. \end{cases} \tag{6.2}$$

Let A_0 be the part of A on $\overline{D(A)}$ which is given by

$$\begin{cases} D(A_0) = \{z \in C^2([0, \pi]; \mathbb{R}) : z(0) = z(\pi) = z''(0) = z''(\pi) = 0\}, \\ A_0z = z''. \end{cases}$$

Then A_0 generates a compact C_0 -semigroup on $\overline{D(A)}$ and Hypothesis (H_1) is satisfied. In the sequel, we make as in [8], the following assumption for the existence of bounded solutions.

(E₃) There exists $d \in (0, 1)$ such that $0 < a \int_{-\infty}^0 \eta(\theta) d\theta < 1 - d$.

Recall that in [8], the authors established the existence of periodic solutions of Eq. (6.2). More precisely, they obtained the following result.

Theorem 6.3. [8, p. 279] *If (E_1) , (E_2) and (E_3) hold, then Eq. (6.2) has a bounded integral solution on \mathbb{R}^+ . Moreover, if F is ω -periodic in t , then Eq. (6.2) has an ω -periodic integral solution.*

Since in this work, F is almost periodic in t uniformly for $x \in [0, \pi]$, it follows that the function f is almost periodic from \mathbb{R} to X . We conclude that all assumptions of Theorem 5.2 are satisfied. We extend Theorem 6.3 to the almost periodic case. More precisely, by Theorem 5.2 we obtain the following result.

Theorem 6.4. *Assume that (E_1) , (E_2) and (E_3) hold. Then, Eq. (6.2) has an almost periodic integral solution.*

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