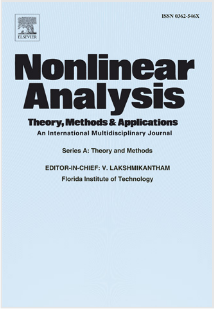

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Behavior near hyperbolic stationary solutions for partial functional differential equations with infinite delay [1]

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Abstract

The aim of this work is to investigate the asymptotic behavior of solutions near hyperbolic stationary solutions for partial functional differential equations with infinite delay. We suppose that the linear part satisfies the Hille-Yosida condition on a Banach space and it is not necessarily densely defined. Firstly, we establish a new variation of constants formula for the nonhomogeneous linear equations. Secondly, we use this formula and the spectral decomposition of the phase space to show the existence of stable and unstable manifolds. The estimations of solutions on these manifolds are obtained. For illustration, we propose to study the stability of stationary solutions for the Lotka-Voltera model with diffusion.

Key words: Semigroup, Hille-Yosida condition, integral solution, variation of constants formula, hyperbolic stationary solution, stable and unstable manifolds.

1. Introduction

In this work, we are concerned with the asymptotic behavior of solutions near hyperbolic stationary solutions for the following partial functional differential equation with infinite delay

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + F(u_t) \text{ for } t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a linear operator, not necessarily densely defined on a Banach space $(X, |.|)$. We suppose that A satisfies the following Hille-Yosida condition

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$(\mathbf{H_1})$ there exist $\omega \in \mathbb{R}$ and $M_0 \geq 1$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M_0}{(\lambda - \omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega, \tag{1.2}$$

where $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda > \omega$. Without loss of generality, we assume that $M_0 = 1$. Otherwise, one can renorm the space X with an equivalent norm for which we get the estimation (1.2) with $M_0 = 1$. $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into X satisfying the fundamental axioms introduced by Hale and Kato in [14]. As usual, for every $t \in \mathbb{R}$, the history function $u_t \in \mathcal{B}$ is defined for $\theta \in (-\infty, 0]$ by

$$u_t(\theta) = u(t + \theta).$$

F is a continuous function from $\mathcal{B}$ into X. In [1], [2], [3] and [9], the authors investigated several results on the existence, regularity, stability and boundedness of solutions for equation (1.1) when $D(A)$ is not necessarily densely defined in X. Recall, in [25], the author gave the basic theory of partial functional differential equation with finite delay, several results on the asymptotic behavior of solutions are given. When the delay is infinite the situation is more complicated, since the properties of solutions depends on the choice of the phase space $\mathcal{B}$. The book [16] contains the fundamental theory related to functional differential equations with infinite delay.

The stability of stationary solutions plays an important role in the qualitative analysis of differential equations. Many results on the existence of stable and unstable manifolds are developed in the context of the following class of differential equations

$$\begin{cases} \dfrac{dv}{dt}(t) = Cv(t) + \chi(v(t)) \text{ for } t \geq 0, \\ v(0) = v_0, \end{cases} \tag{1.3}$$

where C is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space and χ is a smooth function. The solutions of equation (1.3) can be expressed by using the following variation of constants formula

$$v(t) = T(t)v_0 + \int_0^t T(t-s)\chi(v(s)))ds \text{ for } t \geq 0. \tag{1.4}$$

Formula (1.4) and the fixed point theory are the powerful tools to deal with equation (1.3). The book [15] contains more detailed analysis of this problem. In [11], the author proved the existence of stable and unstable manifolds near stationary solutions for functional differential equations of neutral type. Recently, in [4], the authors established new results on the existence of centre manifold for partial functional differential equations with finite delay. They proved the existence of the centre manifold and showed that the flot on the centre manifold is governed by an ordinary differential equation in finite dimensional space. This property allows them the prove some asymptotic stability results in critical case, namely, where the linearized principle cannot be applied..

Recall that the variation of constants formula is the powerful tools to study the existence of stable and unstable manifolds. Several works have been devoted to develop a variation of constants formula adapted to study the qualitative behavior of solutions for partial functional differential equations, about this topics, we refer to [27]. In [17] and [23], the authors established a variation of constants formula for the following nonhomogeneous linear partial functional differential equation with infinite delay

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{B}, \end{cases} \tag{1.5}$$

where L is a bounded linear operator from $\mathcal{B}$ into X and f is a continuous function from $[\sigma, +\infty)$ into X. The authors assumed that A is densely defined in X and satisfies the Hille-Yosida condition $(\mathbf{H_1})$. Which is equivalent, by the Hille-Yosida Theorem, to A is the infinitesimal generator of a strongly continuous semigroup on X. Using this formula, the authors established in [22] the existence of stable and unstable manifolds of equation (1.1). However, this variation of constants formula in [17] and [23], cannot be used when $D(A)$ is not dense in X. In [3], a variation of constants formula for equation (1.5) was obtained only when the phase space $\mathcal{B}$ is a subspace of the space of continuous functions from $(-\infty, 0]$ into X. This formula was used to show the existence of almost periodic solutions.

The goal of this work is to study the existence of stable and unstable manifolds near an hyperbolic stationary solution of equation (1.1) for a general class of phase spaces $\mathcal{B}$. Our results extend the previous works [11], [17], [20], [21] and [23]. To achieve our goal, we construct a new variation of constants formula adapted to our problem. This work is organized as follows. In section 2, we state the fundamental axioms on $\mathcal{B}$ that will be used in the work, and we recall some results on the spectral analysis of the equation (1.5) with $f = 0$. The aim of section 3, is to establish a new variation of constants formula for nonhomogeneous equation (1.5). In Section 4, we use the variation of constants formula in order to investigate the existence of bounded solutions of equation (1.5), respectively on $\mathbb{R}^-$, $\mathbb{R}^+$ and $\mathbb{R}$. In section 5, we establish the existence of stable and unstable manifolds for equation (1.1) near an hyperbolic stationary solution. Finally, we apply the basic theory of this work to study the local stability of stationary solutions for the Lotka-Voltera model with diffusion.

Note that there are many examples where the operator A satisfies the Hille-Yosida condition with non dense domain. In particular, non density occurs in many situations due to restrictions on the space where the equations are considered. For example, periodic continuous functions and Hölder continuous functions are not dense in the space of continuous functions. The boundary conditions may also generate operators with non dense domains: the domain of the Laplacian operator with Dirichlet boundary condition is not dense in the space of continuous functions.

2. Phase space, integral solutions and spectral analysis of the linear equation

The choice of the phase space $\mathcal{B}$ plays an important role in the qualitative analysis of partial functional differential equations with infinite delay. Since properties of solutions depend especially on the choice of $\mathcal{B}$. In this work, we employ the basic theory given by Hale and Kato in [14]. To study our problem, we assume that $(\mathcal{B}, \|\cdot\|)$ is a normed linear space consisting of functions mapping $(-\infty, 0]$ into X satisfying the following fundamental axioms

$(\mathbf{A})$ there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$, $a \in \mathbb{R}$, is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

i) $x_t \in \mathcal{B}$,
ii) $t \to x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$,
iii) $N|x(t)| \leq \|x_t\| \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma)\|x_\sigma\|$.

$(\mathbf{B})$ $\mathcal{B}$ is a Banach space.

As a consequence of axioms $(\mathbf{A})$, we deduce the following result.

Lemma 2.1. [16] *Let* $C_{00}((-\infty,0];X)$ *be the space of continuous functions mapping* $(-\infty,0]$ *into* X *with compact supports. Then,* $C_{00}((-\infty,0];X)\subset\mathcal{B}$*. More precisely, for* $a<0$*, we have*

$$\|\varphi\|\le K(-a)\sup_{\theta\le 0}|\varphi(\theta)|,$$

for any $\varphi\in C_{00}((-\infty,0];X)$ *with the support included in* $[a,0]$*.*

In the whole of this work, we suppose that $\mathcal{B}$ satisfies axioms (**A**) and (**B**).

Definition 2.2. [1] Let $\varphi\in\mathcal{B},\ \sigma\in\mathbb{R}$ and $T>\sigma$. A function $u:(-\infty,T]\longrightarrow X$ is said to be an integral solution for equation (1.1) on $[\sigma,T]$, if the following conditions hold
i) u is continuous on $[\sigma,T]$,
ii) $\int_\sigma^t u(s)ds\in D(A)$ for $t\in[\sigma,T]$,
iii) $u(t)=\begin{cases}\varphi(0)+A\int_\sigma^t u(s)ds+\int_\sigma^t F(u_s)ds & \text{for } \sigma\le t\le T,\\ \varphi(t-\sigma) \text{ for } t\le\sigma.\end{cases}$

For simplicity, in the whole of this work, an integral solution will be called solution. Several results on the existence, uniqueness and stability of solutions of (1.1) are obtained in [1], [2] and [9].

Here, we suppose

($\mathbf{H_2}$) F is Lipschitz continuous. Let c be a positive constant such that

$$|F(\varphi_1)-F(\varphi_2)|\le c\|\varphi_1-\varphi_2\| \text{ for } \varphi_1,\,\varphi_2\in\mathcal{B}.$$

Let A_0 denote the part of the operator A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0)=\{x\in D(A):Ax\in\overline{D(A)}\},\\ A_0x=Ax.\end{cases}$$

Lemma 2.3. [5] *Let* A *satisfy* ($\mathbf{H}_1$)*. Then,* A_0 *generates a strongly continuous semigroup* $(T_0(t))_{t\ge0}$ *on* $\overline{D(A)}$*.*

Let $\alpha>0$ and $\beta\ge1$ be such that

$$|T_0(t)|\le\beta e^{\alpha t} \text{ for } t\ge0.$$

Theorem 2.4. [2] *Assume that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Let* $\sigma\in\mathbb{R}$, $\varphi\in\mathcal{B}$ *be such that* $\varphi(0)\in\overline{D(A)}$*. Then, equation* (1.1) *has a unique integral solution* $u=u(.,\sigma,\varphi)$ *on* $[\sigma,+\infty)$ *which is given by*

$$\begin{cases} u(t)=T_0(t-\sigma)\varphi(0)+\lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)\lambda R(\lambda,A)F(u_s)ds \ \text{ for } t\ge\sigma,\\ u_\sigma=\varphi.\end{cases}$$

Corollary 2.5. *Assume that* ($\mathbf{H_1}$) *holds. Let* $\sigma\in\mathbb{R}$, $\varphi\in\mathcal{B}$ *be such that* $\varphi(0)\in\overline{D(A)}$*. Then, equation* (1.5) *has a unique integral solution* $u=u(.,\sigma,\varphi,L,f)$ *on* $[\sigma,+\infty)$ *which is given by*

$$\begin{cases} u(t)=T_0(t-\sigma)\varphi(0)+\lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)\lambda R(\lambda,A)(L(u_s)+f(s))ds \ \text{ for } t\ge\sigma,\\ u_\sigma=\varphi.\end{cases}$$

Let

$$\mathcal{B}_A = \{\varphi \in \mathcal{B} : \varphi(0) \in \overline{D(A)}\}$$

denote the phase space of equation (1.1). For $t \geq 0$, $V(t)$ is the bounded linear operator on $\mathcal{B}_A$ defined by

$$V(t)\varphi = u_t(.,0,\varphi,L,0) \ \text{ for } \varphi \in \mathcal{B}_A,$$

where $u(.,0,\varphi,L,0)$ is the solution of equation (1.5) with $\sigma = 0$ and $f = 0$.

Proposition 2.6. [2] *Assume that* $(\mathbf{H}_1)$ *holds. Then* $(V(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $\mathcal{B}_A$. *Moreover,* $(V(t))_{t\geq 0}$ *satisfies the translation property*

$$(V(t)\phi)(\theta) = \begin{cases} V(t+\theta)\phi(0) \text{ for } t+\theta \geq 0, \\ \\ \phi(t+\theta) \text{ for } t+\theta \leq 0. \end{cases}$$

In order to study the behavior of solutions, we assume that $\mathcal{B}$ satisfies the following axiom

$(\mathbf{C})$ if a uniformly bounded sequence $(\varphi_n)_{n\geq 0}$ in $C_{00}((-\infty,0];X)$ converges to a function φ compactly on $(-\infty,0]$, then φ is in $\mathcal{B}$ and $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$.

For a Banach space $(Y, |\cdot|)$ and an interval J of $\mathbb{R}$, $BC(J,Y)$ denotes the space of bounded continuous functions from J to Y endowed with the uniform norm topology defined for $\phi \in BC(J,Y)$ by $|\phi|_\infty = \sup\limits_{t\in J}|\phi(t)|$.

Lemma 2.7. [16] *Assume that* $\mathcal{B}$ *satisfies axiom* $(\mathbf{C})$. *Then* $BC(\mathbb{R}^-, X) \hookrightarrow \mathcal{B}$.

As a consequence of the above lemma, we get the following result.

Proposition 2.8. *Assume that* $\mathcal{B}$ *satisfies axiom* $(\mathbf{C})$. *Let* $\varrho : (-\infty, a] \longrightarrow X$, $a \in \mathbb{R}$, *be a function such that* $\varrho_t \in \mathcal{B}$ *for all* $t < a$. *Then* $\varrho \in BC((-\infty,a];X)$ *if and only if the* $\mathcal{B}$*-valued function* $t \to \varrho_t$ *belongs to* $BC((-\infty,a];\mathcal{B})$.

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous solution semigroup associated to the following trivial equation

$$\begin{cases} \dfrac{d}{dt}u(t) = 0 \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}. \end{cases}$$

Then, for $\phi \in \mathcal{B}_0$ and $t \geq 0$, one has

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) \ \text{ for } t+\theta \leq 0, \\ 0 \text{ for } t+\theta \geq 0. \end{cases}$$

Definition 2.9. [16] $\mathcal{B}$ is said to be a fading memory space, if $\mathcal{B}$ satisfies axioms $(\mathbf{A})$, $(\mathbf{B})$, $(\mathbf{C})$ and

$$\|S_0(t)\phi\| \underset{t\to\infty}{\longrightarrow} 0 \ \text{ for } \phi \in \mathcal{B}_0.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if axioms $(\mathbf{A})$, $(\mathbf{B})$, $(\mathbf{C})$ hold and

$$\|S_0(t)\| \underset{t\to\infty}{\longrightarrow} 0.$$

Lemma 2.10. [16] *The following statements hold*

i) if $\mathcal{B}$ *is a fading memory space, then the functions* K *and* M *in axiom* $(\mathbf{A})$ *can be chosen to be constants,*

ii) if $\mathcal{B}$ *is a uniform fading memory space, then the function* K *can be chosen to be a constant function and the function* M *can be chosen such that* $M(t) \to 0$ *as* $t \to \infty$.

We introduce the Kuratowski measure of noncompactness of bounded sets in a Banach space. Let Ω be a bounded subset of a Banach space Y. Then, the Kuratowski measure of noncompactness $\alpha(\Omega)$ of Ω is defined by

$$\alpha(\Omega) = \inf\left\{ \begin{array}{c} d > 0 : \text{ there exists a finite number of sets } \Omega_1, ..., \Omega_n \text{ with} \\ diam\,(\Omega_i) \leq d \text{ such that } \Omega \subseteq \bigcup_{i=1}^{n} \Omega_i \end{array} \right\}.$$

For a bounded linear operator $\mathcal{S}$ on Y, we define the Kuratowski norm $|\mathcal{S}|_\alpha$ by

$$|\mathcal{S}|_\alpha = \inf\left\{d > 0 : \alpha(\mathcal{S}(\Omega)) \leq d\alpha(\Omega) \text{ for any bounded set } \Omega \text{ of } Y\right\}.$$

For semigroup $(V(t))_{t\geq 0}$, we define the essential growth bound $\omega_{ess}(V)$ by

$$\omega_{ess}(V) = \lim_{t\to\infty} \frac{1}{t} \log |V(t)|_\alpha .$$

Let $\mathcal{A}$ denote the infinitesimal generator of the semigroup $(V(t))_{t\geq 0}$ on $\mathcal{B}_A$.

Definition 2.11. Let $\mathcal{C}$ be a densely defined on Y. The essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds

i) $Im(\lambda I - \mathcal{C})$ is not closed,

ii) the generalized eigenspace $M_\lambda(\mathcal{C}) = \bigcup_{k\geq 1} Ker(\lambda I - \mathcal{C})^k$ is of infinite dimension,

iii) λ is a limit point of $\sigma(\mathcal{C}) \setminus \{\lambda\}$.

The essential radius of $\mathcal{C}$ is defined by

$$r_{ess}(\mathcal{C}) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{C})\}.$$

In the sequel, we assume the following compactness condition.

$(\mathbf{H_3})$ $T_0(t)$ is compact whenever $t > 0$.

Lemma 2.12. [7] *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_3)$ *hold. If* $\mathcal{B}$ *is a uniform fading memory space, then* $\omega_{ess}(V) < 0$.

Let $\sigma^+(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A}) : \mathcal{R}e(\lambda) \geq 0\}$. As an immediate consequence of Lemma 2.12, we have the following spectral property of $\mathcal{A}$.

Lemma 2.13. *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then,* $\sigma^+(\mathcal{A})$ *is a finite set of the eigenvalues of* $\mathcal{A}$ *which are not in the essential spectrum. More precisely,* $\lambda \in \sigma^+(\mathcal{A})$ *if and only if there exists* $x \in D(A)\backslash\{0\}$ *solving the following characteristic equation*

$$\Delta(\lambda)x = \lambda x - Ax - L(e^{\lambda\cdot}x) = 0,$$

where $e^{\lambda\cdot}x$ *is the element of* $\mathcal{B}$ *defined for all* $\theta \leq 0$ *by* $(e^{\lambda\cdot}x)(\theta) := e^{\lambda\theta}x$.

Proof. Lemma 2.12 implies that $\omega_{ess}(V) < 0$. By Corollary 2.11, p.258 in [10], we know that $\sigma^+(\mathcal{A})$ is a finite subset of the point spectrum $\sigma_p(\mathcal{A})$. On the other hand, we have

$$r_{ess}(V(t)) = e^{t\omega_{ess}(V)} < 1,$$

and $e^{t\sigma_{ess}(\mathcal{A})} \subset \sigma_{ess}(V(t))$ for $t \geq 0$. It follows that

$$\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}.$$

Consequently, $\sigma^+(\mathcal{A})$ is a finite subset of the point spectrum $\sigma_p(\mathcal{A})$. Let $\lambda \in \sigma^+(\mathcal{A})$. Then, there exists $\varphi \in D(\mathcal{A})$, $\varphi \neq 0$ such that $\mathcal{A}\varphi = \lambda\varphi$, which implies that

$$\lim_{t\to 0^+} \frac{V(t)\varphi - \varphi}{t} = \lambda\varphi.$$

By axiom $(\mathbf{A}) - (ii)$, we deduce that $\lim_{t\to 0^+} \left(\frac{V(t)\varphi - \varphi}{t}\right)(0) = \lambda\varphi(0)$. Moreover,

$$\left(\frac{V(t)\varphi - \varphi}{t}\right)(0) = A\frac{1}{t}\int_0^t (V(s)\varphi)(0)ds + \frac{1}{t}\int_0^t L(V(s)\varphi)ds.$$

Let t go to zero. From the closedness of the operator A, we obtain that

$$\varphi(0) \in D(A) \text{ and } A\varphi(0) + L(\varphi) = \lambda\varphi(0).$$

By the spectral mapping Theorem, p.277 in [10], we have

$$e^{\lambda t} \in \sigma_p(V(t)) \text{ and } V(t)\varphi = e^{\lambda t}\varphi.$$

Using the translation property of the solution semigroup, we obtain that

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) \text{ for } \theta \leq 0.$$

Since $\varphi \neq 0$, then $\varphi(0) \neq 0$ and

$$\ker \Delta(\lambda) \neq \{0\}.$$

Conversely, let $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) \geq 0$. Then for all $x \in X$ we have $e^{\lambda \cdot}x \in \mathcal{B}$. If there exists $a \in D(A) \setminus \{0\}$ such that $\Delta(\lambda)a = 0$, then $\varphi = e^{\lambda \cdot}a \in \mathcal{B}_A$ and $\varphi \in C^1((-\infty, 0]; X)$ with $\varphi'(0) = \lambda a = A\varphi(0) + L(\varphi) \in D(A)$. By Proposition 5 in [2], we conclude that

$$\varphi \in D(\mathcal{A}) \text{ and } \mathcal{A}\varphi = \lambda\varphi. \square$$

Now, we are in a position to present the spectral decomposition of the phase space in the hyperbolic case.

Definition 2.14. We say that the semigroup $(V(t))_{t\geq 0}$ is hyperbolic if

$$\sigma^+(\mathcal{A}) \cap i\mathbb{R} = \emptyset.$$

Consequently, by Theorem 3.7, p. 333 in [10], we get the following spectral decomposition for the phase space $\mathcal{B}_A$.

Theorem 2.15. *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. If the semigroup* $(V(t))_{t\geq 0}$ *is hyperbolic, then* $\mathcal{B}_A$ *is decomposed as follows*

$$\mathcal{B}_A = \mathcal{S} \oplus \mathcal{U},$$

where $\mathcal{S}$ *and* $\mathcal{U}$ *are two closed subspaces of* $\mathcal{B}_A$ *invariant by* $V(t)$. $\mathcal{U}$ *is a finite dimensional space and the restriction of* $V(t)$ *on* $\mathcal{U}$ *becomes a group on* $\mathbb{R}$. *Moreover, there exist positive constants* δ *and* μ *such that the following estimations hold*

$$\begin{aligned} \|V(t)\varphi\| &\leq \delta e^{-\mu t}\|\varphi\| \ \text{ for } \varphi \in \mathcal{S} \ \text{ and } t \geq 0, \\ \|V(t)\varphi\| &\leq \delta e^{\mu t}\|\varphi\| \ \text{ for } \varphi \in \mathcal{U} \ \text{ and } t \leq 0. \end{aligned} \tag{2.1}$$

$\mathcal{S}$ and $\mathcal{U}$ are called respectively the stable and unstable subspaces of $(V(t))_{t\geq 0}$. For the next, P^- and P^+ denote the projection operators respectively on $\mathcal{S}$ and $\mathcal{U}$.

3. Variation of constants formula

For the sake of simplicity, we assume that $\omega > 0$. Recall that in [23], a variation of constants formula for equation (1.5) was established when A is the infinitesimal generator of a strongly continuous semigroup on X. More precisely, one has the following expression.

Theorem 3.1. [23] *Assume that A satisfies the Hille-Yosida condition* $(\mathbf{H}_1)$ *and* $\overline{D(A)} = X$. *Then, the solution* $u(., \sigma, \varphi, L, f)$ *of equation* (1.5) *is given by the following variation of constants formula*

$$u_t(., \sigma, \varphi, L, f) = V(t-\sigma)\varphi + \lim_{n\to\infty}\int_\sigma^t V(t-s)\Gamma^n f(s)ds \quad \textit{for } t \geq \sigma, \tag{3.1}$$

where

$$(\Gamma^n x)(\theta) = \begin{cases} (n\theta+1)x & \textit{for } -\dfrac{1}{n} \leq \theta \leq 0, \\ 0 \ \ \textit{for } \theta < -\dfrac{1}{n}. \end{cases}$$

Moreover, for any $T > \sigma$, the limit in formula (3.1) exists uniformly for $t \in [\sigma, T]$.

The aim of this part is to extend formula (3.1) in the case where $D(A)$ is not necessarily dense in X. We introduce the following sequence of linear operators Θ^n mapping X into $\mathcal{B}$ defined, for $n > \omega$ and $x \in X$, by

$$(\Theta^n x)(\theta) = \begin{cases} n(n\theta+1)R(n, A)x \text{ for } -\dfrac{1}{n} \leq \theta \leq 0, \\ 0 \text{ for } \theta < -\dfrac{1}{n}. \end{cases}$$

For $x \in X$, the function $\Theta^n x$ belongs to $C_{00}((-\infty, 0]; X)$ with the support included in $[-1, 0]$. By Lemma 2.1, we deduce

$$\|\Theta^n x\| \leq \widetilde{N}K(1)|x| \text{ for } x \in X \text{ and } n > \omega,$$

where $\widetilde{N} = \sup\{\lambda|(\lambda I - A)^{-1}| : \lambda > \omega\}$.

Theorem 3.2. *Assume that* $(\mathbf{H}_1)$ *holds. Then, for all $\varphi \in \mathcal{B}_A$, the integral solution $u(., \sigma, \varphi, L, f)$ of equation (1.5) satisfies the following variation of constants formula*

$$u_t(., \sigma, \varphi, L, f) = V(t-\sigma)\varphi + \lim_{n\to\infty}\int_\sigma^t V(t-s)\Theta^n f(s)ds \ \textit{for } t \geq \sigma. \tag{3.2}$$

Moreover, for any $T > \sigma$, the limit in formula (3.2) exists uniformly for $t \in [\sigma, T]$.

Proof. From the uniqueness property of the solution with respect to the initial data, we obtain the following decomposition

$$\begin{aligned} u_t(., \sigma, \varphi, L, f) &= u_t(., \sigma, \varphi, L, 0) + u_t(., \sigma, 0, L, f) \\ &= V(t-\sigma)\varphi + u_t(., \sigma, 0, L, f). \end{aligned}$$

Let $n > \omega$ and $t \geq \sigma$. The integral term $\int_\sigma^t V(t-r)\Theta^n f(r)dr$ is the limit of the following Riemann sum

$$\Lambda_n^p(t, \sigma) = \frac{t-\sigma}{p}\sum_{k=0}^{p-1} V(t-r_k)\Theta^n f(r_k)$$

in $\mathcal{B}$, where $r_k = \sigma + \frac{(t-\sigma)k}{p}$. Moreover, one can see that

$$\Lambda_n^p(t,\sigma)(\theta) = \frac{t-\sigma}{p}\sum_{k=0}^{p-1} u(t+\theta, r_k, \Theta^n f(r_k), L, 0)$$

is a Riemann sum of the integral

$$\xi^n(t,\sigma)(\theta) = \int_\sigma^t u(t+\theta, r, \Theta^n f(r), L, 0)dr.$$

By the uniform continuity of the function $(\theta, r) \mapsto u(t+\theta, r, \Theta^n f(r), L, 0)$ on $(-\infty, 0] \times [\sigma, t]$, the Riemann sum $\Lambda_n^p(t,\sigma)(\theta)$ converges to the above integral uniformly in θ in any compact set of $(-\infty, 0]$. Moreover, $\xi^n(t,\sigma)(\theta)$ is continuous for $\theta \le 0$, with $\xi^n(t,\sigma)(\theta) = 0$ for all $\theta \le \sigma - t - 1$. By Lemma 2.1, we deduce that the function $\xi^n(t,\sigma) \in \mathcal{B}$ and

$$\|\xi^n(t,\sigma) - \Lambda_n^p(t,\sigma)\| \le K(t-\sigma+1) \sup_{\sigma-t-1\le\theta\le 0} |\xi^n(t,\sigma)(\theta) - \Lambda_n^p(t,\sigma)(\theta)|.$$

Therefore, $\Lambda_n^p(t,\sigma)$ converges to $\xi^n(t,\sigma)$ in $\mathcal{B}$ as $p \to \infty$. It follows that

$$\int_\sigma^t V(t-r)\Theta^n f(r)dr = \xi^n(t,\sigma).$$

On the other hand, if $t + \theta \le \sigma - \dfrac{1}{n}$, then

$$\xi^n(t,\sigma)(\theta) = 0.$$

If $\sigma - \dfrac{1}{n} \le t + \theta \le \sigma$, we get

$$\begin{aligned}\xi^n(t,\sigma)(\theta) &= \int_\sigma^t u(t+\theta, r, \Theta^n f(r), L, 0)dr \\ &= \int_\sigma^{\min\{t, t+\theta+\frac{1}{n}\}} [n(t+\theta-r)+1]nR(n,A)f(r)dr.\end{aligned}$$

If $t + \theta \ge \sigma$, we obtain

$$\begin{aligned}\xi^n(t,\sigma)(\theta) &= \int_\sigma^t u(t+\theta, r, \Theta^n f(r), L, 0)dr \\ &= \int_\sigma^{t+\theta} T_0(t+\theta-r)nR(n,A)f(r)dr \\ &\quad + \int_\sigma^{t+\theta} \lim_{\lambda\to+\infty}\left(\int_r^{t+\theta} T_0(t+\theta-s)\lambda R(\lambda,A)L(V(s-r)\Theta^n f(r))ds\right)dr \\ &\quad + \int_{t+\theta}^{\min\{t, t+\theta+\frac{1}{n}\}} [n(t+\theta-r)+1]nR(n,A)f(r)dr.\end{aligned}$$

Set

$$\xi_1^n(t,\sigma)(\theta) = \int_\sigma^{t+\theta} \lim_{\lambda\to+\infty}\left(\int_r^{t+\theta} T_0(t+\theta-s)\lambda R(\lambda,A)L(V(s-r)\Theta^n f(r))ds\right)dr.$$

The limit in λ exists uniformly for $\sigma \le r \le t+\theta$. This implies that

$$\begin{aligned}
\xi_1^n(t,\sigma)(\theta) &= \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta}\int_r^{t+\theta} T_0(t+\theta-s)\lambda R(\lambda,A)L(V(s-r)\Theta^n f(r))dsdr \\
&= \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta}\int_\sigma^{s} T_0(t+\theta-s)\lambda R(\lambda,A)L(V(s-r)\Theta^n f(r))drds \\
&= \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta} T_0(t+\theta-s)\lambda R(\lambda,A)L\left(\int_\sigma^s V(s-r)\Theta^n f(r)dr\right)ds \\
&= \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta} T_0(t+\theta-s)\lambda R(\lambda,A)L\left(\xi^n(s,\sigma)\right)ds.
\end{aligned}$$

Consequently,

$$\xi^n(t,\sigma)(\theta) = \begin{cases} 0 \text{ if } \ t+\theta \le \sigma - \dfrac{1}{n}, \\[2ex] \displaystyle\int_\sigma^{\min\{t,t+\theta+\frac{1}{n}\}}[n(t+\theta-r)+1]nR(n,A)f(r)dr \ \text{ if } \sigma-\dfrac{1}{n}\le t+\theta\le\sigma, \\[2ex] \displaystyle\int_\sigma^{t+\theta}T_0(t+\theta-r)nR(n,A)f(r)dr + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}}[n(t+\theta-r)+1]nR(n,A)f(r)dr \\ \displaystyle +\lim_{\lambda\to+\infty}\int_\sigma^{t+\theta}T_0(t+\theta-r)\lambda R(\lambda,A)L\left(\xi^n(r,\sigma)\right)dr \ \text{ if } \ t+\theta\ge\sigma. \end{cases}$$

Now we claim that $(\xi^n(.,\sigma))_{n>\omega}$ is a Cauchy sequence in $C([\sigma,T];\mathcal{B})$ the space of continuous functions mapping $[\sigma,T]$ into $\mathcal{B}$. Let $n>m>\omega$. Then, by Axiom $(\mathbf{A})$, we obtain

$$\|\xi^n(t,\sigma)-\xi^m(t,\sigma)\| \le K(T-\sigma+1)\sup_{-t+\sigma-\frac{1}{m}\le\theta\le 0}|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)|. \tag{3.3}$$

For $t\ge\sigma$, we denote by $\widehat{f}(\sigma,t)=\sup\limits_{\sigma\le s\le t}|f(s)|$. If $t+\theta\ge\sigma$, we obtain

$$\begin{aligned}
|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)| \le{}& \left|\int_\sigma^{t+\theta}T_0(t+\theta-r)\left(nR(n,A)-mR(m,A)\right)f(r)dr\right| \\
&+\left|\int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}}[n(t+\theta-r)+1]nR(n,A)f(r)dr\right| \\
&+\left|\int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{m}\}}[m(t+\theta-r)+1]mR(m,A)f(r)dr\right| \\
&+\left|\lim_{\lambda\to+\infty}\int_\sigma^{t+\theta}T_0(t+\theta-r)\lambda R(\lambda,A)L\left(\xi^n(r,\sigma)-\xi^m(r,\sigma)\right)dr\right|.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)| \le{}& \sup_{\sigma\le s\le t}\left|\int_\sigma^s T_0(s-r)\left(nR(n,A)-mR(m,A)\right)f(r)dr\right| \\
&+\int_{t+\theta}^{t+\theta+\frac{1}{n}}|nR(n,A)|\widehat{f}(\sigma,T)dr+\int_{t+\theta}^{t+\theta+\frac{1}{m}}|mR(m,A)|\widehat{f}(\sigma,T)dr \\
&+\beta|L|e^{\alpha T}\int_\sigma^t\sup_{\lambda>\omega}|\lambda R(\lambda,A)|\|\xi^n(r,\sigma)-\xi^m(r,\sigma)\|dr,
\end{aligned}$$

which implies that

$$\begin{aligned}|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)| \quad \leq \quad & \sup_{\sigma\leq s\leq T}\left|\int_\sigma^s T_0(s-r)\left(nR(n,A)-mR(m,A)\right)f(r)dr\right| \\ & +\widetilde{N}\beta|L|e^{\alpha T}\int_\sigma^t \sup_{0\leq s\leq r}\|\xi^n(s,\sigma)-\xi^m(s,\sigma)\|dr \\ & +\widetilde{N}\widehat{f}(\sigma,T)\left(\frac{1}{n}+\frac{1}{m}\right).\end{aligned}$$

If $\sigma-\dfrac{1}{n}\leq t+\theta\leq\sigma$, we get

$$|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)|\leq\widetilde{N}\widehat{f}(\sigma,T)\left(\frac{1}{n}+\frac{1}{m}\right).$$

If $\sigma-\dfrac{1}{m}\leq t+\theta\leq\sigma-\dfrac{1}{n}$, we obtain

$$|\xi^n(t,\sigma)(\theta)-\xi^m(t,\sigma)(\theta)|\leq\widetilde{N}\widehat{f}(\sigma,T)\frac{1}{m}.$$

Let $\widehat{M}(T)=\beta Ne^{\alpha T}|L|$ and

$$\varepsilon_{n,m}(T)=\sup_{\sigma\leq s\leq T}\left|\int_\sigma^s T_0(s-r)\left(nR(n,A)-mR(m,A)\right)f(r)dr\right|+\widetilde{N}\widehat{f}(\sigma,T)\left(\frac{1}{n}+\frac{1}{m}\right).$$

Since the limit $\lim\limits_{n\to+\infty}\displaystyle\int_\sigma^s T_0(s+\theta-r)nR(n,A)f(r)dr$ exists uniformly for $\sigma\leq s\leq T$, it follows that

$$\varepsilon_{n,m}(T)\longrightarrow 0 \text{ as } n,m\to+\infty.$$

Consequently, from estimation (3.3), we obtain for $\sigma\leq t\leq T$ that

$$\begin{aligned}\sup_{\sigma\leq s\leq t}\|\xi^n(s,\sigma)-\xi^m(s,\sigma)\| &\leq K(T-\sigma+1)\varepsilon_{n,m}(T) \\ &\quad + K(T-\sigma+1)\widehat{M}(T)\int_\sigma^t \sup_{\sigma\leq s\leq r}\|\xi^n(s,\sigma)-\xi^m(s,\sigma)\|dr.\end{aligned}$$

Gronwall's Lemma gives that

$$\sup_{\sigma\leq s\leq T}\|\xi^n(s,\sigma)-\xi^m(s,\sigma)\|\leq K(T-\sigma+1)\varepsilon_{n,m}(T)\exp\left[\widehat{M}(T)K(T-\sigma+1)T\right]\underset{n,m\to+\infty}{\longrightarrow}0.$$

We conclude that $(\xi^n(.,\sigma))_{n>\omega}$ is a Cauchy sequence in $C([\sigma,T];\mathcal{B})$, for each $T>\sigma$. Consequently, there exists a function $\xi\in C([\sigma,+\infty);\mathcal{B})$ such that

$$\lim_{n\to+\infty}\sup_{\sigma\leq s\leq T}\|\xi^n(s,\sigma)-\xi(s)\|=0 \text{ for any } T>\sigma.$$

Let $y:(-\infty,+\infty)\longrightarrow X$ be the function defined by

$$y(t)=\begin{cases}0 \text{ for } t\leq\sigma,\\ \lim\limits_{\lambda\to+\infty}\displaystyle\int_\sigma^t T_0(t-r)\lambda R(\lambda,A)[L(\xi(r))+f(r)]dr \text{ for } t>\sigma.\end{cases}$$

Then

$$y_t(\theta)=\begin{cases}0 \text{ for } t+\theta\leq\sigma,\\ \lim\limits_{\lambda\to+\infty}\displaystyle\int_\sigma^{t+\theta} T_0(t+\theta-r)\lambda R(\lambda,A)[L(\xi(r))+f(r)]dr \text{ for } t+\theta>\sigma.\end{cases}$$

We claim that

$$\xi(t) = y_t \text{ for } t \geq \sigma.$$

If $t + \theta \leq \sigma - \dfrac{1}{n}$, then $|\xi^n(t,\sigma)(\theta) - y(t+\theta)| = 0$.

If $\sigma - \dfrac{1}{n} \leq t + \theta \leq \sigma$, we have

$$|\xi^n(t,\sigma)(\theta) - y(t+\theta)| \quad = \quad |\xi^n(t,\sigma)(\theta)| \quad \leq \quad \widetilde{N}\widehat{f}(\sigma,t)\frac{1}{n}.$$

If $t + \theta \geq \sigma$, then

$$\begin{aligned}
|\xi^n(t,\sigma)(\theta) - y(t+\theta)| \quad \leq \quad & \left| \int_\sigma^{t+\theta} T_0(t+\theta-r)nR(n,A)f(r)dr \right. \\
& \left. - \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta} T_0(t+\theta-r)\lambda R(\lambda,A)f(r)dr \right| \\
& + \left| \lim_{\lambda\to+\infty}\int_\sigma^{t+\theta} T_0(t+\theta-r)\lambda R(\lambda,A)L[\xi^n(r,\sigma)-\xi(r)]dr \right| \\
& + \left| \int_{t+\theta}^{\min\{t,t+\theta-\frac{1}{n}\}} [n(t+\theta-r)+1]nR(n,A)f(r)dr \right|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\xi^n(t,\sigma)(\theta) - y(t+\theta)| \quad \leq \quad & \sup_{\sigma\leq s\leq t}\left| \int_\sigma^{s} T_0(s-r)nR(n,A)f(r)dr \right. \\
& \left. - \lim_{\lambda\to+\infty}\int_\sigma^{s} T_0(s-r)\lambda R(\lambda,A)f(r)dr \right| \\
& + \widetilde{N}\beta(t-\sigma)e^{\alpha(t-\sigma)}\,|L| \sup_{\sigma\leq s\leq t}\|\xi^n(s,\sigma)-\xi(s)\| + N\widehat{f}(\sigma,t)\frac{1}{n} \\
\leq \quad & \rho_n(t),
\end{aligned}$$

where

$$\begin{aligned}
\rho_n(t) \quad = \quad & \sup_{\sigma\leq s\leq t}\left| \int_\sigma^{s} T_0(s-r)nR(n,A)f(r)dr - \lim_{\lambda\to+\infty}\int_\sigma^{s} T_0(s-r)\lambda R(\lambda,A)f(r)dr \right| \\
& + \widetilde{N}\beta t e^{\alpha t}\,|L| \sup_{\sigma\leq s\leq t}\|\xi^n(s,\sigma)-\xi(s)\| + N\widehat{f}(\sigma,t)\frac{1}{n}.
\end{aligned}$$

By Axiom (**A**), we deduce that

$$\begin{aligned}
\|\xi^n(t,\sigma) - y_t\| \quad & \leq \quad K(t-\sigma+1)\sup_{\sigma-t-1\leq\theta\leq 0}|\xi^n(t,\sigma)(\theta) - y(t+\theta)| \\
& \leq \quad K(t-\sigma+1)\rho_n(t) \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

We conclude, for $t \geq \sigma$, that $\xi(t) = y_t$. From the definition of y, we obtain for $t \geq \sigma$ that

$$y(t) = \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-r)\lambda R(\lambda,A)[L(y_r) + f(r)]dr.$$

Therefore, $y(t) = u(t,\sigma,0,L,f)$ and $\xi(t) = u_t(\cdot,\sigma,0,L,f)$. Consequently,

$$u_t(.,\sigma,\varphi,L,f) = V(t-\sigma) + \lim_{n\to\infty}\int_\sigma^t V(t-r)\Theta^n f(r)dr \text{ for } t \geq \sigma.$$

Moreover, for any $T > \sigma$, the limit exists uniformly for $t \in [\sigma,T]$. $\square$

4. FORMULAS FOR BOUNDED SOLUTIONS ON $\mathbb{R}^-$, $\mathbb{R}^+$ AND $\mathbb{R}$

In the sequel, we assume that the semigroup $(V(t))_{t\geq 0}$ is hyperbolic.

Theorem 4.1. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. If* f *is bounded on* $(-\infty,0]$ *and* u *is a solution of equation* (1.5) *on* $(-\infty,0]$, *then* $u \in BC((-\infty,0];X)$ *if and only if*

$$P^-u_0 = \lim_{n\to+\infty}\int_{-\infty}^{0} V(-s)P^-\left(\Theta^n f(s)\right)ds. \tag{4.1}$$

Moreover, u *is given by*

$$u_t = V(t)P^+u_0 + (\mathcal{K}^-f)(t) \quad \text{for } t\leq 0,$$

where $\mathcal{K}^- : BC((-\infty,0];X) \to BC((-\infty,0];\mathcal{B}_A)$ *is the bounded linear operator defined for each* $f\in BC((-\infty,0];X)$ *and* $t\leq 0$ *by*

$$(\mathcal{K}^-f)(t) = \lim_{n\to+\infty}\int_0^t V(t-s)P^+\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds.$$

Proof. Let u be a bounded solution of equation (1.5) on $(-\infty,0]$. Then, for $a<t\leq 0$, one has

$$u_t = V(t-a)u_a + \lim_{n\to+\infty}\int_a^t V(t-s)\Theta^n f(s)ds.$$

It follows, for $a<t\leq 0$, that

$$P^-u_t = V(t-a)P^-u_a + \lim_{n\to+\infty}\int_a^t V(t-s)P^-\left(\Theta^n f(s)\right)ds. \tag{4.2}$$

Since u is bounded on $(-\infty,0]$, we deduce by Proposition 2.8 that $t\to u_t$ is bounded on $(-\infty,0]$. Letting $a\to-\infty$ in (4.2), we get for $t\leq 0$

$$P^-u_t = \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds.$$

Taking $t=0$, we obtain formula (4.1). On the other hand, for $t\leq 0$, we have

$$\begin{aligned} u_t &= P^+u_t + P^-u_t \\ &= V(t)P^+u_0 + \lim_{n\to+\infty}\int_0^t V(t-s)P^+\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds. \end{aligned}$$

Conversely, assume that formula (4.1) holds and consider the $\mathcal{B}_A$-valued function v defined for $t\leq 0$ by

$$v(t) = V(t)P^+u_0 + (\mathcal{K}^-f)(t).$$

Then, v is a bounded function on $(-\infty,0]$. Moreover, for $a\leq t\leq 0$ one has

$$\begin{aligned} v(t) &= V(t)P^+u_0 + \lim_{n\to+\infty}\int_0^a V(t-s)P^+\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_a^t V(t-s)P^+\left(\Theta^n f(s)\right)ds \\ &\quad + \lim_{n\to+\infty}\int_{-\infty}^a V(t-s)P^-\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_a^t V(t-s)P^-\left(\Theta^n f(s)\right)ds. \end{aligned}$$

Finally, we arrive for $a\leq t\leq 0$ at

$$v(t) = V(t-a)v(a) + \lim_{n\to+\infty}\int_a^t V(t-s)\Theta^n f(s)ds.$$

Moreover,

$$\begin{aligned} v(0) &= P^+u_0 + (\mathcal{K}^-f)(0) \\ &= P^+u_0 + P^-u_0 \\ &= u_0. \end{aligned}$$

Which implies that $u_t = v(t)$ for $t \leq 0$, thus $u \in BC((-\infty, 0]; X)$. $\square$

Theorem 4.2. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. If* f *is bounded on* $[0, +\infty)$ *and* u *is a solution of equation* (1.5) *on* $[0, +\infty)$*, then,* u *is bounded on* $[0, +\infty)$ *if and only if*

$$P^+u_0 = -\lim_{n\to+\infty}\int_0^{+\infty} V(-s)P^+\left(\Theta^n f(s)\right)ds. \tag{4.3}$$

If (4.3) *holds, then* u *is given for* $t \geq 0$ *by*

$$u_t = V(t)P^-u_0 + (\mathcal{K}^+f)(t),$$

where $\mathcal{K}^+ : BC([0, +\infty); X) \longrightarrow BC([0, +\infty); \mathcal{B}_A)$ *is the bounded linear operator defined for* $f \in BC([0, +\infty); X)$ *and* $t \geq 0$ *by*

$$\begin{aligned}(\mathcal{K}^+f)(t) = \;& \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n f(s)\right)ds \\ & -\lim_{n\to+\infty}\int_t^{+\infty} V(t-s)P^+\left(\Theta^n f(s)\right)ds.\end{aligned}$$

Proof. Assume that u is a bounded integral solution of equation (1.5) on $[0, +\infty)$. Then, for $t \geq a \geq 0$,

$$P^+u_t \;=\; V(t-a)P^+u_a + \lim_{n\to+\infty}\int_a^t V(t-s)P^+\left(\Theta^n f(s)\right)ds.$$

Since $(V(t))_{t\geq 0}$ becomes a group on $\mathcal{U}$, we get for $t \geq a \geq 0$ that

$$P^+u_a = V(a-t)P^+u_t - \lim_{n\to+\infty}\int_a^t V(a-s)P^+\left(\Theta^n f(s)\right)ds. \tag{4.4}$$

Since u is a bounded function on $\mathbb{R}^+$ and thanks to Lemma 2.10, we get that $t \to u_t$ is bounded on $[0, +\infty)$. Letting $t \to +\infty$ in (4.4), one has

$$P^+u_a = -\lim_{n\to+\infty}\int_a^{+\infty} V(a-s)P^+\left(\Theta^n f(s)\right)ds.$$

Taking $a = 0$, we get formula (4.3). Moreover, for $t \geq 0$,

$$\begin{aligned}u_t &= P^-u_t + P^+u_t \\ &= V(t)P^-u_0 + \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n f(s)\right)ds - \lim_{n\to+\infty}\int_t^{+\infty} V(t-s)P^+\left(\Theta^n f(s)\right)ds.\end{aligned}$$

Finally, we arrive for $t \geq 0$ at

$$u_t = V(t)P^-u_0 + (\mathcal{K}^+f)(t).$$

Conversely, assume that formula (4.3) is true and consider the $\mathcal{B}_A$-valued function v defined for $t \geq 0$ by

$$v(t) = V(t)P^-u_0 + (\mathcal{K}^+f)(t).$$

Then, v is a bounded function and for all $t \geq a \geq 0$ one has

$$\begin{aligned}v(t) \;=\;& V(t)P^-u_0 + \lim_{n\to+\infty}\int_0^a V(t-s)P^-\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_a^t U(t-s)P^-\left(\Theta^n f(s)\right)ds \\ & -\lim_{n\to+\infty}\int_a^{+\infty} V(t-s)P^+\left(\Theta^n f(s)\right)ds - \lim_{n\to+\infty}\int_t^a V(t-s)P^+\left(\Theta^n f(s)\right)ds.\end{aligned}$$

Consequently,

$$v(t) \;=\; V(t-a)v_a + \lim_{n\to+\infty}\int_t^a V(t-s)\Theta^n f(s)ds.$$

On the other hand,

$$\begin{aligned} v(0) &= P^- u_0 + (\mathcal{K}^+ f)(0) \\ &= P^- u_0 + P^+ u_0 \\ &= u_0, \end{aligned}$$

which implies for $t \geq 0$ that

$$u_t = v(t).$$

Consequently, $u \in BC([0, +\infty); X)$. $\square$

Theorem 4.3. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. If* $f \in BC(\mathbb{R}; X)$, *then equation* (1.5) *has a unique bounded solution* w *on* $\mathbb{R}$ *which is given for* $t \in \mathbb{R}$, *by*

$$w_t = \lim_{n \to +\infty} \int_{-\infty}^{t} V(t-s) P^- \left(\Theta^n f(s) \right) ds - \lim_{n \to +\infty} \int_{t}^{+\infty} V(t-s) P^+ \left(\Theta^n f(s) \right) ds. \tag{4.5}$$

Proof. Let w be given by formula (4.5). Then, for $t \in \mathbb{R}$,

$$\begin{aligned} \|w_t\| &\leq |P^-| \delta K(1) \widetilde{N} |f|_\infty \int_{-\infty}^{t} e^{-\mu(t-s)} ds + |P^+| \delta K(1) \widetilde{N} |f|_\infty \int_{t}^{+\infty} e^{\mu(t-s)} ds \\ &\leq \frac{\delta K(1) \widetilde{N} |f|_\infty}{\mu} (|P^-| + |P^+|), \end{aligned}$$

which implies that w is bounded on $\mathbb{R}$. Moreover, for all $t \geq a$ in $\mathbb{R}$, we have

$$\begin{aligned} w_t &= \lim_{n \to +\infty} \int_{-\infty}^{a} V(t-s) P^- \left(\Theta^n f(s) \right) ds + \int_{a}^{t} V(t-s) P^- \left(\Theta^n f(s) \right) ds \\ &\quad - \lim_{n \to +\infty} \int_{t}^{a} V(t-s) P^+ \left(\Theta^n f(s) \right) ds - \int_{a}^{+\infty} V(t-s) P^+ \left(\Theta^n f(s) \right) ds \\ &= V(t-a) w_a + \int_{a}^{t} V(t-s) \Theta^n f(s) ds. \end{aligned}$$

This implies that w is a solution of equation (1.5) on $\mathbb{R}$. For the uniqueness, we suppose that there is another bounded solution v of equation (1.5). Then $v - w$ is a bounded solution of the homogeneous equation

$$\frac{d}{dt} u(t) = Au(t) + L(u_t) \text{ for } t \in \mathbb{R}.$$

It follows for $t \geq a$ that

$$v_t - w_t = V(t-a)(v_a - w_a),$$

and

$$\begin{aligned} P^+(v_t - w_t) &= V(t-a) P^+(v_a - w_a) \text{ for } t,\, a \in \mathbb{R}, \\ P^-(v_t - w_t) &= V(t-a) P^-(v_a - w_a) \text{ for } t \geq a. \end{aligned}$$

Then,

$$\|P^+(v_t - w_t)\| \leq \delta e^{\mu(t-a)} |P^+| \sup_{s \in \mathbb{R}} \|v_s - w_s\| \text{ for } t \leq a, \tag{4.6}$$

and

$$\|P^-(v_t - w_t)\| \leq \delta e^{-\mu(t-a)} |P^-| \sup_{s \in \mathbb{R}} \|v_s - w_s\| \text{ for } t \geq a. \tag{4.7}$$

Letting a go to $+\infty$ in (4.6), we get that $P^+(v_t - w_t) = 0$. Consequently, $(v_t - w_t) \in \mathcal{S}$. That is $v_t - w_t = P^-(v_t - w_t)$. Passing to the limit as a goes to $-\infty$ in (4.7), we obtain that $v_t = w_t$ for $t \in \mathbb{R}$. This implies that formula (4.5) determines a unique bounded solution on $\mathbb{R}$. $\square$

5. Stable and unstable manifolds

In this section, we study the existence of stable and unstable manifolds near an hyperbolic stationary solution for the following nonlinear partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}u\,(t) = Au(t) + L(u_t) + g(u_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{5.1}$$

where L is a bounded linear operator from $\mathcal{B}$ to X and $g : \mathcal{B} \to X$ is a continuous function. Without loss of generality, we assume that zero is a stationary solution of equation (5.1). That is equivalent to say that $g(0) = 0$. We suppose that

($\mathbf{H}_4$) $g : \mathcal{B} \to X$ is a Lipschitz continuous function and g is differentiable at 0 with $g'(0) = 0$.

Then the linearized equation of (5.1) at 0 is given by

$$\begin{cases} \dfrac{d}{dt}u\,(t) = Au(t) + L(u_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{B}. \end{cases} \tag{5.2}$$

The Lipschitz constant $Lip(g)$ of g is defined by

$$Lip(g) = \sup_{\varphi_1 \neq \varphi_2} \frac{|g(\varphi_1) - g(\varphi_2)|}{\|\varphi_1 - \varphi_2\|}.$$

The stationary solution 0 of equation (5.1) is said to be hyperbolic if the solution semigroup of the linear equation (5.2) is hyperbolic.

The stable manifold $\mathcal{S}(g)$ and unstable manifold $\mathcal{U}(g)$ associated to the stationary solution 0 of equation (5.1) are defined respectively by

$$\mathcal{S}(g) = \left\{ \varphi \in \mathcal{B}_A : u_t(.,\varphi,g) \underset{t \to +\infty}{\longrightarrow} 0 \right\},$$

$$\mathcal{U}(g) = \left\{ \varphi \in \mathcal{B}_A : u_t(.,\varphi,g) \underset{t \to -\infty}{\longrightarrow} 0 \right\},$$

where $u(.,\varphi,g)$ denotes the solution of (5.1) in $(-\infty, 0]$ or $[0, +\infty)$ with $u_0(.,\varphi,g) = \varphi$.

Theorem 5.1. *Assume that* ($\mathbf{H_1}$), ($\mathbf{H_3}$), ($\mathbf{H_4}$) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then, there exists* $\varepsilon > 0$ *such that, for* $Lip(g) < \varepsilon$,

$$\mathcal{S}(g) = \left\{ \varphi \in \mathcal{B}_A : \sup_{t \geq 0} \|u_t(.,\varphi,g)\| < \infty \right\},$$

$$\mathcal{U}(g) = \left\{ \varphi \in \mathcal{B}_A : \sup_{t \leq 0} \|u_t(.,\varphi,g)\| < \infty \right\},$$

and

$$\|u_t(.,\varphi,g)\| \leq 2\delta \|P^- \varphi\| e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}(g),$$

$$\|u_t(.,\varphi,g)\| \leq 2\delta \|P^+ \varphi\| e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{U}(g),$$

where δ *and* μ *are the positive constants defined previously in* (2.1). *Moreover,* $\mathcal{S}(g)$ *and* $\mathcal{U}(g)$ *are respectively positively and negatively invariant.*

Proof. Let $\varphi \in \mathcal{S}(g)$ and $u = u(.,\varphi,g)$ be the solution of equation (5.1) on $[0,\infty)$ with $u_0(.,\varphi,g) = \varphi$. Then u is bounded on $[0,\infty)$ and $\sup\limits_{t\geq 0}|g(u_t)| < \infty$. By Theorem 4.2, we deduce that u is given for $t \geq 0$ by

$$\begin{aligned} u_t = \;& V(t)P^-\varphi + \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n\left(g(u_s)\right)\right)ds \\ & - \lim_{n\to+\infty}\int_t^\infty V(t-s)P^+\left(\Theta^n\left(g(u_s)\right)\right)ds. \end{aligned} \tag{5.3}$$

This implies that

$$\begin{aligned} \|u_t\| \leq \;& \delta e^{-\mu t}\|P^-\varphi\| + \delta|P^-|K(1)\widetilde{N}\,Lip(g)\int_0^t e^{-\mu(t-s)}\|u_s\|ds \\ & + \delta|P^+|K(1)\widetilde{N}\,Lip(g)\int_t^\infty e^{\mu(t-s)}\|u_s\|ds. \end{aligned} \tag{5.4}$$

To complete the proof, we need the following fundamental lemma.

Lemma 5.2. [12, p.110] *Let α', ν', K', l' and N' be positive constants and v be a nonnegative bounded continuous solution of either the inequality*

$$i)\ v(t) \leq K'e^{-\alpha' t} + l'\int_0^t e^{-\alpha'(t-s)}v(s)ds + N'\int_0^{+\infty} e^{-\nu' s}v(t+s)ds \text{ for } t\geq 0,$$

or the inequality

$$ii)\ v(t) \leq K'e^{\alpha' t} + l'\int_t^0 e^{\alpha'(t-s)}v(s)ds + N'\int_{-\infty}^0 e^{\nu' s}v(t+s)ds \quad \text{for } t\leq 0.$$

If $\beta' = \dfrac{l'}{\alpha'} + \dfrac{N'}{\nu'} < 1$, then, in either case,

$$v(t) \leq (1-\beta')^{-1}K'e^{-\left(\left[\alpha'-(1-\beta')^{-1}l'\right]|t|\right.}.$$

Let $\varepsilon > 0$ be chosen such that

$$\frac{4\varepsilon\widetilde{N}\delta K(1)}{\mu}\max(|P^+|,|P^-|) < 1.$$

Then, for $Lip(g) < \varepsilon$, we obtain for $t \geq 0$ that

$$\begin{aligned} \|u_t\| \leq \;& \delta\|P^-\varphi\|e^{-\mu t} + \delta|P^-|K(1)\widetilde{N}\varepsilon\int_0^t e^{-\mu(t-s)}\|u_s\|ds \\ & + \delta|P^+|K(1)\widetilde{N}\varepsilon\int_t^\infty e^{\mu(t-s)}\|u_s\|ds. \end{aligned}$$

Using Lemma 5.2, we get that

$$\|u_t\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t\geq 0. \tag{5.5}$$

Conversely, let u be a bounded solution of equation (5.1) on $[0,\infty)$. Then, by Theorem 4.2, we get that u is given for $t \geq 0$ by formula (5.3) and using Lemma 5.2 we get the estimation (5.5).

We use the same approach in the case of the unstable manifold. In fact, let $\varphi \in \mathcal{U}(g)$ and $u = u(.,\varphi,g)$ be the solution of equation (5.1) in $(-\infty,0]$ with $u_0(.,\varphi,g) = \varphi$. Then u is bounded on $(-\infty,0]$ and $\sup_{t\leq 0}|g(u_t)|<\infty$. By Theorem 4.1, we deduce that u is given for $t\leq 0$ by

$$\begin{aligned} u_t = \;& V(t)P^+\varphi+\lim_{n\to+\infty}\int_0^t V(t-s)P^+\left(\Theta^n\left(g(u_s)\right)\right)ds \\ & +\lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n\left(g(u_s)\right)\right)ds. \end{aligned} \tag{5.6}$$

It follows that

$$\begin{aligned} \|u_t\| \leq \;& \delta e^{\mu t}\|P^+\varphi\|+\delta|P^+|K(1)\widetilde{N}\,Lip(g)\int_t^0 e^{\mu(t-s)}\|u_s\|ds \\ & +\delta|P^-|K(1)\widetilde{N}\,Lip(g)\int_{-\infty}^t e^{-\mu(t-s)}\|u_s\|ds. \end{aligned} \tag{5.7}$$

Let $\varepsilon>0$ be chosen such that

$$\frac{4\varepsilon\widetilde{N}\delta K(1)}{\mu}\max(|P^+|,|P^-|)<1.$$

Then, for $Lip(g)<\varepsilon$, we obtain for $t\leq 0$ that

$$\begin{aligned} \|u_t\| \leq \delta\|P^+\varphi\|e^{\mu t}+\delta|P^+|K(1)\widetilde{N}\varepsilon\int_t^0 e^{\mu(t-s)}\|u_s\|ds \\ +\delta|P^-|K(1)\widetilde{N}\varepsilon\int_{-\infty}^t e^{-\mu(t-s)}\|u_s\|ds. \end{aligned}$$

Using Lemma 5.2, we get that

$$\|u_t\|\leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t\leq 0. \tag{5.8}$$

Conversely, let u be a bounded solution of equation on $(-\infty,0]$. Then, by Theorem 4.1, we get that u is given for $t\leq 0$ by formula (5.6) and using Lemma 5.2 we get the estimation (5.8). About the invariance principle, let $u_t(.,\varphi,g)$ be a bounded solution of equation (5.1) on $\mathbb{R}^+$ such that

$$\sup_{t\geq 0}\|u_t(.,\varphi,g)\|<\infty.$$

Since, equation (5.1) is autonomous, it follows that

$$u_t(.,u_s(.,\varphi,g),g)=u_{t+s}(.,\varphi,g), \text{ for } t,s\geq 0.$$

Which implies that

$$u_s(.,\varphi,g)\in\mathcal{S}(g) \text{ for } s\geq 0.$$

The same argument can be used to show that the unstable manifold is negatively invariant. □

6. Local stable and unstable manifolds

In section 5, the stable and unstable manifolds have been obtained when the function g is Lipschitz continuous and $Lip(g)$ is small enough. In this section, we establish the existence of the local stable and unstable manifolds where g is not globally Lipschitz. More precisely, instead of assumption $(\mathbf{H_4})$, we make the following assumption

$(\mathbf{H_5})$ g is continuously differentiable in $B(0,\rho_0)$, for some $\rho_0>0$ with $g(0)=0$ and $g'(0)=0$.

For $\rho < \rho_0$, we define respectively the local stable and unstable manifolds associated to the zero stationary solution of (5.1) by

$$\mathcal{S}_{loc}(g) = \{\varphi \in B(0,\rho) : \|u_t(.,\varphi,g)\| < \rho \text{ for } t \geq 0\},$$
$$\mathcal{U}_{loc}(g) = \{\varphi \in B(0,\rho) : \|u_t(.,\varphi,g)\| < \rho \text{ for } t \leq 0\}.$$

Theorem 6.1. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_3})$, $(\mathbf{H_5})$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then, there exists* $\rho < \rho_0$ *such that*

$$\|u_t(.,\varphi,g)\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}_{loc}(g),$$

$$\|u_t(.,\varphi,g)\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{U}_{loc}(g).$$

Moreover, any bounded solution by ρ *on* $\mathbb{R}^+$ *of equation (5.1) lies in* $\mathcal{S}_{loc}(g)$ *and any bounded solution by* ρ *on* $\mathbb{R}^-$ *of equation (5.1) lies in* $\mathcal{U}_{loc}(g)$.

Proof. Without loss of generality, one can choose ρ_0 such that $\sup\limits_{\|\varphi\|<\rho_0} |g'(\varphi)| < \infty$. For $0 < \rho < \rho_0$, we introduce the function g_ρ defined on $\mathcal{B}$ by

$$g_\rho(\varphi) = \begin{cases} g(\varphi) \text{ for } \|\varphi\| \leq \rho, \\ g\left(\rho\frac{\varphi}{\|\varphi\|}\right) \text{ for } \|\varphi\| > \rho. \end{cases}$$

Then, we have the following Lipschitz property of the function g_ρ.

Lemma 6.2. [26] *For all* $\rho < \rho_0$, *the function* g_ρ *is Lipschitz continuous and*

$$Lip(g_\rho) \leq 2 \sup_{\|\varphi\|<\rho} |g'(\varphi)|.$$

We consider the following partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + g_\rho(u_t) \text{ for } t \geq 0, \\ u_0 = \varphi. \end{cases} \tag{6.1}$$

Let ε be given in Theorem 5.1. Since $\sup\limits_{\|\varphi\|<\rho} |g'(\varphi)| \to 0$ as $\rho \to 0$, then there exists $\rho < \rho_0$ such that

$$Lip(g_\rho) < \varepsilon.$$

Applying Theorem 5.1 to equation (6.1), we obtain the following estimations

$$\|u_t(.,\varphi,g_\rho)\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}(g_\rho),$$

$$\|u_t(.,\varphi,g_\rho)\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{U}(g_\rho).$$

Since $g = g_\rho$ in $B(0,\rho)$, then

$$u(.,\varphi,g) = u(.,\varphi,g_\rho) \text{ for } \varphi \in \mathcal{S}_{loc}(g),$$

$$u(.,\varphi,g) = u(.,\varphi,g_\rho) \text{ for } \varphi \in \mathcal{U}_{loc}(g).$$

Consequently, we get the asymptotic behavior of solutions

$$\begin{aligned} &\|u_t(.,\varphi,g)\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}_{loc}(g), \\ &\|u_t(.,\varphi,g)\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{U}_{loc}(g). \end{aligned} \tag{6.2}$$

Let $u(.,\varphi,g)$ be a solution of equation (5.1) which is bounded by ρ. Then $u_t(.,\varphi,g) = u_t(.,\varphi,g_\rho)$ for $t \geq 0$. By Theorem 5.1 we get that $\varphi \in \mathcal{S}(g_\rho)$. Since $\mathcal{S}(g_\rho)$ is positively invariant, then $u_t(.,\varphi,g_\rho) \in \mathcal{S}(g_\rho)$ for $t \geq 0$, this implies that $u_t(.,\varphi,g) \in \mathcal{S}_{loc}(g)$ for $t \geq 0$. The same argument can be used in the case of the unstable manifold.□

Definition 6.3. Let $\mathcal{O}$ be a subset of $\mathcal{B}_A$ which contains the origin 0. We say that $\mathcal{O}$ is tangent to $\mathcal{S}$ (respectively $\mathcal{U}$) at 0 if

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \longrightarrow 0 \text{ (respectively } \frac{\|P^-\varphi\|}{\|P^+\varphi\|} \longrightarrow 0) \text{ as } \varphi \to 0 \text{ in } \mathcal{O}.$$

Theorem 6.4. *$\mathcal{S}_{loc}(g)$ (respectively $\mathcal{U}_{loc}(g)$) is tangent to $\mathcal{S}$ (respectively to $\mathcal{U}$) at 0.*

Proof. Let $\varphi \in \mathcal{S}_{_{loc}}(g)$. Then the corresponding solution $u(.,\varphi,g)$ of equation (5.1) is bounded on $[0,\infty)$. From Theorem 4.2, we get that

$$\varphi - P^-\varphi = P^+\varphi = -\lim_{n\to+\infty} \int_0^\infty V(-s)P^+\left(\Theta^n\left(g(u_s)\right)\right) ds. \tag{6.3}$$

Thus,

$$\|P^+\varphi\| \leq \frac{\delta \widetilde{N} K(1)|P^+|}{\mu} \sup_{\|\psi\|<\rho} |g'(\psi)| \sup_{t\geq 0} \|u_t\|.$$

By estimation (6.2), we obtain that

$$\|P^+\varphi\| \leq 2\frac{\delta^2 \widetilde{N} K(1)|P^+|}{\mu} \sup_{\|\psi\|<\rho} |g'(\psi)| \|P^-\varphi\|.$$

We can choose ρ sufficiently small such that

$$\frac{\delta^2 \widetilde{N} K(1)|P^+|}{\mu} \sup_{\|\psi\|<\rho} |g'(\psi)| < \frac{1}{4}.$$

Then,

$$\|P^+\varphi\| \leq \frac{1}{2}\|P^-\varphi\|. \tag{6.4}$$

Consequently,

$$\|\varphi\| \geq \frac{1}{2}\|P^-\varphi\|.$$

Let $\varphi \in \mathcal{S}_{loc}(g)$ such that $\varphi \neq 0$. Then, (6.3) and expression (6.4) implies that $\|u_t(\cdot,\varphi,g)\| \leq 2\delta(2\|\varphi\|) = 4\delta\|\varphi\|$; hence

$$\|P^+\varphi\| \leq \frac{2\delta^2 \widetilde{N} K(1)|P^+|}{\mu} \sup_{\|\psi\|<4\delta\|\varphi\|} \left|g'(\psi)\right| \|P^-\varphi\|.$$

Then,

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \leq \frac{2\delta^2 \widetilde{N} K(1)|P^+|}{\mu} \sup_{\|\psi\|<4\delta\|\varphi\|} \left|g'(\psi)\right|.$$

Hence,

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \to 0 \text{ as } \|\varphi\| \to 0 \text{ in } \mathcal{S}_{loc}(g).$$

That is $\mathcal{S}_{loc}(g)$ is tangent to $\mathcal{S}$ at 0. One can use the same reasoning to show that $\mathcal{U}_{loc}(g)$ is tangent to $\mathcal{U}$ at 0. □

For $\rho < \rho_0$, we define the following sets

$$\widetilde{\mathcal{S}}_\rho(g) = \{\varphi \in B(0,\rho) : \|P^-\varphi\| < \frac{\rho}{2\delta} \text{ and } \|u_t(.,\varphi,g)\| < \rho \text{ for } t \geq 0\},$$

$$\widetilde{\mathcal{U}}_\rho(g) = \{\varphi \in B(0,\rho) : \|P^+\varphi\| < \frac{\rho}{2\delta} \text{ and } \ \|u_t(.,\varphi,g)\| < \rho, \text{ for } t \leq 0\}.$$

We suppose furthermore that

$(\mathbf{H_6})$ g' is bounded in $B(0,\rho_0)$.

Theorem 6.5. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_3})$, $(\mathbf{H_5})$, $(\mathbf{H_6})$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then, there exists* $\rho < \rho_0$ *such that* P^- *is a diffeomorphism from* $\widetilde{\mathcal{S}}_\rho(g)$ *to* $B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$ *and* P^+ *is a diffeomorphism from* $\widetilde{\mathcal{U}}_\rho(g)$ *to* $B(0, \frac{\rho}{2\delta}) \cap \mathcal{U}$.

Proof. Let $\rho < \rho_0$ be obtained in Theorem 6.1. Set

$$B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho) = \left\{ y \in BC(\mathbb{R}^+,\mathcal{B}) : \sup_{t\geq 0}|y(t)| < \rho \right\}.$$

Consider the Nemitsky operator G defined, for $u \in B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho)$ and $t \geq 0$ by

$$(Gu)(t) = g(u(t)).$$

Let H be the operator defined from $B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho) \times (B(0,\frac{\rho}{2\delta}) \cap \mathcal{S})$ to $BC(\mathbb{R}^+,\mathcal{B})$, for $t \geq 0$ by

$$H(u,\varphi)(t) = u(t) - V(t)\varphi - [\mathcal{K}^+(Gu)](t).$$

Using the same argument as in Lemma 1.1, Appendix IV in [8], one can show that G has the same properties as g. More precisely, we have the following interesting result.

Lemma 6.6. *G is continuously differentiable. Moreover, for* $z \in B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho)$, $h \in BC(\mathbb{R}^+,\mathcal{B})$ *and* $t \geq 0$

$$(G'(z)h)(t) = g'(z(t))h(t).$$

Consequently H is continuously differentiable in a neighborhood of $(0,0)$. Moreover,

$$H(0,0) = 0 \text{ and } \frac{\partial H}{\partial u}(0,0) = I.$$

By the implicit functions Theorem, we deduce that there exists $\rho < \rho_0$ such that for $\varphi \in B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$, there exists a unique $u^*(\varphi) \in B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho)$ satisfying

$$H(u^*(\varphi),\varphi) = 0. \tag{6.5}$$

Moreover, the mapping

$$\varphi \to u^*(\varphi)$$

is a diffeomorphism from $B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$ to $B_{BC(\mathbb{R}^+,\mathcal{B})}(0,\rho)$. On the other hand, $u^*(\varphi)$ satisfies for $t \geq 0$

$$\begin{aligned} u^*(\varphi)(t) = V(t)\varphi &+ \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds \\ &- \lim_{n\to+\infty}\int_t^\infty V(t-s)P^+\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds. \end{aligned}$$

We introduce the following function

$$v^*(t,\varphi) = \left\{ \begin{array}{l} [u^*(\varphi)(t)](0) \text{ for } t \geq 0, \\ \varphi(t) - \left(\displaystyle\lim_{n\to+\infty}\int_0^\infty V(-s)P^+\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds\right)(t) \text{ for } t \leq 0. \end{array}\right.$$

Then $v^*(.,\varphi)$ is a bounded solution of (5.1) on $\mathbb{R}^+$ and

$$P^- v_0^*(.,\varphi) = \varphi.$$

Using Theorem 6.1, we get for $t \geq 0$,

$$\|v_t^*(.,\varphi)\| \leq 2\delta e^{-\frac{\mu}{2}t}\|P^-\varphi\|.$$

Since $\|\varphi\| < \dfrac{\rho}{2\delta}$, we deduce for $t \geq 0$, that

$$\|v_t^*(.,\varphi)\| < \rho.$$

Let $Q(\varphi) = v_0^*(.,\varphi)$, for $\varphi \in B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$. Then Q is a diffeomorphism from $B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$ to $\widetilde{\mathcal{S}}_\rho(g)$. Furthermore, for $\varphi \in B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$, we have

$$P^- Q(\varphi) = \varphi.$$

This implies that

$$P^- = Q^{-1},$$

which implies that P^- is a diffeomorphism from $\widetilde{\mathcal{S}}_\rho(g)$ to $B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$. The same reasoning can be used in the case of the local unstable manifold. □

As an immediate consequence of Theorem 6.1, we have the following well known linearized stability principle which was established in [2].

Corollary 6.7. *Assume that $\mathcal{B}$ is a uniform fading memory space and* $(\mathbf{H_1})$, $(\mathbf{H_3})$ *and* $(\mathbf{H_5})$ *hold. If $\sigma^+(\mathcal{A}) = \emptyset$, then the zero solution of equation* (5.1) *is locally exponentially stable, that is, there exist $\rho_1,\ \mu_1,\ M_1 > 0$ such that,*

$$\|u_t(.,\varphi,g)\| \leq M_1 e^{-\mu_1 t}\|\varphi\| \text{ for } t \geq 0 \text{ and } \varphi \in B(0,\rho_1). \tag{6.6}$$

Proof. If $\sigma^+(\mathcal{A}) = \emptyset$, then the unstable manifold is reduced to zero and the estimation (6.6) is obtained from theorem 6.1. □

7. Application

Now, we propose to apply the previous stability results to the following Lotka-Voltera model with diffusion

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,\xi) = & \dfrac{\partial^2}{\partial \xi^2}v(t,\xi) + \displaystyle\int_{-\infty}^0 K_1(\theta)v(t+\theta,\xi)d\theta + \int_{-\infty}^0 K_2(\theta,v(t+\theta,\xi))d\theta \text{ for } \ t \geq 0 \text{ and } \ 0 \leq \xi \leq \pi, \\ v(t,0) = & v(t,\pi) = 0 \ \text{ for } \ t \geq 0, \\ v(\theta,\xi) = & v_0(\theta,\xi) \ \text{ for } \ -\infty < \theta \leq 0 \text{ and } 0 \leq \xi \leq \pi, \end{cases} \tag{7.1}$$

where $K_1 : (-\infty,0] \to \mathbb{R}$, $K_2 : (-\infty,0] \times \mathbb{R} \to \mathbb{R}$ and $v_0 : (-\infty,0] \times [0,\pi] \longrightarrow \mathbb{R}$ are continuous functions. Let $X = C([0,\pi];\mathbb{R})$ be the space of continuous functions from $[0,\pi]$ into $\mathbb{R}$ endowed with the uniform norm topology. We consider the operator $A : D(A) \subseteq X \longrightarrow X$ defined by

$$\begin{cases} D(A) = & \{y \in C^2(]0,\pi[;\mathbb{R}) \cap C^1([0,\pi];\mathbb{R}) : y(0) = y(\pi) = 0\}, \\ Ay = & y''. \end{cases}$$

Lemma 7.1. [2] *A satisfies the Hille-Yosida condition in X, namely*

$$(0,+\infty) \subseteq \rho(A) \text{ and } |R(\lambda,A)| \leq \frac{1}{\lambda} \text{ for } \lambda > 0.$$

Moreover

$$\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\} \neq X.$$

For the phase space $\mathcal{B}$, we choose the well known space C_γ, with $\gamma > 0$, defined by

$$C_\gamma = \{\phi : (-\infty, 0] \to X \text{ continuous such that } \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\},$$

endowed with the following norm

$$\|\phi\|_\gamma = \sup_{\theta \leq 0} e^{\gamma\theta}|\phi(\theta)| \text{ for } \phi \in C_\gamma.$$

Lemma 7.2. [16] *C_γ is a normed space satisfying* (**A**), (**B**) *and* (**C**). *Moreover, C_γ is a uniform fading memory space.*

In order to rewrite equation (7.1) in an abstract form, we introduce the following notations. For $t \geq 0,\ \theta \leq 0,\ \xi \in [0, \pi]$

$$u(t)(\xi) = v(t, \xi) \text{ and } \varphi(\theta)(\xi) = v_0(\theta, \xi).$$

For $\phi \in C_\gamma$

$$L(\phi)(\xi) = \int_{-\infty}^{0} K_1(\theta)\phi(\theta)(\xi)d\theta \text{ and } g(\phi)(\xi) = \int_{-\infty}^{0} K_2(\theta, \phi(\theta)(\xi))d\theta.$$

Then, equation (7.1) takes the following form

$$\begin{cases} \dfrac{du}{dt} = & Au(t) + L(u_t) + g(u_t) \text{ for } t \geq 0, \\ u_0 = & \varphi \in C_\gamma. \end{cases} \tag{7.2}$$

In order to study the existence of solutions of equation (7.2), we suppose the following

($\mathbf{E_1}$) $\lim_{\theta \to -\infty} e^{\gamma\theta} v_0(\theta, \xi)$ exists uniformly for $\xi \in [0, \pi]$ and for $\theta \leq 0$, $v_0(\theta, 0) = v_0(\theta, \pi) = 0$.

($\mathbf{E_2}$) For all $\theta \leq 0$ and $\xi_1,\ \xi_2 \in \mathbb{R}$, one has

$$|K_1(\theta)| + \sup_{\xi_1 \neq \xi_2} \frac{|K_2(\theta, \xi_1) - K_2(\theta, \xi_2)|}{|\xi_1 - \xi_2|} \leq k_0(\theta),$$

where k_0 is a positive measurable function on $(-\infty, 0]$ such that

$$\int_{-\infty}^{0} e^{-\gamma\theta} k_0(\theta)d\theta < \infty.$$

Assumption ($\mathbf{E_1}$) implies that $\varphi \in C_\gamma$ and $\varphi(0) \in \overline{D(A)}$. Moreover, ($\mathbf{E_2}$) implies that L is a bounded linear operator on C_γ and g is Lipschitz continuous. In fact, for all $\phi_1,\ \phi_2 \in C_\gamma$, one has

$$|g(\phi_1) - g(\phi_2)| = \sup_{\xi \in [0,\pi]} |g(\phi_1)(\xi) - g(\phi_2)(\xi)| \leq \left(\int_{-\infty}^{0} e^{-\gamma\theta} k_0(\theta)d\theta\right) \|\phi_1 - \phi_2\|_\gamma.$$

Theorem 2.4 implies that the existence and uniqueness of solutions for equation (7.2). Now, we propose to study the asymptotic behavior of solutions. We make the following assumptions.

($\mathbf{E_3}$) For $\theta \leq 0$, $K_2(\theta, 0) = 0$, the function $y \to K_2(\theta, y)$ is continuously differentiable for $y \in \mathbb{R}$, $\dfrac{\partial}{\partial y} K_2(\theta, 0) = 0$ and there exists a positive measurable function k_1 on $(-\infty, 0]$ such that

$\int_{-\infty}^{0} e^{-\gamma\theta} k_1(\theta)d\theta < \infty$ and

$$\left|\frac{\partial}{\partial y}K_2(\theta, y_1) - \frac{\partial}{\partial y}K_2(\theta, y_2)\right| \leq k_1(\theta)|y_1 - y_2| \text{ for } \theta \leq 0 \text{ and } y_1, y_2 \in \mathbb{R}.$$

Assumption ($\mathbf{E_3}$) implies that g is continuously differentiable in C_γ. Moreover, for $\phi \in C_\gamma$, $\psi \in C_\gamma$ and $x \in [0, \pi]$, the following formula holds

$$g'(\phi)(\psi)(x) = \int_{-\infty}^{0} \frac{\partial}{\partial y}K_2(\theta, \phi(\theta)(x))\psi(\theta)(x)d\theta.$$

It follows that $g'(0) = 0$ and the linearized equation at 0 is given by

$$\begin{cases} \dfrac{du}{dt} = & Au(t) + L(u_t) \text{ for } t \geq 0, \\ u_0 = & \varphi \in C_\gamma. \end{cases} \tag{7.3}$$

Proposition 7.3. *Assume that the assumptions* ($\mathbf{E}_1$), ($\mathbf{E}_2$), ($\mathbf{E}_3$) *are true and*

$$\int_{-\infty}^{0} |K_1(\theta)|d\theta < 1.$$

Then, the solution semigroup of equation (7.3) is exponentially stable. Consequently, the zero solution of equation (7.2) is locally exponentially stable.

Proof. To study the stability of equation (7.3), we need to compute the characteristic equation. In fact, for $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) \geq 0$, the operator $\Delta(\lambda)$ associated to the linear equation (7.3) is defined, for $\vartheta \in D(A)$, by

$$\Delta(\lambda)\vartheta = \lambda\vartheta - A\vartheta - \left(\int_{-\infty}^{0} K_1(\theta)e^{\lambda\theta}d\theta\right)\vartheta.$$

Let $\lambda \in \mathbb{C}$ be such that $\mathcal{R}e(\lambda) \geq 0$. Then λ is a characteristic value of equation (7.3) if there exists $\vartheta \in D(A)\backslash\{0\}$ such that $\Delta(\lambda)\vartheta = 0$, this is equivalent to

$$\left(\lambda - \int_{-\infty}^{0} K_1(\theta)e^{\lambda\theta}d\theta - A\right)\vartheta = 0. \tag{7.4}$$

Since the spectrum $\sigma(A)$ is reduced to the point spectrum $\sigma_p(A)$ and $\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Then λ is a solution of the characteristic equation (7.4) with $\mathcal{R}e(\lambda) \geq 0$ if and only if λ satisfies

$$\lambda - \int_{-\infty}^{0} K_1(\theta)e^{\lambda\theta}d\theta = -n^2 \text{ for some } n \in \mathbb{N}^*. \tag{7.5}$$

By Corollary 6.7, it is enough to show that all roots of the characteristic equation (7.5) have negative real parts. We proceed by contradiction and assume that equation (7.5) has, for some $n \in \mathbb{N}^*$, at least one root λ with $\mathcal{R}e(\lambda) \geq 0$. Then

$$\begin{aligned} \mathcal{R}e(\lambda) &= \int_{-\infty}^{0} K_1(\theta)e^{\mathcal{R}e(\lambda)\theta}\cos(\mathrm{Im}(\lambda)\theta)d\theta - n^2 \\ &\leq \int_{-\infty}^{0} |K_1(\theta)|d\theta - n^2. \end{aligned}$$

Since $\int_{-\infty}^{0} |K_1(\theta)|d\theta < 1$, then a contradiction is obtained with the fact that $\mathcal{R}e(\lambda) \geq 0$. Consequently, Corollary 6.7 implies that 0 is locally exponentially stable for (7.2). □

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