

SQUARES AND DIFFERENCE SETS IN FINITE FIELDS

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Abstract

For infinitely many primes $p = 4k + 1$ we give a slightly improved upper bound for the maximal cardinality of a set $B \subset \mathbb{Z}_p$ such that the difference set $B - B$ contains only quadratic residues. Namely, instead of the "trivial" bound $|B| \leq \sqrt{p}$ we prove $|B| \leq \sqrt{p} - 1$, under suitable conditions on p . The new bound is valid for approximately three quarters of the primes $p = 4k + 1$.

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1. Introduction

Let q be a prime-power, say $q = p^k$. We will be interested in estimating the maximal cardinality $s(q)$ of a set $B \subset \mathbb{F}_q$ such that the difference set $B - B$ contains only

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squares. While our main interest is in the case $k = 1$, we find it instructive to compare the situation for different values of k .

This problem makes sense only if -1 is a square; to ensure this we assume $q \equiv 1 \pmod{4}$. The universal upper bound $s(q) \leq \sqrt{q}$ can be proved by a pigeonhole argument or by simple Fourier analysis, and it has been re-discovered several times (see [8, Theorem 3.9], [12, Problem 13.13], [4, Proposition 4.7], [3, Chapter XIII, Theorem 14], [11, Theorem 31.3], [10, Proposition 4.5], [7, Section 2.8] for various proofs). For even k we have equality, since \mathbb{F}_{p^k} can be constructed as a quadratic extension of $\mathbb{F}_{p^{k/2}}$, and then every element of the embedded field $\mathbb{F}_{p^{k/2}}$ will be a square. It is known that every case of equality can be obtained by a linear transformation from this one, [2].

Such problems and results are often formulated in terms of the Paley graph P_q , which is the graph with vertex set \mathbb{F}_q and an edge between x and y if and only if $x - y = a^2$ for some non-zero $a \in \mathbb{F}_q$.

Paley graphs are self-complementary, vertex and edge transitive, and $(q, (q - 1)/2, (q - 5)/4, (q - 1)/4)$ -strongly regular (see [3] for these and other basic properties of P_q). Paley graphs have received considerable attention over the past decades because they exhibit many properties of random graphs $G(q, 1/2)$ where each edge is present with probability $1/2$. Indeed, P_q form a family of *quasi-random* graphs, as shown in [5].

With this terminology $s(q)$ is the *clique number* of P_q . The general lower bound $s(q) \geq (\frac{1}{2} + o(1)) \log_2 q$ is established in [6], while it is proved in [9] that $s(p) \geq c \log p \log \log \log p$ for infinitely many primes p . The “trivial” upper bound $s(p) \leq \sqrt{p}$ is notoriously difficult to improve, and it is mentioned explicitly in the selected list of problems [7]. The only improvement we are aware of concerns the special case $p = n^2 + 1$ for which it is proved in [13] that $s(p) \leq n - 1$ (the same result was proved independently by T. Sanders – unpublished, personal communication). It is more likely, heuristically, that the lower bound is closer to the truth than the upper bound. Numerical data [16, 15] up to $p < 10000$ suggest (very tentatively) that the correct order of magnitude for the clique number of P_p is $c \log^2 p$ (see the discussion and the plot of the function $s(p)$ at [17]).

In this note we prove the slightly improved upper bound $s(p) \leq \sqrt{p} - 1$ for the *majority* of the primes $p = 4k + 1$ (we will often suppress the dependence on p , and just write s instead of $s(p)$).

We will denote the set of nonzero quadratic residues by Q , and that of nonzero non-residues by NQ . Note that $0 \notin Q$ and $0 \notin NQ$.

2. The improved upper bound

Theorem 2.1. *Let q be a prime-power, $q = p^k$, and assume that k is odd and $q \equiv 1 \pmod{4}$. Let $s = s(q)$ be the maximal cardinality of a set $B \subset \mathbb{F}_q$ such that the difference set $B - B$ contains only squares.*

- (i) *If $\lfloor \sqrt{q} \rfloor$ is even then $s^2 + s - 1 \leq q$,*
- (ii) *if $\lfloor \sqrt{q} \rfloor$ is odd then $s^2 + 2s - 2 \leq q$.*

Proof. The claims hold if $s < \lfloor \sqrt{q} \rfloor$. Hence we may assume that $s \geq \lfloor \sqrt{q} \rfloor$.

Lemma 2.2. *Let $D \subset \mathbb{F}_q$ be a set such that*

$$D \subset NQ, \quad D - D \subset Q \cup \{0\}.$$

With $r = |D|$ we have

$$s(q) \leq 1 + \frac{q-1}{2r}. \tag{1}$$

Proof. Let B be a maximal set such that $B - B \subset Q \cup \{0\}$, $|B| = s(q) = s$. Consider the equation

$$b_1 - b_2 = zd, \quad b_1, b_2 \in B, \quad d \in D, \quad z \in NQ.$$

This equation has exactly $s(s-1)r$ solutions; indeed, every pair of distinct $b_1, b_2 \in B$ and a $d \in D$ determines z uniquely. On the other hand, given b_1 and z , there can be at most one pair b_2 and d to form a solution. Indeed, if there were another pair b'_2, d' , then by subtracting the equations

$$b_1 - b_2 = zd, \quad b_1 - b'_2 = zd'$$

we get $(b'_2 - b_2) = z(d - d')$, a contradiction, as the left hand side is a square and the right hand side is not. This gives $s(s-1)r \leq s(q-1)/2$ as wanted. \square

We try to construct such a set D in the form $D = (B - t) \cap NQ$ with a suitable t . The required property then follows from $D - D \subset B - B$.

Let χ denote the quadratic multiplicative character, i.e. $\chi(t) = 1$ according to whether $t \in Q$ or $t \in NQ$ (and $\chi(0) = 0$). Let

$$\varphi(t) = \sum_{b \in B} \chi(b - t). \tag{2}$$

Clearly

$$\varphi(t) = |(B - t) \cap Q| - |(B - t) \cap NQ|,$$

and hence for $t \notin B$ we have

$$|(B - t) \cap NQ| = \frac{s - \varphi(t)}{2}.$$

To find a large set in this form we need to find a negative value of φ .

We list some properties of this function. For $t \in B$ we have $\varphi(t) = s - 1$, and otherwise

$$\varphi(t) \leq s - 2, \varphi(t) \equiv s \pmod{2}$$

(the inequality expresses the maximality of B). Furthermore,

$$\sum_t \varphi(t) = 0,$$

and, since translations of the quadratic character have the quasi-orthogonality property

$$\sum_t \chi(t+a)\chi(t+b) = -1$$

for $a \neq b$, we conclude

$$\sum_t \varphi(t)^2 = s(q-1) - s(s-1) = s(q-s).$$

By subtracting the contribution of $t \in B$ we obtain

$$\sum_{t \notin B} \varphi(t) = -s(s-1),$$

$$\sum_{t \notin B} \varphi(t)^2 = s(q-s) - s(s-1)^2 = s(q-s^2+s-1).$$

These formulas assume an even nicer form by introducing the function $\varphi_1(t) = \varphi(t) + 1$:

$$\sum_{t \notin B} \varphi_1(t) = q - s^2, \tag{3}$$

$$\sum_{t \notin B} \varphi_1(t)^2 = (s+1)(q-s^2). \tag{4}$$

As a byproduct, the second equation shows the familiar estimate $s \leq \sqrt{q}$, so we have $s = \lfloor \sqrt{q} \rfloor < \sqrt{q}$ (recall that we assume that $s \geq \lfloor \sqrt{q} \rfloor$, the theorem being trivial otherwise).

Now we consider separately the cases of odd and even s . If s is even, then, since $\sum_{t \notin B} \varphi(t) < 0$ and each summand is even, we can find a t with $\varphi(t) \leq -2$. This gives us an r with $r \geq (s+2)/2$, and on substituting this into (1) we obtain the first case of the theorem.

If s is odd, we claim that there is a t with $\varphi(t) \leq -3$. Otherwise we have $\varphi(t) \geq -1$, that is, $\varphi_1(t) \geq 0$ for all $t \notin B$. We also know $\varphi(t) \leq s-2$, $\varphi_1(t) \leq s-1$ for $t \notin B$. Consequently

$$\sum_{t \notin B} \varphi_1(t)^2 \leq (s-1) \sum_{t \notin B} \varphi_1(t) = (s-1)(q-s^2),$$

a contradiction to (4). (Observe that to reach a contradiction we need that $q - s^2$ is strictly positive. In case of an even k it can happen that $q = s^2$ and the function φ_1 vanishes outside B .)

This t provides us with a set D with $r \geq (s + 3)/2$, and on substituting this into (1) we obtain the second case of the theorem. \square

Remark 2.3. An alternative proof for the case $q = p$ and s being odd is as follows. Assume by contradiction that φ_1 is even-valued and nonnegative. Then by (3) it must be 0 for at least

$$q - |B| - \frac{q - s^2}{2} = \frac{q + s^2 - 2s}{2}$$

values of t . Let $\tilde{\chi}, \tilde{\varphi}, \tilde{\varphi}_1$ denote the images of χ, φ, φ_1 in \mathbb{F}_q (i.e. the functions are evaluated mod p). By the previous observation $\tilde{\varphi}_1$ has at least $(q + s^2 - 2s)/2$ zeroes. On the other hand, we have $\tilde{\chi}(x) = x^{\frac{q-1}{2}}$, and hence $\tilde{\varphi}_1$ is a polynomial of degree $(q - 1)/2$; its leading coefficient is $s = [\sqrt{q}] \not\equiv 0 \pmod{p}$ (This last fact may fail if $q = p^k$, even if k is odd. Therefore this proof is restricted in its generality. Nevertheless we include it here, because we believe that it has the potential to lead to stronger results if $q = p$.) Consequently $\tilde{\varphi}_1$ can have at most $(q - 1)/2$ zeros, a contradiction. In the case of even k we can have $s = \sqrt{q} \equiv 0 \pmod{p}$ and so the polynomial $\tilde{\varphi}_1$ can vanish, as it indeed does when B is a subfield.

Remark 2.4. It is clear from (1) that any improved lower bound on r will lead to an improved upper bound on s . If one thinks of elements of \mathbb{Z}_p as being quadratic residues randomly with probability $1/2$, then we expect that $r \geq \frac{s}{2} + c\sqrt{s}$. This would lead to an estimate $s \leq \sqrt{p} - cp^{1/4}$. This seems to be the limit of this method. In order to get an improved lower bound on r one can try to prove non-trivial upper bounds on the third moment $\sum_{t \in \mathbb{Z}_p} \varphi^3(t)$. To do this, we would need that the distribution of numbers $\frac{b_1 - b_2}{b_1 - b_3}$ is approximately uniform on Q as b_1, b_2, b_3 ranges over B . This is plausible because if $s \approx \sqrt{p}$ then the distribution of $B - B$ must be close to uniform on NQ . However, we could not prove anything rigorous in this direction.

Remark 2.5. Theorem 2.1 gives the bound $s \leq [\sqrt{p}] - 1$ for about three quarters of the primes $p = 4k + 1$. Indeed, part (ii) gives this bound for almost all p such that $n = [\sqrt{p}]$ is odd, with the only exception when $p = (n + 1)^2 - 3$. Part (i) gives the improved bound $s \leq n - 1$ if $n^2 + n - 1 > p$. This happens for about half of the primes $p = 4k + 1$ for which n is even. To make these statements rigorous we note that $\sqrt{p}/2$ is uniformly distributed modulo one, when p ranges over primes of the form $p = 4k + 1$: this is a special case of a result of Balog, [1, Theorem 1].

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