

SPECTRAL BOUNDS FOR THE INDEPENDENCE RATIO AND THE CHROMATIC NUMBER OF AN OPERATOR

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ABSTRACT. We define the independence ratio and the chromatic number for bounded, self-adjoint operators on an L^2 -space by extending the definitions for the adjacency matrix of finite graphs. In analogy to the Hoffman bounds for finite graphs, we give bounds for these parameters in terms of the numerical range of the operator. This provides a theoretical framework in which many packing and coloring problems for finite and infinite graphs can be conveniently studied with the help of harmonic analysis and convex optimization. The theory is applied to infinite geometric graphs on Euclidean space and on the unit sphere.

1. INTRODUCTION

The independence number α and the chromatic number χ are invariants of finite graphs which are computationally difficult to determine in general. A by now classical result due to Hoffman [19, (1.6), (4.2)] gives a relatively easy way to provide upper and lower bounds in terms of the graph's spectrum. Hoffman gave a bound for the chromatic number of a graph G having at least one edge in terms of the smallest eigenvalue $m(A)$ and largest eigenvalue $M(A)$ of the adjacency matrix A of G :

$$(1) \quad \chi(G) \geq \frac{M(A) - m(A)}{-m(A)}.$$

From the proof of this inequality it is clear that the argument also works if one replaces the adjacency matrix with any symmetric matrix whose support is contained in the support of A . One can therefore maximize the bound by adjusting the entries in the support. This maximum exists and is the Lovász ϑ -number of the complement of G [24]. As an application of the ϑ -number, Lovász gave the following spectral bound for the independence ratio $\bar{\alpha}$ of a regular graph on n vertices [24, Theorem 9]:

$$(2) \quad \bar{\alpha}(G) = \frac{\alpha(G)}{n} \leq \frac{-m(A)}{M(A) - m(A)}.$$

Since its first appearance in 1979 the ϑ -number became a fundamental tool in combinatorial optimization. Lovász [24] gave a proof (from the book, see Aigner and

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Ziegler [1]) that the Shannon capacity of the pentagon equals $\sqrt{5}$. McEliece, Rodemich, and Rumsey [26] and independently Schrijver [29] showed that a strengthening of the ϑ -number gives Delsarte's linear programming bound in coding theory [9]. Grötschel, Lovász, and Schrijver [17] used the ϑ -number and the related ϑ -body to characterize perfect graphs and to give polynomial time algorithms to solve the independent set problem and the coloring problem for perfect graphs. Recently, Bachoc, Pêcher, and Thiéry [5] extended these results to the circular chromatic number and circular perfect graphs. Gouveia, Parrilo, and Thomas [15] extended the theory of ϑ -bodies from graphs to polynomial ideals. Computing the ϑ -number of a graph is a basic subroutine in the design of many approximation algorithms, see e.g. Karger, Motwani, and Sudan [20], Kleinberg, Goemans [21], Alon, Makarychev, Makarychev, and Naor [2], and Briët, Oliveira, and Vallentin [8]. Bachoc, Nebe, Oliveira, and Vallentin [6] generalized the ϑ -number from finite graphs to infinite graphs whose vertex sets are compact metric spaces. Recent striking applications in extremal combinatorics are the proof of an Erdős-Ko-Rado-type theorem for permutations by Ellis, Friedgut, and Pilpel [11] and the proof of the Simonovits-Sós conjecture of triangle intersecting families of graphs by Ellis, Friedgut, and Filmus [10].

By looking at the adjacency matrix of a finite graph $G = (V, E)$ as an operator on $L^2(V)$, we extend the definitions of independence number and chromatic number from finite graphs to bounded, self-adjoint operators on $L^2(V)$ where V is now a measure space. In this paper we develop a theory for bounding these parameters in terms of the spectrum of the operator, thereby extending Hoffman's and Lovász' results from finite to infinite graphs.

The main body of the theory is presented in Section 2. There are several recent results which can be conveniently interpreted as examples of our theory: In Section 3 we compute spectral bounds for graphs defined on the Euclidean space which are invariant under translations. Thereby we can recover results from Oliveira and Vallentin [27], Kolountzakis [23], and Steinhardt [32]. In Section 4 we determine spectral bounds for distance graphs defined on the unit sphere and generalize results from Bachoc, Nebe, Oliveira, and Vallentin [6].

2. SPECTRAL BOUNDS FOR BOUNDED, SELF-ADJOINT OPERATORS

2.1. Some graph theory. Let $G = (V, E)$ be a finite, undirected graph on n vertices, with vertex set V and edge set $E \subseteq \binom{V}{2}$. The *adjacency matrix* of G is the symmetric matrix $A \in \mathbb{R}^{V \times V}$ defined by

$$A(v, w) = \begin{cases} 1 & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of A are real; we order them decreasingly

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A).$$

The smallest and largest eigenvalues of A are most important for us, so we set $m(A) = \lambda_n(A)$, and $M(A) = \lambda_1(A)$.

An *independent set* of G is a subset of the vertex set in which no two vertices are adjacent. The *independence number* $\alpha(G)$ of G is the cardinality of a largest independent set. The *chromatic number* $\chi(G)$ of G is the smallest number k so that one can partition the vertex set V into k independent sets. These independent sets are often called *color classes*.

An observation that will be key to our generalization in the next section is the following: A set $I \subseteq V$ is independent if and only if

$$(3) \quad \sum_{v \in V} \sum_{w \in V} A(v, w) f(v) f(w) = 0$$

holds for all $f \in \mathbb{R}^V$ supported on I , i.e., for all f such that $f(v) = 0$ whenever $v \notin I$.

Hoffman [19] gave the following spectral bound for χ :

$$\chi(G) \geq \frac{M(A) - m(A)}{-m(A)}.$$

A possibly stronger bound is given by the Lovász ϑ -number of the complement of the graph. One of many possible definitions (see Knuth [22] for a survey) of the ϑ -number of a graph G , denoted by $\vartheta(G)$, is as the optimal value of the following semidefinite program:

$$(4) \quad \begin{aligned} \max \quad & \sum_{v, w \in V} K(v, w) \\ & \sum_{v \in V} K(v, v) = 1, \\ & K(v, w) = 0 \quad \text{whenever } \{v, w\} \in E, \\ & K \in \mathbb{R}^{V \times V} \text{ is positive semidefinite.} \end{aligned}$$

Lovász [24] proved the sandwich theorem

$$\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G}),$$

where \bar{G} is the complement of G . Sometimes, $\chi(\bar{G})$ is called the *clique cover number* of G . He also pointed out the relation of the ϑ -number to Hoffman's bound for the chromatic number, namely that

$$\chi(G) \geq \vartheta(\bar{G}) \geq \frac{M(A) - m(A)}{-m(A)},$$

by proving that $\vartheta(\bar{G})$ is the maximum of

$$\frac{M(B) - m(B)}{-m(B)},$$

where B ranges over all symmetric matrices in $\mathbb{R}^{V \times V}$ such that $B(v, w) = 0$ whenever $\{v, w\} \notin E$.

Lovász [24, Theorem 9] gave the following spectral bound for the independence number of an r -regular graph:

$$\alpha(G) \leq \vartheta(G) \leq n \frac{-m(A)}{M(A) - m(A)},$$

where the second inequality is an equality in the case of edge-transitive graphs. Notice that r -regularity of a graph is equivalent to the property that the all-one vector $1_V = (1, \dots, 1)^T$ is an eigenvector of the adjacency matrix with eigenvalue r . In this case r is the largest eigenvalue.

The ϑ -number is actually a lower bound for the *fractional chromatic number* $\chi^*(\bar{G})$ of the graph \bar{G} , also called the *fractional clique covering number* of G . This is the minimal value of the sum $\lambda_1 + \dots + \lambda_k$ so that $\lambda_1, \dots, \lambda_k$ are nonnegative numbers such that there are independent sets C_1, \dots, C_k in \bar{G} with

$$\lambda_1 1_{C_1} + \dots + \lambda_k 1_{C_k} = 1_V,$$

where $1_S \in \mathbb{R}^V$ denotes the characteristic vector of the set $S \subseteq V$. We have $\chi(\overline{G}) \geq \chi^*(\overline{G}) \geq \vartheta(G)$.

Schrijver [29] introduced a variant of the ϑ -number called the ϑ' -number by adding to (4) the constraint that K has only nonnegative entries. Galtman [14] noticed that $\vartheta'(\overline{G})$ is related to Hoffman's bound, being the maximum of

$$\frac{M(B) - m(B)}{-m(B)},$$

where B ranges over all symmetric *nonnegative* matrices in $\mathbb{R}^{V \times V}$ such that $B(v, w) = 0$ whenever $\{v, w\} \notin E$. The proof of this result uses Lovász's original argument together with Perron-Frobenius theory. One way to interpret this result is that ϑ' provides the best spectral bound for all weighted adjacency matrices where the weights are given by a probability distribution on the edge set of G .

2.2. Some Hilbert space theory. We recall some definitions and facts from Hilbert space theory. For background we refer the reader to the Hilbert space problem book [18] by Halmos.

Let (V, Σ, μ) be a measure space consisting of a set V , a σ -algebra Σ on V , and a measure μ . We consider the Hilbert space

$$L^2(V) = \left\{ f: V \rightarrow \mathbb{C} : f \text{ measurable, } \int_V |f|^2 d\mu < \infty \right\}$$

of complex-valued square-integrable functions where we identify two functions which are equal μ -almost everywhere. The inner product is defined by

$$(f, g) = \int_V f(x)\overline{g(x)} d\mu(x).$$

A linear operator $A: L^2(V) \rightarrow L^2(V)$ is *bounded* if there is a nonnegative real number M so that for all $f \in L^2(V)$ the inequality

$$\|Af\| \leq M\|f\|$$

holds. The infimum of the numbers M having this property is the *norm* of the operator A , denoted by $\|A\|$.

The operator A is *self-adjoint* if we have

$$(Af, g) = (f, Ag)$$

for all $f, g \in L^2(V)$. The *numerical range* of A is defined as

$$W(A) = \{(Af, f) : \|f\| = 1\}.$$

If A is self-adjoint, then $(Af, f) = (f, Af) = \overline{(Af, f)}$ for each $f \in L^2(V)$, which implies that the numerical range of A is a subset of \mathbb{R} . In this case, the numerical range is always an interval. Indeed, given $f, g \in L^2(V)$ with $\|f\| = \|g\| = 1$ and $g \neq -f$, define for each $t \in [0, 1]$ the element

$$h_t = \frac{tf + (1-t)g}{\|tf + (1-t)g\|}$$

of $L^2(V)$. Then $t \mapsto (Ah_t, h_t)$ is a continuous function $[0, 1] \rightarrow \mathbb{R}$ mapping 0 to (Af, f) and 1 to (Ag, g) .

If moreover A is bounded, then $W(A)$ is contained in the interval $[-\|A\|, \|A\|]$, by the Cauchy-Schwarz inequality. In this case, we denote the endpoints of $W(A)$

by $m(A) = \inf\{(Af, f) : \|f\| = 1\}$ and $M(A) = \sup\{(Af, f) : \|f\| = 1\}$. The two endpoints $m(A)$ and $M(A)$ may or may not belong to $W(A)$.

2.3. The independence ratio of an operator. In (3) we saw how to characterize the independent sets of a finite graph in terms of the adjacency matrix. This motivates the following definition.

Definition 2.1. *Let $A: L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator. A measurable set $I \subseteq V$ is called an independent set of A if $(Af, f) = 0$ for each $f \in L^2(V)$ which vanishes almost everywhere outside of I .*

Notice that sets of μ -measure zero are always independent. In a similar vein, if I is a measurable set and N has measure zero, then I is independent if and only if $I \cup N$ is also independent. This says that we may speak of a measurable set being independent even if we only know it up to nullsets.

In order to measure the size of an independent set it is convenient to assume that the measure μ is a probability measure, although we will extend these ideas to the Lebesgue upper density of subsets of \mathbb{R}^n in Section 3.

The *independence ratio* of a bounded, self-adjoint operator $A: L^2(V) \rightarrow L^2(V)$ is defined as

$$\bar{\alpha}(A) = \sup\{\mu(I) : I \text{ independent set of } A\}.$$

In the case of a finite graph $G = (V, E)$ with adjacency matrix A , the independence ratio of A is equal to $\alpha(G)/|V|$.

Let 1_V denote the all-one function in $L^2(V)$. The following theorem can be used to upper bound the independence ratio of an operator.

Theorem 2.2. *Let (V, μ) be a probability space and let $A: L^2(V) \rightarrow L^2(V)$ be a nonzero, bounded, self-adjoint operator. Fix a real number R and set $\varepsilon = \|A1_V - R1_V\|$. Suppose there exists a set $I \subseteq V$ with $\mu(I) > 0$ which is independent for A . Then, if $R - m(A) - \varepsilon > 0$, we have*

$$\bar{\alpha}(A) \leq \frac{-m(A) + 2\varepsilon}{R - m(A) - \varepsilon}.$$

When A is the adjacency matrix of a finite regular graph (here μ is the uniform probability measure on V), then 1_V is the eigenvector corresponding to the maximum eigenvalue $M(A)$. Then, taking $R = M(A)$, we have $\varepsilon = 0$, and we recover (2) from the theorem.

Proof. Let $I \subseteq V$ be an independent set with $\mu(I) > 0$ and let 1_I denote its characteristic function. Decompose 1_I orthogonally as $\beta 1_V + g$, with β a scalar and g orthogonal to 1_V . Notice the identities

$$\mu(I) = (1_I, 1_V) = \beta$$

and

$$\mu(I) = (1_I, 1_I) = \beta^2 + \|g\|^2.$$

By independence and self-adjointness one therefore has

$$\begin{aligned} 0 &= (A1_I, 1_I) = (A(\beta 1_V + g), \beta 1_V + g) \\ &= \beta^2 (A1_V, 1_V) + \beta (A1_V, g) + \beta (g, A1_V) + (Ag, g). \end{aligned}$$

Let $\eta = A1_V - R1_V$. We consider the first three summands:

$$(A1_V, 1_V) = (R1_V + \eta, 1_V) = R + (\eta, 1_V),$$

and

$$\begin{aligned} (A1_V, g) &= (R1_V + \eta, 1_I - \beta 1_V) = R(1_V, 1_I) - R\beta(1_V, 1_V) + (\eta, 1_I) - \beta(\eta, 1_V) \\ &= (\eta, 1_I) - \beta(\eta, 1_V), \end{aligned}$$

and similarly

$$(g, A1_V) = (1_I, \eta) - \beta(1_V, \eta) = (1_I, \eta) - \beta(\eta, 1_V).$$

Putting it together, using Cauchy-Schwarz, the inequality $\beta \leq 1$, and $(g, g) = \beta - \beta^2$, yields

$$\begin{aligned} 0 &= \beta^2(A1_V, 1_V) + \beta(A1_V, g) + \beta(g, A1_V) + (Ag, g) \\ &= \beta^2(R + (\eta, 1_V)) + \beta((\eta, 1_I) + (1_I, \eta) - 2\beta(\eta, 1_V)) + (Ag, g) \\ &= \beta^2 R - \beta^2(\eta, 1_V) + \beta((\eta, 1_I) + (1_I, \eta)) + (Ag, g) \\ &\geq \beta^2 R - \beta^2 \varepsilon - 2\beta \varepsilon + m(A)(\beta - \beta^2). \end{aligned}$$

Now we divide by β and rearrange the terms and get the desired result

$$\frac{-m(A) + 2\varepsilon}{R - m(A) - \varepsilon} \geq \beta. \quad \square$$

Notice that the hypothesis $\mu(I) > 0$ is really needed; if A is the operator $L^2(V) \rightarrow L^2(V)$ defined by $A1_V = 1_V$ and $Ag = \frac{1}{2}g$ for all g orthogonal to 1_V , then clearly the empty set is independent for A , but $-m(A)/(M(A) - m(A)) = -1$.

2.4. The chromatic number of an operator. Let $A: L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator. The *chromatic number* of A , denoted by $\chi(A)$, equals the smallest number k such that one can partition V into k independent sets.

We can lower bound the chromatic number of A when we know the two endpoints $m(A)$ and $M(A)$ of the numerical range $W(A)$. This is completely analogous to Hoffman's bound (1) for finite graphs when A is the adjacency matrix of the graph.

Theorem 2.3. *Let $A: L^2(V) \rightarrow L^2(V)$ be a nonzero, bounded, self-adjoint operator. If $\chi(A) < \infty$, then*

$$\chi(A) \geq \frac{M(A) - m(A)}{-m(A)}.$$

Proof. Let C_1, \dots, C_k be a partition of V into independent sets. Recall that the union of an independent set with a nullset is also independent. Hence we may assume that $\mu(C_i) > 0$ for all $i = 1, \dots, k$.

We decompose the Hilbert space $L^2(V)$ into an orthogonal direct sum

$$L^2(V) = \bigoplus_{i=1}^k L^2(C_i).$$

Fix $\varepsilon > 0$. Let $f \in L^2(V)$ be such that $\|f\| = 1$ and such that $(Af, f) \geq M(A) - \varepsilon$. Decompose f as

$$f = \sum_{i=1}^k \alpha_i f_i, \quad \text{with } f_i \in L^2(C_i) \text{ and } \|f_i\| = 1,$$

and consider the k -dimensional Euclidean space $U \subseteq L^2(V)$ with orthonormal basis f_1, \dots, f_k . By $P: L^2(V) \rightarrow U$ we denote the orthogonal projection of $L^2(V)$ onto U .

We will consider the finite-dimensional self-adjoint operator $B: U \rightarrow U$ defined by $B = PA$. The numerical range of B , in this case the interval between the smallest and largest eigenvalues, lies in $[m(A), M(A)]$ because for a unit vector $u \in U$ we have

$$(Bu, u) = (PAu, u) = (Au, u) \in [m(A), M(A)].$$

Furthermore, the largest eigenvalue $\lambda_1(B)$ of B is at least $M(A) - \varepsilon$ because

$$\lambda_1(B) \geq (Bf, f) = (Af, f) \geq M(A) - \varepsilon.$$

The trace of B equals zero because

$$\text{trace } B = \sum_{i=1}^k (Bf_i, f_i) = \sum_{i=1}^k (Af_i, f_i) = 0,$$

where $(Af_i, f_i) = 0$ since $f_i \in L^2(C_i)$. Now for the sum of the eigenvalues of B the following holds:

$$M(A) - \varepsilon + (k-1)m(A) \leq \sum_{i=1}^k \lambda_i(B) = \text{trace } B = 0,$$

and hence

$$k \geq 1 - \frac{M(A) - \varepsilon}{m(A)}.$$

The last inequality holds for all $\varepsilon > 0$ and so the theorem follows. \square

From the proof it follows that if $A \neq 0$ and if $\chi(A) < \infty$, then $m(A) < 0$ and $M(A) > 0$, because $[m(B), M(B)] \subseteq [m(A), M(A)]$ and $m(B) < 0$ and $M(B) > 0$ since $\text{trace } B = 0$ and $B \neq 0$.

Finally, we remark that the above proof is very close to the proof of Hoffman's bound in the book [7, Chapter VIII.2, Theorem 7, page 265] by Bollobás. In fact, we only had to include the epsilon.

2.5. The fractional chromatic number of an operator. When (V, μ) is a probability space we can give a bound for the fractional chromatic number of an operator. Let $A: L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator. The *fractional chromatic number* of A , denoted by $\chi^*(A)$, is the infimum over all sums $\lambda_1 + \dots + \lambda_k$ of nonnegative numbers $\lambda_1, \dots, \lambda_k$ such that there are independent sets C_1, \dots, C_k of A with

$$\lambda_1 1_{C_1} + \dots + \lambda_k 1_{C_k} = 1_V.$$

Theorem 2.4. *Let (V, μ) be a probability space and let $A: L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator which is not zero. If $\chi^*(A) < \infty$, then*

$$\chi^*(A) \geq \frac{(A1_V, 1_V) - m(A)}{-m(A)}.$$

Proof. Consider a fractional coloring $\lambda_1 1_{C_1} + \dots + \lambda_k 1_{C_k} = 1_V$ with $\gamma = \lambda_1 + \dots + \lambda_k$. Then,

$$\begin{aligned}
0 &\leq \sum_{i=1}^k \lambda_i ((A - m(A)I)(\gamma 1_{C_i} - 1_V), \gamma 1_{C_i} - 1_V) \\
&= \gamma^2 \sum_{i=1}^k \lambda_i ((A - m(A)I)1_{C_i}, 1_{C_i}) - 2\gamma \sum_{i=1}^k \lambda_i ((A - m(A)I)1_{C_i}, 1_V) \\
&\quad + \sum_{i=1}^k \lambda_i ((A - m(A)I)1_V, 1_V) \\
&= -\gamma^2 m(A) - \gamma((A - m(A)I)1_V, 1_V) \\
&= -\gamma^2 m(A) - \gamma((A1_V, 1_V) - m(A)),
\end{aligned}$$

and so

$$\gamma \geq \frac{(A1_V, 1_V) - m(A)}{-m(A)}. \quad \square$$

The above proof is very close to the proof of the sandwich theorem given in Schrijver [30, Theorem 67.1].

2.6. Relation to the ϑ -number. In Section 2.1 we quickly discussed the relation between Hoffman's bound and the Lovász ϑ -number. We now attempt to develop an analogous theory that relates the spectral bounds we presented for the chromatic number and independence ratio of operators with suitable generalizations of the Lovász ϑ -number for some classes of infinite graphs. This theory is based on transferring arguments, mainly due Lovász, from the finite into our infinite setting. So in a sense, our contribution here is to come up with appropriate definitions.

In what follows we shall work with measurable graphs. Let V be a topological space and μ be a Borel probability measure on V . A graph $G = (V, E)$ is *measurable* if E is measurable as a subset of the product space $V \times V$.

The *independence ratio* of G is defined as

$$\bar{\alpha}(G) = \sup\{\mu(I) : I \text{ a measurable independent set of } G\}.$$

The *measurable chromatic number* of G , denoted by $\chi_m(G)$, is the minimum k such that V can be partitioned into k measurable independent sets.

We say that an operator $A: L^2(V) \rightarrow L^2(V)$ *respects* G if measurable independent sets of G are also independent sets of A . Notice that in this case

$$\bar{\alpha}(G) \leq \bar{\alpha}(A) \quad \text{and} \quad \chi(A) \leq \chi_m(G).$$

So from any operator that respects G we may obtain bounds for the independence ratio and the measurable chromatic number of G . In fact, as long as $\bar{\alpha}(G) > 0$, we get from Theorem 2.2 that

$$(5) \quad \bar{\alpha}(G) \leq \inf_A \frac{-m(A)}{M_1(A) - m(A)},$$

where the infimum is taken over all nonzero, bounded, and self-adjoint operators $A: L^2(V) \rightarrow L^2(V)$ that respect G and are such that $A1_V = M_1(A)1_V$ and $M_1(A) - m(A) > 0$. (Here, we take $R = (A1_V, 1_V)$ and hence $\varepsilon = 0$ in the statement of the theorem.)

Similarly, from Theorem 2.3 we see that, if $\chi_m(G) < \infty$, then

$$(6) \quad \chi_m(G) \geq \sup_A \frac{M(A) - m(A)}{-m(A)},$$

where the supremum is taken over all nonzero, bounded, and self-adjoint operators $A: L^2(V) \rightarrow L^2(V)$ that respect G .

For some vertex-transitive measurable graphs we will see that both the infimum in (5) and the supremum in (6) correspond to natural generalizations of the Lovász ϑ -number to measurable graphs. We will also show that the product of the infimum and the supremum in this case equals 1, showing that this property of the ϑ -number of finite graphs carries over to this setting.

An *automorphism* of G is a measure preserving bijection $\varphi: V \rightarrow V$ (i.e., both φ and its inverse are measure preserving) that preserves the adjacency relation, that is, $\{\varphi(v), \varphi(w)\} \in E$ if and only if $\{v, w\} \in E$.

We say that G is *vertex-transitive* if there is a subgroup \mathcal{T} of the automorphism group of G that is a topological group, acts continuously on V (i.e., the “action map” $(T, v) \mapsto T \cdot v$ is a continuous function), and acts transitively on V . We call any such group \mathcal{T} a *transitivity group* of G .

In what follows, if $A: L^2(V) \rightarrow L^2(V)$ is a bounded, self-adjoint operator, we write $A \succeq 0$ if A is a *positive operator*, that is, if $(Af, f) \geq 0$ for all $f \in L^2(V)$, or equivalently, if the numerical range of A is nonnegative. We will denote by I the identity operator and by $J: L^2(V) \rightarrow L^2(V)$ the Hilbert-Schmidt operator such that

$$Jf = (f, 1_V)1_V$$

for all $f \in L^2(V)$.

Theorem 2.5. *Let $G = (V, E)$ be a measurable graph with positive independence ratio. The infimum in (5) is at least*

$$(7) \quad \inf \lambda \quad \begin{array}{l} \lambda I + Z - J \succeq 0, \\ Z: L^2(V) \rightarrow L^2(V) \text{ is a bounded, self-adjoint} \\ \text{operator that respects } G, \end{array}$$

with equality when G is vertex-transitive with a compact transitivity group and the infimum above is < 1 .

When G is a finite graph, problem (7) is one of the formulations for $\vartheta(G)$. Namely, it is the dual of the semidefinite programming problem (4).

To prove the second part of the theorem, namely that when G has a compact transitivity group, then we have equality, we will need to symmetrize an operator with respect to the transitivity group. This operation will be used again later, so we present it now.

Suppose the measurable graph $G = (V, E)$ has a compact transitivity group. Let \mathcal{T} be such a group, and denote by ν the Haar measure on \mathcal{T} , normalized such that $\nu(\mathcal{T}) = 1$. For $T \in \mathcal{T}$, denote the right action on $L^2(V)$ by

$$(f^T)(x) = f(T \cdot x) \quad \text{with } f \in L^2(V).$$

If $A: L^2(V) \rightarrow L^2(V)$ is a bounded operator, we denote by $\mathcal{R}_{\mathcal{T}}(A)$ the operator such that

$$(\mathcal{R}_{\mathcal{T}}(A)f)(x) = \int_{\mathcal{T}} (Af^T)^{T^{-1}}(x) d\nu(T).$$

Note $\mathcal{R}_{\mathcal{T}}(A)$ is a bounded operator, and that $\mathcal{R}_{\mathcal{T}}(A)$ is self-adjoint if A is self-adjoint.

Two properties we use of the symmetrized operator are the following. First, if A respects G , then also does $\mathcal{R}_{\mathcal{T}}(A)$, as can be easily seen from the fact that \mathcal{T} is a subgroup of the automorphism group of G . Second, $\mathcal{R}_{\mathcal{T}}(A)$ always has 1_V as an eigenfunction. Indeed, if $x, y \in V$, then for some $U \in \mathcal{T}$ we have $x = U \cdot y$. Then, using the invariance of the Haar measure and using $1_V^U = 1_V$,

$$\begin{aligned} (\mathcal{R}_{\mathcal{T}}(A)1_V)(x) &= (\mathcal{R}_{\mathcal{T}}(A)1_V)(U \cdot y) \\ &= \int_{\mathcal{T}} (A1_V^T)(T^{-1}U \cdot y) d\nu(T) \\ &= \int_{\mathcal{T}} (A(1_V^U)^T)(T^{-1} \cdot y) d\nu(T) \\ &= \int_{\mathcal{T}} (A1_V^T)(T^{-1} \cdot y) d\nu(T) \\ &= (\mathcal{R}_{\mathcal{T}}(A)1_V)(y), \end{aligned}$$

as we wanted.

Proof of Theorem 2.5. Let $A: L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator that respects G . Assume moreover $A1_V = M_1(A)1_V$ and $M_1(A) - m(A) > 0$. Since

$$0 < \bar{\alpha}(G) \leq \bar{\alpha}(A) \leq \frac{-m(A)}{M_1(A) - m(A)},$$

we see that $m(A) < 0$. So we may assume that $m(A) = -1$.

We claim that $\lambda = (M_1(A) - m(A))^{-1}$ and $Z = \lambda A$ form a feasible solution of problem (7). Obviously, Z respects G . We show that $\lambda I + Z - J \succeq 0$.

For this, write $X = \lambda I + Z$. Since $m(A) = -1$, we have $X \succeq 0$. Now take $f \in L^2(V)$ and say $f = \beta 1_V + w$ for some w orthogonal to 1_V . Then

$$\begin{aligned} ((\lambda I + Z - J)f, f) &= (X(\beta 1_V + w), \beta 1_V + w) - (J(\beta 1_V + w), \beta 1_V + w) \\ &= \beta^2 (X1_V, 1_V) + 2\beta (X1_V, w) + (Xw, w) \\ &\quad - (\beta^2 (J1_V, 1_V) + 2\beta (J1_V, w) + (Jw, w)) \\ &= \beta^2 + (Xw, w) - \beta^2 \\ &\geq 0. \end{aligned}$$

Here, we used the fact that 1_V is an eigenfunction of X with eigenvalue 1 and that $(1_V, w) = 0$. We also used that $X \succeq 0$, since then $(Xw, w) \geq 0$.

So we have that the optimal value of (7) is at most $\lambda = -m(A)/(M_1(A) - m(A))$ since $m(A) = -1$, proving the inequality we wanted.

Now we show that equality holds when G is vertex-transitive. We start by observing that, in general, any feasible solution of (7) gives an upper bound to $\bar{\alpha}(G)$. Indeed, let I be a measurable independent set of G with $\mu(I) > 0$. Then

$$0 \leq ((\lambda I + Z - J)1_I, 1_I) = \lambda\mu(I) - \mu(I)^2,$$

and dividing by $\mu(I)$ we obtain $\mu(I) \leq \lambda$.

So let $\lambda < 1$ and Z be a feasible solution of (7). From the above observation, we have $\lambda > 0$. Since $\lambda < 1$, we know that Z is nonzero. We may also assume that 1_V is an eigenfunction of Z , as otherwise λ and $\mathcal{R}_{\mathcal{T}}(Z)$, where \mathcal{T} is a compact transitivity group of G , are also feasible for (7) and $\mathcal{R}_{\mathcal{T}}(Z)$ has 1_V as an eigenfunction.

Then Z is a nonzero, bounded, self-adjoint operator that respects G . If $Z1_V = M_1(Z)1_V$, then we have

$$M_1(Z) = (Z1_V, 1_V) \geq (J1_V, 1_V) - \lambda(I1_V, 1_V) = 1 - \lambda.$$

So we see that $M_1(Z) > 0$, as $\lambda < 1$.

We also have $m(Z) \geq -\lambda$. Indeed, let $f \in L^2(V)$ and write $f = \alpha 1_V + w$, where w is orthogonal to 1_V . Then, since 1_V is an eigenfunction of Z , we get

$$(Zf, f) = (Z(\alpha 1_V + w), \alpha 1_V + w) = \alpha^2(Z1_V, 1_V) + (Zw, w) \geq (Zw, w),$$

and we get from $\lambda I + Z - J \succeq 0$ that

$$(Zf, f) \geq (Zw, w) \geq -\lambda(w, w) \geq -\lambda.$$

Notice that we must have $m(Z) < 0$. Indeed, if $m(Z) > 0$, then $(Z1_I, 1_I) > 0$ for any measurable independent set I of G with $\mu(I) > 0$, a contradiction since Z respects G . If $m(Z) = 0$, then $M_1(Z) - m(Z) = M_1(Z) > 0$, and we have by (5) that $\bar{\alpha}(G) \leq 0$, a contradiction since we assume G has positive independence ratio.

Hence, since $M_1(Z) \geq 1 - \lambda$ and $0 < -m(Z) \leq \lambda$ we have

$$\frac{M_1(Z) - m(Z)}{-m(Z)} = 1 + \frac{M_1(Z)}{-m(Z)} \geq \frac{1}{\lambda},$$

showing that the infimum in (5) is at most λ . \square

Given $a \in L^\infty(V)$, we denote by $D_a: L^2(V) \rightarrow L^2(V)$ the multiplication operator $(D_a f)(x) = a(x)f(x)$ for $f \in L^2(V)$. Note D_a is a bounded operator. The following theorem connects (6) with the Lovász ϑ -number.

Theorem 2.6. *Let $G = (V, E)$ be a measurable graph with finite measurable chromatic number. The supremum in (6) is at most*

$$(8) \quad \begin{aligned} & \sup \quad ((D_a + K)1_V, 1_V) \\ & \quad D_a + K \succeq 0, \\ & \quad a \in L^\infty(V) \text{ is nonnegative and } \int_V a(x) d\mu(x) = 1, \\ & \quad K: L^2(V) \rightarrow L^2(V) \text{ is a bounded, self-adjoint} \\ & \quad \quad \text{operator that respects } G, \end{aligned}$$

with equality when G is vertex-transitive with a compact transitivity group and the supremum above is > 1 .

Again, when G is a finite graph, problem (8) corresponds to the semidefinite programming problem (4) applied to the complement of G . So for a finite graph G the optimization problem above gives us $\vartheta(\bar{G})$.

Proof. Let $A: L^2(V) \rightarrow L^2(V)$ be a nonzero, bounded, self-adjoint operator that respects G . Recall from Section 2.4 that $M(A) > 0 > m(A)$. So we may assume that $m(A) = -1$.

Fix $\varepsilon > 0$ and let $f \in L^\infty(V)$ be such that $(Af, f) > M(A) - \varepsilon$ and $\|f\| = 1$ (such an f exists, because $L^\infty(V)$ is dense in $L^2(V)$ and A is bounded). Let $a(x) = |f(x)|^2$ and set $K = D_f^* A D_f$. Notice K is a bounded and self-adjoint operator and $\int_V a(x) d\mu(x) = 1$. Also, by construction K respects G and $D_a + K \succeq 0$, since $D_a + K = D_f^*(I + A)D_f$ and $I + A \succeq 0$ as $m(A) = -1$.

So a and K form a feasible solution of (8). Moreover

$$\begin{aligned} ((D_a + K)1_V, 1_V) &= (D_a 1_V, 1_V) + (D_f^* A D_f 1_V, 1_V) \\ &= 1 + (A D_f 1_V, D_f 1_V) \\ &= 1 + (A f, f) \\ &\geq 1 + M(A) - \varepsilon, \end{aligned}$$

and taking $\varepsilon \rightarrow 0$ we obtain that the optimal of (8) is at least

$$1 + M(A) = \frac{M(A) - m(A)}{-m(A)},$$

as we wanted.

Now we show that when G is vertex-transitive, then equality holds, as long as the optimal value of (8) is > 1 . Indeed, let a and K be a feasible solution of (8) with K nonzero. Such solution must exist, since the optimal value of our problem is greater than 1.

Then K is a nonzero, bounded, self-adjoint operator that respects G . We may assume that $a = 1_V$, so that $D_a = I$. Indeed, let \mathcal{T} be a compact transitivity group of G . Then $\bar{a} = 1_V$ and $\bar{K} = \mathcal{R}_{\mathcal{T}}(K)$ form a feasible solution of (8) (note $\mathcal{R}_{\mathcal{T}}(D_a) = I$, since G is vertex transitive). Also, $((D_{\bar{a}} + \bar{K})1_V, 1_V) = ((D_a + K)1_V, 1_V)$, so we may take the symmetrized solution instead of the original one.

Then, since $a = 1_V$, we have $(D_a f, f) \leq 1$ for all $f \in L^2(V)$. This, together with $D_a + K \succeq 0$, implies that $m(K) \geq -1$. Then since $M(K) \geq (K 1_V, 1_V)$, we have

$$\frac{M(K) - m(K)}{-m(K)} \geq 1 + M(K) \geq 1 + (K 1_V, 1_V) = ((D_a + K)1_V, 1_V),$$

proving the equality. \square

When $G = (V, E)$ is a finite graph, we have $\vartheta(G)\vartheta(\bar{G}) \geq |V|$, with equality when G is vertex-transitive. We finish this section by observing that the same holds for the optimal values of (7) and (8), showing that this property carries on to our setting.

Theorem 2.7. *Let $G = (V, E)$ be a measurable graph with positive independence ratio and finite measurable chromatic number. Let θ be the optimal value of (7) and $\tilde{\theta}$ be the optimal value of (8). Then $\theta \cdot \tilde{\theta} \geq 1$, with equality when G has a compact transitivity group.*

Proof. Let λ and Z be a feasible solution of (7). Then $\lambda > 0$, since G has a positive independence ratio. Set $a = 1_V$, so that $D_a = I$, and $K = \lambda^{-1}Z$. Then we have that K respects G and that $D_a + K \succeq 0$, so a and K are a feasible solution of (8). Moreover

$$\begin{aligned} ((D_a + K)1_V, 1_V) &= ((I + \lambda^{-1}Z - \lambda^{-1}J)1_V, 1_V) + (\lambda^{-1}J 1_V, 1_V) \\ &\geq \lambda^{-1}(J 1_V, 1_V) \\ &= \lambda^{-1}, \end{aligned}$$

so that $\tilde{\theta} \geq \lambda^{-1}$, and we see that $\theta \cdot \tilde{\theta} \geq 1$.

To see the reverse inequality when G has a compact transitivity group \mathcal{T} , let a and K be a feasible solution of (8). Notice $\bar{a} = 1_V$ and $\bar{K} = \mathcal{R}_{\mathcal{T}}(K)$ are also feasible for (8), and $((D_{\bar{a}} + \bar{K})1_V, 1_V) = ((D_a + K)1_V, 1_V)$.

Notice $D_{\bar{\alpha}} = I$. Set $\beta = ((I + \bar{K})1_V, 1_V)$. We claim $\lambda = \beta^{-1}$ and $Z = \beta^{-1}\bar{K}$ form a feasible solution for (7).

Indeed, notice Z respects G by construction. We show that $\lambda I + Z - J \succeq 0$. For this, write $X = I + \bar{K}$ and let $f \in L^2(V)$. Write $f = \alpha 1_V + w$, where w is orthogonal to f . Then

$$\begin{aligned} ((\lambda I + Z - J)f, f) &= \beta^{-1}(Xf, f) - (Jf, f) \\ &= \beta^{-1}(X(\alpha 1_V + w), \alpha 1_V + w) - (J(\alpha 1_V + w), \alpha 1_V + w) \\ &= \beta^{-1}(\alpha^2(X1_V, 1_V) + 2\alpha(X1_V, w) + (Xw, w)) \\ &\quad - (\alpha^2(J1_V, 1_V) + 2\alpha(J1_V, w) + (Jw, w)) \\ &= \beta^{-1}\alpha^2\beta + (Xw, w) - \alpha^2 \\ &\geq 0, \end{aligned}$$

where we use the fact that $X \succeq 0$, proving the claim.

So we see that $\theta \leq \lambda = \beta^{-1}$, and so $\theta \cdot \theta \leq 1$, as we wanted. \square

3. GRAPHS ON EUCLIDEAN SPACE

In this section we consider translation invariant graphs defined on the Euclidean space $V = \mathbb{R}^n$. Let $N \subseteq \mathbb{R}^n$ be a bounded, Lebesgue measurable set which does not contain the origin in its topological closure. Say two vertices $x, y \in \mathbb{R}^n$ are adjacent whenever $x - y \in N$. To make the adjacency relation symmetric we require that N is centrally symmetric, i.e. $N = -N$. We denote this graph by $G(\mathbb{R}^n, N)$.

Our aim is to determine lower bounds for the measurable chromatic number $\chi_m(G(\mathbb{R}^n, N))$ of this graph. This is the smallest number of colors one needs to paint all points of \mathbb{R}^n so that two points which are adjacent receive different colors and all points having the same color form measurable sets. This number is finite since N is bounded and since every point is in an open set of positive measure which does not contain any of its neighbors in the graph.

In general, finding $\chi_m(G(\mathbb{R}^n, N))$ is a notoriously difficult problem: It has been intensively studied for the unit sphere $N = S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ and there even in the planar case the number is only known to lie between five and seven; see Soifer [31] and Székely [34] for the history of this problem.

The technique we present is a common extension of the technique of Steinhardt [32] who used it to show that

$$\lim_{N \rightarrow \infty} \chi_m \left(G \left(\mathbb{R}^2, \bigcup_{k=0}^N (2k+1)S^1 \right) \right) = \infty,$$

and an extension of the technique of Oliveira and Vallentin [27] who gave an upper bound for the measurable chromatic number of graphs of the form

$$G(\mathbb{R}^n, d_1 S^{n-1} \cup d_2 S^{n-1} \cup \dots \cup d_N S^{n-1}),$$

which also led to the best known lower bounds for the measurable chromatic number of $G(\mathbb{R}^n, S^{n-1})$ in dimensions $3, \dots, 24$. We revisit these two examples in Section 3.3 and Section 3.4.

3.1. Computing lower bounds for measurable chromatic numbers via Fourier analysis. Let ν be a signed Borel measure (i.e. which does not take the values $\pm\infty$) with support contained in N , and which is centrally symmetric, i.e. $\nu(-S) = \nu(S)$ for all measurable sets S . The convolution operator $A_\nu: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ given by

$$(9) \quad (A_\nu f)(x) = (f * \nu)(x) = \int_{\mathbb{R}^n} f(x-y) d\nu(y), \quad \text{for } f \in L^2(\mathbb{R}^n),$$

is a bounded operator, by Minkowski's integral inequality (see e.g. [12, Proposition 8.49]). The fact that ν is centrally symmetric implies that A_ν is self-adjoint:

$$\begin{aligned} (A_\nu f, g) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) d\nu(y) \overline{g(x)} dx = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \overline{g(x+y)} d\nu(y) dx \\ &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \overline{g(x-y)} d\nu(y) dx = (f, A_\nu g). \end{aligned}$$

If I is a measurable independent set of $G(\mathbb{R}^n, N)$ then it is also an independent set of any such convolution operator. In fact, let $f \in L^2(\mathbb{R}^n)$ be a function which vanishes almost everywhere outside of I . For $x \in I$ and $y \in N$ we cannot have $x-y \in I$, and therefore $f(x-y)\overline{f(x)} = 0$ for Lebesgue almost every x . Hence,

$$(A_\nu f, f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \overline{f(x)} d\nu(y) dx = 0.$$

So A_ν respects $G(\mathbb{R}^n, N)$ and $\chi(A_\nu) \leq \chi_m(G(\mathbb{R}^n, N))$. Theorem 2.3 then gives

$$\frac{M(A_\nu) - m(A_\nu)}{-m(A_\nu)} \leq \chi(A_\nu) \leq \chi_m(G(\mathbb{R}^n, N))$$

for every signed Borel measure supported on N which is centrally symmetric.

To determine the numerical range of A_ν we apply the Fourier transform which by Plancherel's theorem is a unitary operator on $L^2(\mathbb{R}^n)$:

$$(A_\nu f, f) = (\widehat{A_\nu f}, \widehat{f}) = (\widehat{f * \nu}, \widehat{f}) = (\widehat{\nu} \widehat{f}, \widehat{f}),$$

where

$$\widehat{\nu}(u) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot u} d\nu(x)$$

is the Fourier transform of the measure ν .

So it suffices to determine the numerical range of the multiplication operator $g \mapsto \widehat{\nu}g$. If $g \in L^2(\mathbb{R}^n)$ with $\|g\| = 1$, then clearly

$$(\widehat{\nu}g, g) = \int |g(x)|^2 \widehat{\nu}(x) dx$$

lies between $\inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)$ and $\sup_{u \in \mathbb{R}^n} \widehat{\nu}(u)$, since $\widehat{\nu}$ is continuous and bounded. If $\epsilon > 0$, then choose $x_0 \in \mathbb{R}^n$ with $\widehat{\nu}(x_0) \leq \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u) + \epsilon$. If $B_r(x_0)$ denotes the ball of radius r centered at x_0 and $\text{vol}(B_r(x_0))$ its volume, then by

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^n} \widehat{\nu}(x) \left| \frac{1_{B_r(x_0)}}{\sqrt{\text{vol}(B_r(x_0))}} \right|^2 dx = \lim_{r \rightarrow 0} \frac{1}{\text{vol}(B_r(x_0))} \int_{B_r(x_0)} \widehat{\nu}(x) dx = \widehat{\nu}(x_0),$$

whence $m(A_\nu) \leq \widehat{\nu}(x_0) \leq \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u) + \epsilon$, and since ϵ was arbitrary we get $m(A_\nu) = \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)$. A similar argument shows $M(A_\nu) = \sup_{u \in \mathbb{R}^n} \widehat{\nu}(u)$.

So we finally get

$$(10) \quad \frac{\sup_{u \in \mathbb{R}^n} \widehat{\nu}(u) - \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)}{-\inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)} \leq \chi_m(G(\mathbb{R}^n, N)).$$

To get the best possible bound from this approach, one can optimize over all measures ν having the required properties.

3.2. Computing upper bounds for upper densities via Fourier analysis.

Independent sets in $G(\mathbb{R}^n, N)$ might have infinite Lebesgue measure. So, while it does not make sense to look for upper bounds for the measure of independent sets, we might look for upper bounds for their upper density, a measure of the fraction of space they cover. Given a measurable set $S \subseteq \mathbb{R}^n$, its *upper density* is

$$\bar{\delta}(S) = \limsup_{r \rightarrow \infty} \frac{\text{vol}(S \cap B_r)}{\text{vol}(B_r)},$$

where B_r is the ball of radius r centered at the origin.

By using the upper density, we may extend the definition of independence ratio also to the graph $G(\mathbb{R}^n, N)$. We simply put

$$\bar{\alpha}(G(\mathbb{R}^n, N)) = \sup\{\bar{\delta}(I) : I \text{ independent set of } G(\mathbb{R}^n, N)\}.$$

Now let ν be a signed Borel measure with support in N that is centrally symmetric. Recall from the previous section that every independent set of $G(\mathbb{R}^n, N)$ is also an independent set of the convolution operator A_ν defined in (9).

For $r > 0$, let B_r be the ball of radius r centered at the origin. We view B_r as a measure space equipped with the normalized Lebesgue measure. Denote by $\langle f, g \rangle$ the inner product in $L^2(B_r)$. Consider the operator $A_\nu^r: L^2(B_r) \rightarrow L^2(B_r)$ given by

$$A_\nu^r f = (A_\nu f)|_{B_r},$$

where $A_\nu f$ is the operator A_ν applied to the extension of f by zeros to all of \mathbb{R}^n .

Then we have that

$$\limsup_{r \rightarrow \infty} \bar{\alpha}(A_\nu^r) \geq \bar{\alpha}(G(\mathbb{R}^n, N)).$$

Using Theorem 2.2 we may upper bound $\bar{\alpha}(A_\nu^r)$ for every r . For a given $r > 0$, we apply the theorem with $R = R(r) = \langle A_\nu^r \mathbf{1}_{B_r}, \mathbf{1}_{B_r} \rangle$ and $\varepsilon = \varepsilon(r) = \|A_\nu^r \mathbf{1}_{B_r} - R(r) \mathbf{1}_{B_r}\|$. We then obtain

$$\frac{-m(A_\nu^r) + 2\varepsilon(r)}{R(r) - m(A_\nu^r) - \varepsilon(r)} \geq \bar{\alpha}(A_\nu^r).$$

We claim:

$$\lim_{r \rightarrow \infty} m(A_\nu^r) = m(A_\nu) = \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u), \quad \lim_{r \rightarrow \infty} R(r) = \widehat{\nu}(0), \quad \text{and} \quad \lim_{r \rightarrow \infty} \varepsilon(r) = 0.$$

Now we prove the first identity. For $f \in L^2(\mathbb{R}^n)$, we write $f^r = f|_{B_r}$. Then,

$$\langle A_\nu^r f^r, f^r \rangle = \frac{1}{\text{vol } B_r} \langle A_\nu f^r, f^r \rangle \geq \frac{1}{\text{vol } B_r} m(A_\nu) \langle f^r, f^r \rangle = m(A_\nu) \langle f^r, f^r \rangle,$$

and so $m(A_\nu^r) \geq m(A_\nu)$.

For the reverse inequality pick $f \in L^2(\mathbb{R}^n)$ of unit norm such that $(A_\nu f, f)$ is close to $m(A_\nu)$. Then, by taking r large enough we can make the norm of f^r as close to one as we want. Notice that by Cauchy-Schwarz

$$m(A_\nu^r) \leq \frac{\langle A_\nu^r f^r, f^r \rangle}{\langle f_r, f_r \rangle} = \frac{\langle A_\nu f^r, f^r \rangle}{\langle f_r, f_r \rangle} \xrightarrow{r \rightarrow \infty} (A_\nu f, f).$$

Now we prove the second identity. Since ν is supported on a bounded set, we have

$$(A_\nu^r 1_{B_r})(x) = \int_{\mathbb{R}^n} 1_{B_r}(x-y) d\nu(y) \xrightarrow{r \rightarrow \infty} \widehat{\nu}(0),$$

pointwise in x . Then

$$\lim_{r \rightarrow \infty} \langle A_\nu^r 1_{B_r}, 1_{B_r} \rangle = \lim_{r \rightarrow \infty} \frac{1}{\text{vol } B_r} \int_{B_r} (A_\nu^r 1_{B_r})(x) dx = \widehat{\nu}(0).$$

Indeed, let D be the diameter of N . Then if $(A_\nu^r 1_{B_r})(x) \neq \widehat{\nu}(0)$, we must have $x \in B_r \setminus B_{r-D}$. Hence

$$\begin{aligned} & \frac{1}{\text{vol } B_r} \int_{B_r} |(A_\nu^r 1_{B_r})(x) - \widehat{\nu}(0)| dx \\ & \leq \frac{1}{\text{vol } B_r} \|A_\nu^r 1_{B_r} - \widehat{\nu}(0)\|_\infty \text{vol}(B_r \setminus B_{r-D}), \\ & \leq \frac{\text{vol}(B_r \setminus B_{r-D})}{\text{vol } B_r} \|\nu\| \rightarrow 0 \end{aligned}$$

as $r \rightarrow \infty$.

This argument also proves the third identity in the claim, giving us

$$(11) \quad \frac{-\inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)}{\widehat{\nu}(0) - \inf_{u \in \mathbb{R}^n} \widehat{\nu}(u)} \geq \bar{\alpha}(G(\mathbb{R}^n, N)).$$

3.3. The odd distance graph. A notoriously difficult problem in discrete geometry is whether the odd distance graph in the plane

$$G\left(\mathbb{R}^n, \bigcup_{k=0}^{\infty} (2k+1)S^1\right)$$

has a finite chromatic number, a question due to Rosenfeld. Ardal, Mañuch, Rosenfeld, Shelah, and Stacho [4] showed that the chromatic number is at least five. It follows from a theorem of Furstenberg, Katznelson, and Weiss [13] that the measurable chromatic number of the odd distance graph is infinite. Steinhardt [32] gave the following alternative proof of this fact which can be seen as a nice example of the method described above.

For a positive number $\beta > 1$ Steinhardt defined the probability measure

$$\nu = \frac{\beta - 1}{\beta} \sum_{k=0}^{\infty} \beta^{-k} \omega_{2k+1}$$

on \mathbb{R}^2 where ω_{2k+1} is the rotationally invariant probability measure on the circle of radius $2k+1$ centered at 0. Then he showed that

$$\lim_{\beta \rightarrow 1} \inf_{u \in \mathbb{R}^2} \frac{\beta - 1}{\beta} \sum_{k=0}^{\infty} \beta^{-k} \int_{S^1} e^{-2\pi i x \cdot u} d\omega_{2k+1}(x) = 0.$$

When we define the measure with bounded support

$$\nu_{\beta, N} = \frac{\beta - 1}{\beta} \sum_{k=0}^N \beta^{-k} \omega_{2k+1},$$

we see that

$$\sup_{u \in \mathbb{R}^2} \widehat{\nu}_{\beta, N}(u) = \widehat{\nu}_{\beta, N}(0) \xrightarrow{\beta \rightarrow 1, N \rightarrow \infty} 1,$$

since $\nu_{\beta,N}$ is positive, and

$$\inf_{u \in \mathbb{R}^2} \widehat{\nu}_{\beta,N}(u) \xrightarrow{\beta \rightarrow 1, N \rightarrow \infty} 0,$$

and so by (10)

$$\lim_{N \rightarrow \infty} \chi_m \left(G \left(\mathbb{R}^2, \bigcup_{k=0}^N (2k+1)S^1 \right) \right) \geq \lim_{\beta \rightarrow 1} \lim_{N \rightarrow \infty} \frac{\widehat{\nu}_{\beta,N}(0) - \inf_{u \in \mathbb{R}^2} \widehat{\nu}(u)}{-\inf_{u \in \mathbb{R}^2} \widehat{\nu}(u)} = \infty.$$

In a similar way, (11) can be used to prove a quantitative version of the theorem of Furstenberg, Katznelson, Weiss; see Oliveira and Vallentin [27, Theorem 5.1] and also Kolountzakis [23] for a similar result and a similar proof in the case of arbitrary norms whose unit ball is not polytopal. To obtain this result Kolountzakis chooses a probability measure ν supported on the boundary of the unit norm ball and studies the decay of the function $u \rightarrow \widehat{\nu}(u)$ when u becomes large.

3.4. The unit distance graph. Let N be the unit sphere S^{n-1} . The orthogonal group acts transitively on N . Then the measure that optimizes the bound in (1) for the unit distance graph $G(\mathbb{R}^n, S^{n-1})$ is the rotationally invariant probability measure ω on S^{n-1} . Its Fourier transform can be explicitly computed:

$$\widehat{\omega}(u) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot u} d\omega(x) = \Omega_n(\|u\|),$$

where

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{t}\right)^{(n-2)/2} J_{(n-2)/2}(t) \quad \text{with } \Omega(0) = 1,$$

and $J_{(n-2)/2}$ is the Bessel function of the first kind with parameter $(n-2)/2$. The global minimum of Ω is at $j_{n/2,1}$ which is the first positive zero of the Bessel function $J_{n/2}$. Hence,

$$\chi_m(G(\mathbb{R}^n, S^{n-1})) \geq \frac{\Omega_n(j_{n/2,1}) - 1}{\Omega_n(j_{n/2,1})} \quad \text{and} \quad \bar{\alpha}(G(\mathbb{R}^n, S^{n-1})) \leq \frac{\Omega_n(j_{n/2,1})}{\Omega_n(j_{n/2,1}) - 1}$$

recovering the result of Oliveira and Vallentin [27, Section 3].

4. GRAPHS ON THE UNIT SPHERE

In this section we consider distance graphs defined on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$. To define the edge set we use a Borel subset D of the interval $[-1, 1]$ where 1 does not lie in the topological closure of D . Then two vertices are adjacent whenever $x \cdot y \in D$. We denote this graph by $G(S^{n-1}, D)$. Again, we aim at lower bounding the measurable chromatic number $\chi_m(G(S^{n-1}, D))$. Here we extend the technique of Bachoc, Nebe, Oliveira and Vallentin [6], who gave a formulation for the ϑ -number of $G(S^{n-1}, D)$. However, they showed how to compute it only in the case that D is finite.

The techniques in this section are quite similar to those presented in the previous section. The orthogonal group, which is a compact, non-commutative group, is a transitivity group of the measurable graph $G(S^{n-1}, D)$. So one can interpret the results in this section as the compact, non-abelian case whereas those in the previous section as the locally-compact, abelian case. In principle there is no technical difficulty to extend the results of this section from graphs on the sphere to graphs on compact, connected, rank-one symmetric spaces, see Oliveira and Vallentin [28].

4.1. Computing spectral bounds via spherical harmonics. Let ν be a signed Borel measure which is supported on the set D . For $t \in (-1, 1)$ define the operator $A_t: C(S^{n-1}) \rightarrow C(S^{n-1})$ by

$$(A_t f)(\xi) = \int_{S^{n-1}} f(\eta) d\omega_{\xi,t}(\eta),$$

where $\omega_{\xi,t}(\eta)$ is the rotationally invariant probability measure on the $(n-2)$ -dimensional sphere $\{\eta \in S^{n-1} : \xi \cdot \eta = t\}$.

We choose an orthonormal basis of $C(S^{n-1})$, and so of $L^2(S^{n-1})$, consisting of spherical harmonics $S_{k,l}$ where $k = 0, 1, \dots$ and $l = 1, \dots, c_{k,n}$ with $c_{k,n} = \binom{k+n-2}{k} + \binom{k+n-3}{k-1}$. The degree of $S_{k,l}$ equals k .

For $\delta > 0$ we have

$$(A_t f)(\xi) = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \int_{S^{n-1}} f(\eta) d\omega_{\xi,u}(\eta) \frac{(1-u^2)^{(n-3)/2}}{(1-t^2)^{(n-3)/2}} du$$

and by the Funk-Hecke formula (see [3, Theorem 9.7.1]¹) we see that the spherical harmonics $S_{k,l}$ are eigenfunctions of A_t with eigenvalue

$$\lambda_k(t) = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \overline{P}_k^{(\alpha,\alpha)}(u) \frac{(1-u^2)^{(n-3)/2}}{(1-t^2)^{(n-3)/2}} du = \overline{P}_k^{(\alpha,\alpha)}(t),$$

where $\alpha = (n-3)/2$ and $\overline{P}_k^{(\alpha,\alpha)}$ is a Jacobi polynomial of degree k , normalized so that $\overline{P}_k^{(\alpha,\alpha)}(1) = 1$. Jacobi polynomials are orthogonal polynomials defined on the interval $[-1, 1]$ with respect to the measure $(1-u^2)^{(n-3)/2} du$.

Now we can extend A_t to $L^2(S^{n-1})$ by setting

$$(A_t f)(\xi) = \sum_{k=0}^{\infty} \lambda_k(t) \sum_{l=1}^{c_{k,l}} (f, S_{k,l}) S_{k,l}(\xi),$$

where $(f, g) = \int_{S^{n-1}} f(x) \overline{g(x)} d\omega(x)$ with ω being the rotationally invariant probability measure on the sphere. Since $\lambda_k(t)$ lies in the interval $[-1, 1]$ the operator A_t is a bounded, self-adjoint operator. One can estimate the growth of the eigenvalues following Szegő [33, (4.1.1) and Theorem 8.21.8] by

$$\lambda_k(t) = O(k^{-1/2-(n-3)/2}),$$

so the operator is even compact when $n \geq 3$.

Now we can define $A_\nu: L^2(S^{n-1}) \rightarrow L^2(S^{n-1})$ by

$$(A_\nu f)(\xi) = \sum_{k=0}^{\infty} \int_{-1}^1 \lambda_k(t) d\nu(t) \sum_{l=1}^{c_{k,l}} (f, S_{k,l}) S_{k,l}(\xi),$$

and it is clear that

$$m(A_\nu) = \inf_{k \in \mathbb{N}} \int_{-1}^1 \lambda_k(t) d\nu(t) \quad \text{and} \quad M(A_\nu) = \sup_{k \in \mathbb{N}} \int_{-1}^1 \lambda_k(t) d\nu(t).$$

¹In [3] the Funk-Hecke formula is only stated for continuous functions $f \in C([-1, 1])$ but in fact it is also valid when $f \in L^1([-1, 1])$, see Groemer [16, Chapter 3.4]. Notice also that we use a more convenient normalization of the surface measure here so that their term ω_{n-1} disappears.

4.2. The single inner product graph. Let D be a Borel subset of the interval $[-1, 1]$ where 1 does not lie in the topological closure of D . Let ν be a Borel measure whose support lies in D . It is easy to see that the operator A_ν respects the graph $G(S^{n-1}, D)$. So one can apply (5) and (6) to this graph.

The case when D consists only of the single inner product t is particularly simple as no optimization over ν is necessary to find the optimal spectral bound:

$$\bar{\alpha}(G(S^{n-1}, \{t\})) \leq \frac{-\inf_{k \in N} \lambda_k(t)}{1 - \inf_{k \in N} \lambda_k(t)}, \quad \chi_m(G(S^{n-1}, \{t\})) \geq \frac{1 - \inf_{k \in N} \lambda_k(t)}{-\inf_{k \in N} \lambda_k(t)},$$

where $\lambda_k(t) = \overline{P}_k^{(\alpha, \alpha)}(t)$. This result also follows from [6, Theorem 6.2, Section 10]. The two bounds multiply to one which also follows from Theorem 2.7.

The bound is tight in the planar case $n = 2$ for $t = \cos(p/q\pi)$, with $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, when q is even, and when $t = \cos(x\pi)$ when x is not rational. It is also tight when $n = 3$ and $t = -1/3$. The latter was first proved by Lovász [25] using topological methods.

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