SIEGEL MODULAR FORMS, GRASSMANNIAN DESIGNS, AND UNIMODULAR LATTICES

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ABSTRACT. Siegel theta series with harmonic coefficients are vector-valued Siegel modular forms. We use them to show that certain sections of lattices form designs in Grassmannian space.

1. Introduction

In [2], a notion of t-design on the Grassmann manifold $\mathcal{G}_{m,n}$ is introduced, generalizing the so-called (antipodal) spherical designs. Many examples of such designs arise from lattices, the most famous ones being the designs associated to the root lattice E_8 and the Leech lattice. In both cases, these designs can be explained by properties of the representations afforded by their automorphism groups. In the case of the spherical designs, another proof, due to Boris Venkov, uses the theta series of these lattices as modular forms. Such an argument has been applied successfully to other families of lattices (see [15] and [3]).

In this paper, we prove a similar connection between the Grassmannian designs and certain vector-valued Siegel modular forms associated to a lattice. By using the explicit description of certain spaces of vector-valued Siegel modular forms, we can prove the existence of Grassmannian designs in the family of the extremal even unimodular lattices of dimension 32.

2. Grassmannian designs

2.1. **Definitions.** We briefly recall here the notion of Grassmannian designs. For a more detailled presentation, the reader is referred to [2].

Let $\mathcal{G}_{m,n}$ denote the real Grassmannian space of m-dimensional subspaces of $\mathbb{R}^n$, together with the transitive action of the real orthogonal group $\mathrm{O}(n,\mathbb{R})$. The starting point is the decomposition of the Hilbert-space of complex-valued absolutely squared integrable functions $L^2(\mathcal{G}_{m,n})$ under the action of $\mathrm{O}(n,\mathbb{R})$. As an $\mathrm{O}(n,\mathbb{R})$-module:

$$L^2(\mathcal{G}_{m,n}) = \overline{\oplus_\mu \mathrm{H}^\mu_{m,n}} \tag{1}$$

where the sum is over the partitions $\mu = \mu_1 \geq \cdots \geq \mu_m \geq 0$ with even parts $\mu_i \equiv 0 \mod 2$. The spaces $\mathrm{H}^\mu_{m,n}$ are isomorphic to the irreducible

Date: January 8, 2003.
1991 *Mathematics Subject Classification.* 11H06,

representation V_n^μ (see [6]) of $\mathrm{O}(n,\mathbb{R})$ canonically associated to μ. The degree of the partition μ is by definition $\deg(\mu) := \sum_i \mu_i$.

Definition 2.1. *A finite subset X of $\mathcal{G}_{m,n}$ is called a t-design if one of the following equivalent properties is satisfied:*

1. *For all $f \in \mathrm{H}_{m,n}^\mu$ and all μ with $0 \le \deg(\mu) \le t$,* $\int_{\mathcal{G}_{m,n}} f(p)dp = \frac{1}{|X|}\sum_{x\in X} f(x)$.
2. *For all $f \in \mathrm{H}_{m,n}^\mu$ and all μ with $2 \le \deg(\mu) \le t$, $\sum_{x\in X} f(x) = 0$.*

There is a nice characterization of the designs in terms of the zonal functions of $\mathcal{G}_{m,n}$: It is a classical fact that the orbits under the action of $\mathrm{O}(n,\mathbb{R})$ of the pairs (p,p') of elements of $\mathcal{G}_{m,n}$ are characterized by their so-called principal angles $(\theta_1,\dots,\theta_m) \in [0,\pi/2]^m$. We denote $y_i := \cos^2(\theta_i)$. The polynomial functions on $\mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$ which are invariant under the diagonal action of $\mathrm{O}(n,\mathbb{R})$ are polynomials in the variables $(y_1,\dots,y_m)$. They form an algebra isomorphic to the algebra $\mathbb{C}[Y_1,\dots,Y_m]^{S_m}$ of symmetric polynomials in m variables. Moreover, there is a unique sequence of polynomials $p_\mu(Y_1,\dots,Y_m)$ indexed by the partitions into even parts, such that $\mathbb{C}[Y_1,\dots,Y_m]^{S_m} = \sum_\mu \mathbb{C}p_\mu$, $p_\mu(1,\dots,1) = 1$, and the function : $p \in \mathcal{G}_{m,n} \to p_\mu(y_1(p,p'),\dots,y_m(p,p'))$ defines, for all $p' \in \mathcal{G}_{m,n}$, an element of $\mathrm{H}_{m,n}^\mu$. These polynomials have degree $\deg(\mu)/2$. They are explicitely calculated in [8].

Theorem 2.2. *(see* [2, Proposition 4.2]*) Let $X \subset \mathcal{G}_{m,n}$ be a finite set. Then,*

1. $\sum_{p,p'\in X} p_\mu(y_1(p,p'),\dots,y_m(p,p')) \ge 0$.
2. *The set $X \subset \mathcal{G}_{m,n}$ is a t-design if and only if for all μ, $2 \le \deg(\mu) \le t$,* $\sum_{p,p'\in X} p_\mu(y_1(p,p'),\dots,y_m(p,p')) = 0$.

2.2. **Some subsets of $\mathcal{G}_{m,n}$ associated to a lattice.** Let $L \subset \mathbb{R}^n$ be a lattice. We define certain natural finite subsets of $\mathcal{G}_{m,n}$ associated to L, in the following way. The spaces of $m \times m$ real symmetric, real positive definite, and real positive semi-definite matrices are denoted by $\mathrm{S}_m(\mathbb{R})$, $\mathrm{S}_m^{>0}(\mathbb{R})$, $\mathrm{S}_m^{\ge 0}(\mathbb{R})$, respectively.

Definition 2.3. *Let $S \in \mathrm{S}_m^{>0}(\mathbb{R})$. We denote L_S the set of $p \in \mathcal{G}_{m,n}$ such that $p \cap L$ is a lattice, having a basis $(v_1,\dots,v_m)$ with $v_i \cdot v_j = S_{i,j}$.*

Clearly, the sets L_S are finite sets. In the case $m = 1$, the sets L_S are the sets of lines supporting the primitive lattice vectors of fixed norm. It is worth noticing that these sets are unions of orbits under the automorphism group of the lattice.

We introduce a few more notations. An m-tuple of vectors of $\mathbb{R}^n$ is denoted by $v^{(m)}$ and the Gram matrix of its vectors by $\mathrm{gram}(v^{(m)})$. The real vector space spanned by these vectors is $\mathbb{R}v^{(m)}$. If the vectors of $v^{(m)}$ belong to the lattice L, and consist of a $\mathbb{Z}$-basis of $L\cap\mathbb{R}v^{(m)}$, $v^{(m)}$ is called *primitive*.

One of the aims of this paper is to study the design properties of the sets L_S. Therefore, we have to consider sums of the type $\sum_{p\in L_S} f(p)$ where f runs over the spaces $\mathrm{H}^{\mu}_{m,n}$.

Lemma 2.4. *The following assertions are equivalent:*

1. *For all* $S \in \mathrm{S}_m^{>0}(\mathbb{R})$, $\sum_{p\in L_S} f(p) = 0$
2. *For all* $S \in \mathrm{S}_m^{>0}(\mathbb{R})$, $\sum_{\substack{v^{(m)}\in L^m, primitive\\ \mathrm{gram}(v^{(m)})=S}} f(\mathbb{R}v^{(m)}) = 0$
3. *For all* $S \in \mathrm{S}_m^{>0}(\mathbb{R})$, $\sum_{\substack{v^{(m)}\in L^m\\ \mathrm{gram}(v^{(m)})=S}} f(\mathbb{R}v^{(m)}) = 0$

Proof. Two bases of the lattice $L\cap p$ with the same Gram matrix are exchanged by an element of the automorphism group of $L\cap p$, so the second sum differs from the first by a multiplicative factor. In the third sum, the non primitive $v^{(m)}$ contribute to subsums of the type $\sum_{p\in L_{S'}} f(p)$ with $\det(S') < \det(S)$ so we can conclude by induction on $\det(S)$. □

Remark 2.5. *With the help of representation theory of the automorphism group, one finds examples of lattices L such that all the (non empty) sets L_S (with rank(S)≤ $\frac{dim(L)}{2}$) are Grassmannian k-designs (see* [2]*). For the root lattices D_4, E_6, E_7 one can take $k = 4$, for E_8 and the Barnes-Wall lattice BW_{16}, $k = 6$ and even $k = 10$ for the Leech lattice Λ_{24}.*

It turns out that the sums of Lemma 2.4(3) can be interpreted in terms of certain vector-valued modular forms. The next section recalls the basic properties of these modular forms.

3. Vector-valued Siegel modular forms

Let H_m denote the Siegel space

$$H_m := \{Z \in M^{m\times m}(\mathbb{C}) \mid Z^t = Z, Z = X + iY \text{ and } Y > 0\} \tag{2}$$

endowed with the usual action of the symplectic group $\mathrm{Sp}(m,\mathbb{R})$. If $M := \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(m,\mathbb{R})$ and $Z \in H_m$ then $M.Z := (AZ + B)(CZ + D)^{-1}$.

Let (ρ, V_ρ) be a finite dimensional complex representation of $\mathrm{GL}(m,\mathbb{C})$. A V_ρ-valued Siegel modular form for the modular group $\Gamma_m := \mathrm{Sp}(m,\mathbb{Z})$ is a holomorphic function $f : H_m \to V_\rho$ satisfying the transformation formula $f\mid_\rho M = f$ for all $M \in \Gamma_m$, where

$$(f\mid_\rho M)(Z) := \rho(CZ + D)^{-1} f(MZ)$$

(plus a condition on the growth of f in the case $m = 1$). Such a modular form has got a Fourier expansion of the type:

$$f(Z) = \sum_S a_f(S)e(SZ) \tag{3}$$

where $e(SZ) := e^{i\pi\,\mathrm{trace}(SZ)}$ and S runs over the set of even symmetric positive semi-definite matrices $\mathrm{S}_m^{even} := \{S \in \mathrm{S}_m^{\geq 0}(\mathbb{R}) \mid S_{i,j} \in \mathbb{Z}$ and $S_{i,i} \equiv 0 \mod 2\}$.

One can restrict without loss of generality to the case when the representation ρ is irreducible. Then, it is characterized by its highest weight, an m-tuple $\mu := (\mu_1, \dots, \mu_m)$ with $\mu_1 \geq \cdots \geq \mu_m$, and we may denote (ρ_μ, V_μ) this representation.

The vector space $[\Gamma_m, \rho]$ of these modular forms is finite dimensional. The classical case of complex-valued Siegel modular forms corresponds to the one-dimensional representations; the spaces may be denoted $[\Gamma_m, \det^k]$ or more briefly $[\Gamma_m, k]$. The direct sum $A(\Gamma_m) := \oplus_{k\equiv 0 \mod 2}[\Gamma_m, k]$ is a $\mathbb{C}$-algebra, the structure of which is completely understood only for $m = 1, 2, 3$. For an arbitrary representation ρ, the sum $A(\Gamma_m, \rho) := \oplus_{k\equiv 0 \mod 2}[\Gamma_m, \det^k \otimes \rho]$ is a module over the previous algebra. Its structure is completely described in the cases $m = 2$ and $\rho = [2,0], [4,0], [6,0]$ (see [12], [7]).

Such modular forms can be constructed from lattices, in the following way (see [4] and [5] for detailed proofs). Let L be again an n-dimensional lattice contained in $\mathbb{R}^n$. The theta series of degree $m \leq n/2$ associated to L is:

$$\theta_L^{(m)} := \sum_{\substack{v^{(m)} \in L^m \\ S := \mathrm{gram}(v^{(m)})}} e(SZ) = \sum_{S \in \mathrm{S}_m^{\mathrm{even}}} a_L(S) e(SZ) \tag{4}$$

where $a_L(S)$ counts the number of $v^{(m)} \in L^m$ with $\mathrm{gram}(v^{(m)}) = S$. Then $\theta_L^{(m)}$ is a Siegel modular form for some congruence subgroup, which can be taken to be the full modular group $\Gamma_m = \mathrm{Sp}(m, \mathbb{Z})$, if the lattice L is even unimodular. The weight of $\theta_L^{(m)}$ is equal to $n/2$ (i.e. they are modular forms for the representation $\rho = \det^{n/2}$).

More generally, one can construct vector-valued modular forms from a lattice L and some spaces of harmonic polynomials.

Let $\mathbb{C}[\underline{X}]$ denote the polynomial algebra in the matrix variables $(X_{i,j})_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}$ with the action of $\mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ given by $(g, h).P = P(g^t X h)$. The decomposition of this space is well-known to be:

$$\mathbb{C}[\underline{X}] \simeq \oplus_\mu F_m^\mu \otimes F_n^\mu \tag{5}$$

where F_m^μ denotes the irreducible $\mathrm{GL}(m, \mathbb{R})$-module canonically associated to the partition $\mu = (\mu_1, \dots, \mu_m)$ with $\mu_1 \geq \cdots \geq \mu_m \geq 0$. The harmonic polynomials are the polynomials belonging to the intersection of the kernels of the operators

$$\Delta_{i,j} := \sum_{k=1}^{n} \frac{\partial^2}{\partial X_{i,k} \partial X_{j,k}}. \tag{6}$$

Their space is denoted $\mathcal{H}_{m,n}$ and is stable under the action of $\mathrm{GL}(m, \mathbb{R}) \times \mathrm{O}(n, \mathbb{R})$. Its decomposition is given by

$$\mathcal{H}_{m,n} \simeq_{\mathrm{GL}(m,\mathbb{R})\times \mathrm{O}(n,\mathbb{R})} \bigoplus_{\mu} F_m^{\mu} \otimes V_n^{\mu} \tag{7}$$

Equivalently, the polynomial functions: $P : M^{m\times n}(\mathbb{C}) \to F_m^{\mu}$ satisfying $\rho_{\mu}(u)P(X) = P(u^t X)$ for all $u \in \mathrm{GL}(m,\mathbb{R})$ span a vector space, $\mathrm{O}(n,\mathbb{R})$-isomorphic to V_n^{μ}. We shall denote it $\mathrm{Harm}_{m,n}^{\mu}$.

Definition 3.1. *Let $L \subset \mathbb{R}^n$ be a lattice and $P \in \mathrm{Harm}_{m,n}^{\mu}$. For $m \leq \frac{n}{2}$ let*

$$\theta_{L,P}^{(m)} := \sum_{\substack{v^{(m)}\in L^m \\ S:=\mathrm{gram}(v^{(m)})}} P(v^{(m)}) e(SZ) \tag{8}$$

where $P(v^{(m)})$ stands for the value of P on the $m \times n$ matrix $X_{v^{(m)}}$, the rows of which are equal to the m vectors of $v^{(m)}$. $\theta_{L,P}^{(m)}$ is called the harmonic Siegel theta series of L with coefficients P.

Proposition 3.2. *([5]) If $L \subset \mathbb{R}^n$ is an even unimodular lattice and $P \in \mathrm{Harm}_{m,n}^{\mu}$, then $\theta_{L,P}^{(m)} \in [\Gamma_m, \det^{n/2} \otimes \rho_{\mu}]$ is a vector valued Siegel modular form for the full modular group.*

4. Harmonic theta series and Grassmannian designs

In this section we show how harmonic Siegel theta series can be used to show that certain sets L_S of sections of a lattice L provide Grassmannian designs.

Theorem 4.1. *Let $L \subset \mathbb{R}^n$ be an even lattice, and let $m \leq n/2$. Assume that, for all $P \in \mathrm{Harm}_{m,n}^{\mu}$ and all even μ with $2 \leq \deg(\mu) \leq t$, $\theta_{L,P}^{(m)} = 0$. Then, for all $m_0 \leq m$ and all $S \in \mathrm{S}_{m_0}^{>0}(\mathbb{R})$, the non empty sets L_S are t-designs.*

Proof. The space $(F_m^{\mu})^{\mathrm{O}(m,\mathbb{R})}$ of $\mathrm{O}(m,\mathbb{R})$-invariant elements in F_m^{μ} is one-dimensional if and only if μ is even. We denote v_{μ} an arbitrary non-zero vector of this space. We choose on F_m^{μ} an $\mathrm{O}(m,\mathbb{R})$-invariant hermitian form, denoted by $<,>$, and we can assume v_{μ} to be of norm 1 for this form. If $P \in \mathrm{Harm}_{m,n}^{\mu}$, let $P_0 : M^{m\times n}(\mathbb{C}) \to \mathbb{C}$ be defined by: $P_0(X) :=< P(X), v_{\mu} >$. By construction, the function P_0 is $\mathrm{O}(m,\mathbb{R})$-invariant and therefore defines an element $\tilde{P}_0$ of $L^2(\mathcal{G}_{m,n})$ by: $\tilde{P}_0(p) := P_0(X_p)$, where X_p is the matrix of any orthonormal basis of p. The mapping $P \to \tilde{P}_0$ is an isomorphism of $\mathrm{O}(n,\mathbb{R})$-modules from $\mathrm{Harm}_{m,n}^{\mu}$ to $\mathrm{H}_{m,n}^{\mu}$.

Let $S \in \mathrm{S}_m^{even}$ be of rank m. There exists $U \in \mathrm{GL}(m,\mathbb{R})$ such that $S = UU^t$. From the hypothesis, we have, for all $P \in \mathrm{Harm}_{m,n}^{\mu}$,

$$\sum_{\substack{v^{(m)}\in L^m \\ S:=\mathrm{gram}(v^{(m)})}} P(v^{(m)}) = 0.$$

Since $\rho_\mu((U^{-1})^t)P(v^{(m)}) = P(U^{-1}X_{v^{(m)}})$, and since $U^{-1}X_{v^{(m)}}$ is the matrix of an orthonormal basis of the space $\mathbb{R}v^{(m)}$, we conclude that

$$\sum_{\substack{v^{(m)} \in L^m \\ S := \mathrm{gram}(v^{(m)})}} \tilde{P}_0(\mathbb{R}v^{(m)}) = 0.$$

From Lemma 2.4, the set L_S is a t-design. The assertion on the other $m_0 < m$ derives from the same argument applied to the successive images of $\theta^{(m)}_{L,P}$ by the Φ-operator. □

In order to apply the previous theorem to concrete situations, we need to study the spaces of vector-valued modular forms. The next proposition shows that in general we only need to study the cusp forms. The space of cusp forms is the space of forms $f \in [\Gamma_m, \mu]$ for which $a_f(S) = 0$ for all the matrices S of rank smaller than m, and is denoted $[\Gamma_m, \mu]_0$.

Proposition 4.2. *Assume that, for all $m_0 \leq m$, and for all $S \in \mathrm{S}^{>0}_{m_0}(\mathbb{R})$, the non empty sets L_S are t-designs. Then, the modular forms $\theta^{(m+1)}_{L,P}$ are cusp forms, when P is associated to a partition μ with either $\mu_{m+1} > 0$ or $\sum_{i=1}^m \mu_i \leq t$.*

Proof. If $S \in \mathrm{S}^{even}_{m+1}$ is such that $S_{m+1,m+1} = 0$, and if $S = UU^t$, then the last row vector u_{m+1} of U equals 0. One has $AU = U$, with A the diagonal matrix with 1 on the diagonal except the last coefficient equals 0. If $P \in \mathrm{H}^\mu_{m+1,n}$, $P(U) = P(AU) = \rho_\mu(A)P(U) = 0$ if $\mu_{m+1} > 0$ (in that case, det divides ρ_μ). On the other hand, the polynomial P restricted to the matrix variables $X_{i,j}$ with $X_{m+1,j} = 0$ belongs to a subspace isomorphic as a $\mathrm{GL}(m,\mathbb{R})$-module to $F_m^{(\mu_1,\dots,\mu_m)}$, and is harmonic in these variables. Hence the design property implies that the coefficients of $\theta^{(m+1)}_{L,P}$ corresponding to matrices S with $S_{m+1,m+1} = 0$ and of rank m are equal to zero. We can iterate the same argument to obtain the nullity of the coefficients associated to matrices S of lower rank.

5. Even unimodular extremal lattices

Let L be an even unimodular lattice of dimension $n = 24q+8r$ ($r = 0,1,2$). Since its theta series θ_L belongs to the space $[\Gamma_1, n/2]$, and since, as is well-known, the algebra of modular forms $A(\Gamma_1) = \mathbb{C}[E_4, E_6]$, the following bound holds for the minimum of L:

$$\min(L) \leq 2[n/24] + 2 \qquad (9)$$

A lattice is called (analytically) *extremal*, if its minimum attains this bound. This notion can be defined for other families of lattices, see the nice survey paper [10]. Extremal even unimodular lattices are only known for $n = 8, 16, 24, 32, 40, 48, 56, 64, 80$ and are completely determined only up to

$n = 24$ (where the unique Leech lattice satisfies this bound). In dimension 32, they form a huge family, among which 5 of them are constructed from extremal binary codes. In dimension 48, which is the first dimension for the minimum 6, only three of them are known. The question of the existence of such a lattice in dimension 72 (hence of minimum 8) is still opened.

Let $S \in \mathrm{S}_m^{>0}(\mathbb{R})$, we denote $\min(S) := \min\{xSx^t, x \in \mathbb{Z}^m, x \neq 0\}$. Let f be a non zero cusp form; we define $m(f) := \frac{1}{2}\min\{\min(S) \mid a_f(S) \neq 0\}$. We set $m(0) = +\infty$. For example, if $f = \theta_{L,P}^{(m)}$, clearly $m(f) \geq \min(L)/2$. In the case of degree one, due to the explicit description of $[\Gamma_1, k]$, it is easy to see that:

$$\text{if } f \neq 0, \quad m(f) \leq k/12 \tag{10}$$

where k is the weight of f. Applied to the forms $f = \theta_{L,P}^{(1)}$, it leads to the result, due to Boris Venkov, that the sets L_S associated to extremal lattices (here $L_S = L_{(a)}$ is the set of lines supporting lattice vectors of given norm a) support designs of strength $10 - 4r$. We introduce the following notation:

$$\min([\Gamma_m, \mu]_0) := \max\{\min(f) \mid f \in [\Gamma_m, \mu]_0\}. \tag{11}$$

We now consider the question of the generalization of this result to the higher degree Grassmannian designs contained in extremal even unimodular lattices. For the E_8 lattice and the Leech lattice, the properties of their automorphism groups prove that they do contain respectively 6- and 10-Grassmannian designs (see [2]). So, the first interesting case is the case of dimension 32.

We now restrict to the case $m = 2$, and give the numerical results obtained by the explicit calculations of the spaces $[\Gamma_2, \mu]_0$ for $\rho_\mu = \det^{16} \otimes \rho_\nu$, where ν runs over partitions of small degree. A formula for the dimensions of these spaces is given in [13].

$\|\nu\|$	0	2	4	6	8	10
ν	(0)	$(2,0)$	$(4,0)$	$(6,0)$	$(8,0)$	$(10,0)$
$\dim([\Gamma_m, \mu]_0)$	2	2	3	5	7	8
$\min([\Gamma_m, \mu]_0)$		2	2	2	2(?)	2(?)
ν			$(2,2)$	$(4,2)$	$(6,2)$	$(8,2)$
$\dim([\Gamma_m, \mu]_0)$			2	2	4	7
$\min([\Gamma_m, \mu]_0)$			2	2	2	4
ν					$(4,4)$	$(6,4)$
$\dim([\Gamma_m, \mu]_0)$					3	3
$\min([\Gamma_m, \mu]_0)$					4	2

Corollary 5.1. *For all 32-dimensional even unimodular lattices of minimum 4 and all S of rank ≤ 2 the non-empty sets L_S are 6-designs.*

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