Kernel density estimation in adaptive tracking

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Outline

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Goal

Let (X_n) be a sequence of **iid** random variables with **unknown density function f**.

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Estimate the density function f by a **kernel density estimator**.

Let K be a **nonnegative, bounded, Lipschitz** function called **Kernel**, such that

$$\int_{\mathbb{R}} K(x)\,dx = 1, \qquad \int_{\mathbb{R}} xK(x)\,dx = 0,$$

$$\int_{\mathbb{R}} K^2(x)\,dx = \tau^2.$$

Choices for the kernel

- **Uniform kernel**

$$K_a(x) = \begin{cases} \dfrac{1}{2a} & \text{if } |x| \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

- **Epanechnikov kernel**

$$K_b(x) = \begin{cases} \dfrac{3}{4b}\left(1 - \dfrac{x^2}{b^2}\right) & \text{if } |x| \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

- **Gaussian kernel**

$$K_c(x) = \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{x^2}{2c^2}\right).$$

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Parzen-Rosenblatt or Wolverton-Wagner

Let (h_n) be a sequence of positive real numbers decreasing to zero called **bandwidth**. We can estimate the density f by the **Parzen-Rosenblatt** estimator given for all $x \in \mathbb{R}$ by

$$\widetilde{f}_n(x) = \frac{1}{nh_h}\sum_{i=1}^{n} K\Big(\frac{X_i - x}{h_n}\Big).$$

We can also estimate the density f by the **Wolverton-Wagner** estimator given for all $x \in \mathbb{R}$ by

$$\widehat{f}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i} K\left(\frac{X_i - x}{h_i}\right).$$

On Wolverton-Wagner

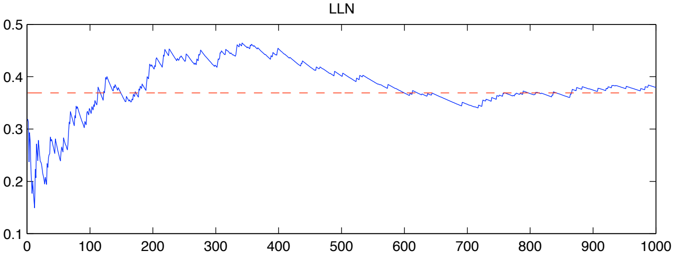
Theorem

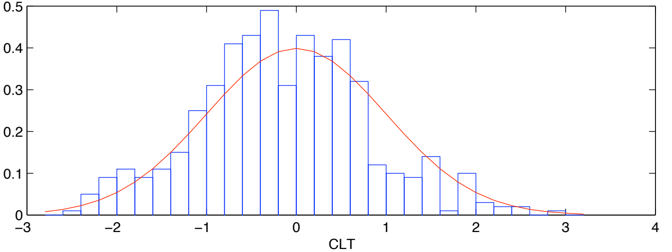
Assume that f is derivable with bounded derivative. If the bandwidth $h_n = 1/n^{\alpha}$ *with* $0 < \alpha < 1$*, we have*

(LLN) $$\lim_{n \to \infty} \widehat{f}_n(x) = f(x) \quad a.s.$$

In addition, if $1/5 < \alpha < 1$*, we also have*

(CLT) $$\sqrt{nh_n}(\widehat{f}_n(x) - f(x)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\tau^2 f(x)}{1+\alpha}\right).$$


LLN
0.5
0.4
0.3
0.2
0.1
0
100
200
300
400
500
600
700
800
900
1000


0.5
0.4
0.3
0.2
0.1
0
−3
−2
−1
0
1
2
3
4
CLT

Outline

Consider the d-dimensional **ARMAX(p,q,r)** model given by

$$A(R)X_n = B(R)U_n + C(R)\varepsilon_n$$

where R is the shift-back operator, X_n is the **system output**, U_n is the **system input** and ε_n is the **driven noise**,

- $A(R) = I_d - A_1 R - \cdots - A_p R^p$,
- $B(R) = B_1 R + B_2 R^2 + \cdots + B_q R^q$,
- $C(R) = I_d - C_1 R - \cdots - C_r R^r$

where A_i, B_j, and C_k are unknown matrices. We assume that the **high frequency gain** B_1 is known with $B_1 = I_d$.

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The unknown parameter of the model is given by

$$\theta^t = (A_1, \dots, A_p, B_2, \dots, B_q, C_1, \dots, C_r).$$

The **ARMAX(p,q,r)** model can be rewritten as

$$\boldsymbol{X_{n+1} = \theta^t \Psi_n + U_n + \varepsilon_{n+1},}$$

where $\Psi_n = \left(X_n^p, U_n^q, \varepsilon_n^r\right)^t$ with

$$\begin{aligned} X_n^p &= (X_n^t, \dots, X_{n-p+1}^t), \\ U_n^q &= (U_{n-1}^t, \dots, U_{n-q+1}^t), \\ \varepsilon_n^r &= (\varepsilon_n^t, \dots, \varepsilon_{n-r+1}^t). \end{aligned}$$

Causality and Passivity

Definition

The matrix polynomial B is **causal** if for all $z \in \mathbb{C}$ with $|z| \leqslant 1$

$$\det(z^{-1}B(z)) \neq 0.$$

Definition

The matrix polynomial C is **passif** if for all $z \in \mathbb{C}$ with $|z| = 1$

$$\det(C(z)) \neq 0 \quad \text{and} \quad C^{-1}(z) > \frac{1}{2} I_d$$

Extended least squares

We estimate θ by the **extended least squares** estimator

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + S_n^{-1}\Phi_n(X_{n+1} - U_n - \widehat{\theta}_n^t\Phi_n)^t,$$

$$\widehat{\varepsilon}_{n+1} = X_{n+1} - U_n - \widehat{\theta}_n^t\Phi_n,$$

where the vector $\Phi_n = \left(X_n^p, U_n^q, \widehat{\varepsilon}_n^r\right)^t$ with $\widehat{\varepsilon}_n^r = (\widehat{\varepsilon}_n^t, \ldots, \widehat{\varepsilon}_{n-r+1}^t)$,

$$S_n = \sum_{i=0}^{n} \Phi_i \Phi_i^t + S,$$

where S is a positive definite and deterministic matrix.

Adaptive Control

The role played by U_n is to force X_n to track step by step a given trajectory (x_n). We make use of the **adaptive tracking control**

$$U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n.$$

Then, the closed-loop system is given by

$$X_{n+1} - x_{n+1} = \pi_n + \varepsilon_{n+1}$$

where the prediction error

$$\pi_n = (\theta - \widehat{\theta}_n)^t \Phi_n.$$

We assume that (ε_n) is a sequence of **iid** random vectors with **unknown density *f***. If (ε_n) were observable, we could estimate f by

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h_i^d} K\left(\frac{\varepsilon_i - x}{h_i}\right).$$

However, ε_{n+1} is unobservable but it can estimated by

$$\widehat{\varepsilon}_{n+1} = X_{n+1} - U_n - \widehat{\theta}_n^t \Phi_n = X_{n+1} - x_{n+1}.$$

Consequently, we can use the **Wolverton-Wagner** estimator

$$\widehat{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h_i^d} K\left(\frac{X_i - x_i - x}{h_i}\right).$$

Outline

Uniform law of large numbers

Theorem

Assume that f is positive and differentiable with **bounded gradient** *and that* (ε_n) *has finite moment of order* > 2*. If the bandwidth* $\boldsymbol{h_n = 1/n^{\alpha}}$ *with* $\alpha \in]0, 1/d[$*, then*

(LLN) $$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} |\widehat{f}_n(x) - f(x)| = 0 \quad a.s.$$

Central limit theorem

Theorem

Assume that f is positive and differentiable with **bounded gradient** *and that* (ε_n) *has finite moment of order* > 2. *If the bandwidth* $\boldsymbol{h_n = 1/n^{\alpha}}$ *with* $\alpha \in]1/(d+2), 1/d[$, *then*

$$G_n(x) = \sqrt{nh_n^d}(\widehat{f}_n(x) - f(x)) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\tau^2 f(x)}{1+\alpha d}\right) = G(x).$$

In addition, for N distinct points $x_1, \ldots, x_N$ *of* $\mathbb{R}^d$, *we also have*

$$\text{(MCLT)} \quad \Big(G_n(x_1), \cdots, G_n(x_N)\Big) \xrightarrow{\mathcal{L}} \Big(G(x_1), \cdots, G(x_N)\Big)$$

where $G(x_1), \ldots, G(x_N)$ *are independent.*

Outline

goodness of fit test

We wish to test

$$\mathcal{H}_0 \ : \ \langle\langle\, f = f_0 \,\rangle\rangle \qquad \textbf{versus} \qquad \mathcal{H}_1 \ : \ \langle\langle\, f \neq f_0 \,\rangle\rangle$$

where f_0 is a given density function. Our statistical test is

$$T_n(N) = \frac{1}{\tau^2 \ell_h} \sum_{i=1}^{N} \frac{(\widehat{f}_n(x_i) - f_0(x_i))^2}{\widehat{f}_n(x_i)}$$

where $x_1, \ldots, x_N$ are N distinct points of $\mathbb{R}^d$ and

$$\ell_h = \frac{1}{1 + \alpha d}.$$

Theorem

Assume that f is positive and differentiable with **bounded gradient** *and that (ε_n) has finite moment of order > 2. If the bandwidth* $\boldsymbol{h_n = 1/n^{\alpha}}$ *with $\alpha \in]1/(d+2), 1/d[$, then under $\mathcal{H}_0$*

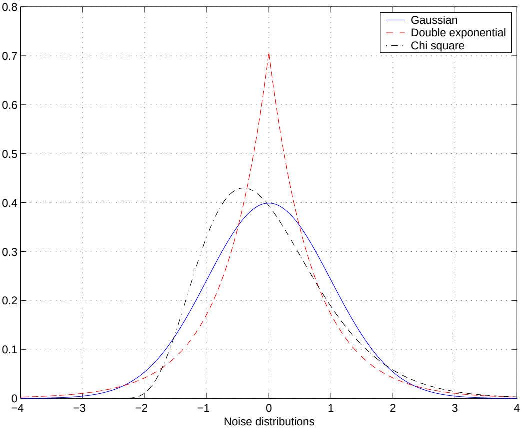
$$\boldsymbol{n h_n^d T_n(N) \xrightarrow{\mathcal{L}} \chi^2(N).}$$

In addition, under $\mathcal{H}_1$ and if one can find some point x of $\mathbb{R}^d$ in $\{x_1, x_2, \ldots, x_N\}$ such that $f(x) \neq f_0(x)$, then $T_n(N) \to \sigma^2$ a.s.

$$\boldsymbol{\sqrt{n h_n^d}(T_n(N) - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \lambda^2)}$$

$$\sigma^2 = \frac{1}{\tau^2 \ell_h} \sum_{i=1}^{N} \frac{(f(x_i) - f_0(x_i))^2}{f(x_i)}, \qquad \lambda^2 = \frac{1}{\tau^2 \ell_h} \sum_{i=1}^{N} \frac{(f^2(x_i) - f_0^2(x_i))^2}{f^3(x_i)}.$$

Simulations



Noise distributions

- **Gaussian**

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big),$$

- **Double exponential**

$$f_1(x) = \frac{1}{\sqrt{2}} \exp\Big(-\sqrt{2}|x|\Big),$$

- **Chi square**

$$f_2(x) = \begin{cases} \frac{9}{5}(x+\sqrt{6})^5 \exp\Big(-\sqrt{6}(x+\sqrt{6})\Big) & \text{if } |x| \geqslant -\sqrt{6}, \\ 0 & \text{otherwise.} \end{cases}$$

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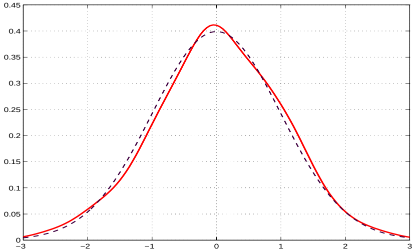
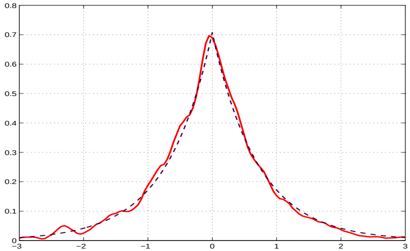
Law of large numbers





ARX Goodness of fit test

$$X_{n+1} = \theta X_n + U_n + \varepsilon_{n+1}$$

Table: Results under $\mathcal{H}_0$ and $\mathcal{H}_1$ with test level 5%.

	$n = 200, N = 8$				$n = 1000, N = 22$	
	$\mathcal{H}f_0$	$\mathcal{H}f_1$	$\mathcal{H}f_2$	$\mathcal{H}f_0$	$\mathcal{H}f_1$	$\mathcal{H}f_2$
$\mathcal{G}f_0$	**3.8%**	35.7%	28%	**3.7%**	99.7%	98.2%
$\mathcal{G}f_1$	45.8%	**5.5%**	71.5%	100%	**5%**	100%
$\mathcal{G}f_2$	21.2%	54.5%	**3.2%**	96.7%	100%	**5.1%**

NARX Goodness of fit test

$$X_{n+1} = \theta X_n^2 + U_n + \varepsilon_{n+1}$$

Table: Results under $\mathcal{H}_0$ and $\mathcal{H}_1$ with test level 5%.

	$n = 200,\ N = 8$				$n = 1000,\ N = 22$	
	$\mathcal{H}f_0$	$\mathcal{H}f_1$	$\mathcal{H}f_2$	$\mathcal{H}f_0$	$\mathcal{H}f_1$	$\mathcal{H}f_2$
$\mathcal{G}f_0$	**3%**	37.1%	28.5%	**4.3%**	99.5%	98.6%
$\mathcal{G}f_1$	44.6%	**5.2%**	72%	100%	**5.1%**	100%
$\mathcal{G}f_2$	19.8%	58.3%	**3.7%**	97.2%	100%	**5%**