Asymptotic results for empirical measures of weighted sums of independent random variables

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Workshop on Limit Theorems, University Paris 1

Paris, January 14, 2008
Motivation
- Circulant random matrices
- Empirical periodogram
- Almost sure central limit theorem

Main results
- The sequence of weights
- Uniform strong law
- Central limit theorem
- Large deviation principle
Outline

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Let \((X_n)\) be a sequence of random variables and consider the **symmetric circulant random** matrix

\[
A_n = \frac{1}{\sqrt{n}} \begin{pmatrix}
X_1 & X_2 & X_3 & \cdots & X_{n-1} & X_n \\
X_2 & X_3 & X_4 & \cdots & X_n & X_1 \\
X_3 & X_4 & X_5 & \cdots & X_1 & X_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
X_n & X_1 & X_2 & \cdots & X_{n-2} & X_{n-1}
\end{pmatrix}.
\]

**Goal.** Asymptotic behavior of the empirical spectral distribution

\[
F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{\lambda_k \leq x\}}.
\]
We shall make use of
\[ r_n = \left[ \frac{n - 1}{2} \right] \]
and of the trigonometric weighted sums
\[
S_{n,k} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t \cos \left( \frac{2\pi kt}{n} \right),
\]
\[
T_{n,k} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t \sin \left( \frac{2\pi kt}{n} \right).
\]
Lemma (Bose-Mitra)

The eigenvalues of $A_n$ are given by

$$
\lambda_0 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t,
$$

if $n$ is even

$$
\lambda_{n/2} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (-1)^{t-1} X_t,
$$

and for all $1 \leq k \leq r_n$

$$
\lambda_k = -\lambda_{n-k} = \sqrt{S_{n,k}^2 + T_{n,k}^2}.
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**Lemma (Bose-Mitra)**

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$$
Theorem (Bose-Mitra, 2002)

Assume that \((X_n)\) is a sequence of iid random variables such that \(E[X_1] = 0, E[X_1^2] = 1, E[|X_1|^3] < \infty\). Then, for each \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} E[(F_n(x) - F(x))^2] = 0
\]

where \(F\) is given by

\[
F(x) = \frac{1}{2} \begin{cases} 
\exp(-x^2) & \text{if } x \leq 0, \\
2 - \exp(-x^2) & \text{if } x \geq 0.
\end{cases}
\]

Remark. \(F\) is the symmetric Rayleigh CDF.
Let \((X_n)\) be a sequence of random variables and consider the **empirical periodogram** defined, for all \(\lambda \in [-\pi, \pi]\), by

\[
I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} e^{-it\lambda} X_t \right|^2.
\]

At the **Fourier frequencies** \(\lambda_k = \frac{2\pi k}{n}\), we clearly have

\[
I_n(\lambda_k) = S_{n,k}^2 + T_{n,k}^2.
\]

**Goal.** Asymptotic behavior of the empirical distribution

\[
F_n(x) = \frac{1}{r_n} \sum_{k=1}^{r_n} \mathbf{1}\{I_n(\lambda_k) \leq x\}.
\]
Theorem (Kokoszka-Mikosch, 2000)

Assume that $(X_n)$ is a sequence of iid random variables such that $E[X_1] = 0$ and $E[X_1^2] = 1$. Then, for each $x \in \mathbb{R}$,

$$\lim_{n \to \infty} E[(F_n(x) - F(x))^2] = 0$$

where $F$ is the exponential CDF.

Remark. $(F_n)$ also converges uniformly in probability to $F$. 
Almost sure central limit theorem

Let \((X_n)\) be a sequence of iid random variables such that 
\[\mathbb{E}[X_n] = 0 \text{ and } \mathbb{E}(X_n^2) = 1\] 
and denote 
\[S_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t.\]

**Theorem (Lacey-Phillip, 1990)**

The sequence \((X_n)\) satisfies an **ASCLT** which means that for any \(x \in \mathbb{R}\)
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} 1\{S_t \leq x\} = \Phi(x) \quad \text{a.s.}
\]

where \(\Phi\) stands for the standard normal CDF.
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Assumptions

Let \((U^{(n)})\) be a family of real rectangular \(r_n \times n\) matrices with \(1 \leq r_n \leq n\), satisfying for some constants \(C, \delta > 0\)

\[(A_1)\quad \max_{1 \leq k \leq r_n, 1 \leq t \leq n} |u_{k, t}^{(n)}| \leq \frac{C}{(\log(1 + r_n))^{1+\delta}},\]

\[(A_2)\quad \max_{1 \leq k, l \leq r_n} \left| \sum_{t=1}^{n} u_{k, t}^{(n)} v_{l, t}^{(n)} - \delta_{k,l} \right| \leq \frac{C}{(\log(1 + r_n))^{1+\delta}}.\]

Let \((V^{(n)})\) be such a family and assume that \((U^{(n)}, V^{(n)})\) satisfies

\[(A_3)\quad \max_{1 \leq k, l \leq r_n} \left| \sum_{t=1}^{n} u_{k, t}^{(n)} v_{l, t}^{(n)} \right| \leq \frac{C}{(\log(1 + r_n))^{1+\delta}}.\]
Trigonometric weights

$(A_1)$ to $(A_3)$ are fulfilled in many situations. For example, if

$$r_n \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

and for the trigonometric weights

$$u_{k,t}^{(n)} = \sqrt{\frac{2}{n}} \cos \left( \frac{2\pi kt}{n} \right),$$

$$v_{k,t}^{(n)} = \sqrt{\frac{2}{n}} \sin \left( \frac{2\pi kt}{n} \right).$$

We shall make use of the sequences of weighted sums

$$S_{n,k} = \sum_{t=1}^{n} u_{k,t}^{(n)} X_t$$

and

$$T_{n,k} = \sum_{t=1}^{n} v_{k,t}^{(n)} X_t.$$
The sequence of weights

Uniform strong law

Central limit theorem

Large deviation principle

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**Uniform strong law**

Theorem (Bercu-Bryc, 2007)

Assume that \((X_n)\) is a sequence of independent random variables such that \(\mathbb{E}[X_n] = 0, \mathbb{E}[X_n^2] = 1, \sup \mathbb{E}[|X_n|^3] < \infty\).

If \((A_1)\) and \((A_2)\) hold, we have

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{r_n} \sum_{k=1}^{r_n} 1\{S_{n,k} \leq x\} - \Phi(x) \right| = 0 \quad \text{a.s.}
\]

In addition, under \((A_3)\), we also have for all \((x, y) \in \mathbb{R}^2\),

\[
\lim_{n \to \infty} \frac{1}{r_n} \sum_{k=1}^{r_n} 1\{S_{n,k} \leq x, T_{n,k} \leq y\} = \Phi(x) \Phi(y) \quad \text{a.s.}
\]
Corollary

The result of Bose and Mitra on empirical spectral distributions of symmetric circulant random matrices holds with probability one

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = 0 \quad \text{a.s.} \]

Corollary

The result of Kokoszka and Mikosch on empirical distributions of periodograms at Fourier frequencies holds with probability one if \( \sup \mathbb{E}[|X_n|^3] < \infty \)

\[ \lim_{n \to \infty} \sup_{x \geq 0} |F_n(x) - (1 - \exp(-x))| = 0 \quad \text{a.s.} \]
In all the sequel, we only deal with \textbf{trigonometric weights}. We shall make use of Sakhanenko’s strong approximation lemma.

\textbf{Definition.} A sequence \((X_n)\) of independent random variables satisfies Sakhanenko’s condition if \(\mathbb{E}[X_n] = 0, \mathbb{E}[X_n^2] = 1\) and for some constant \(a > 0,\)

\[
(S) \quad \sup_{n \geq 1} a \mathbb{E}[|X_n|^3 \exp(a|X_n|)] \leq 1.
\]

\textbf{Remark.} Sakhanenko’s condition is stronger than Cramér’s condition as it implies for all \(|t| \leq a/3\)

\[
\sup_{n \geq 1} \mathbb{E}[\exp(tX_n)] \leq \exp(t^2).
\]
A keystone lemma

Lemma (Sakhanenko, 1984)

Assume that \((X_n)\) is a sequence of independent random variables satisfying \((S)\). Then, one can construct a sequence \((Y_n)\) of iid \(N(0, 1)\) random variables such that, if

\[
S_n = \sum_{t=1}^{n} X_t \quad \text{and} \quad T_n = \sum_{t=1}^{n} Y_t
\]

then, for some constant \(c > 0\),

\[
\mathbb{E} \left[ \exp \left( ac \max_{1 \leq t \leq n} |S_t - T_t| \right) \right] \leq 1 + na.
\]
Theorem (Bercu-Bryc, 2007)

Assume that \((X_n)\) is a sequence of independent random variables satisfying \((S)\). Suppose that \((\log n)^2 r_n^3 = o(n)\).

Then, for all \(x \in \mathbb{R}\),

\[
\frac{1}{\sqrt{r_n}} \sum_{k=1}^{r_n} (\mathbf{1}_{\{S_n,k \leq x\}} - \Phi(x)) \xrightarrow{L} \mathcal{N}(0, \Phi(x)(1 - \Phi(x))).
\]

In addition, we also have

\[
\sqrt{r_n} \sup_{x \in \mathbb{R}} \left| \frac{1}{r_n} \sum_{k=1}^{r_n} \mathbf{1}_{\{S_n,k \leq x\}} - \Phi(x) \right| \xrightarrow{L} +\infty \text{ as } n \to +\infty \text{ \(\mathcal{K}\)}
\]

where \(\mathcal{K}\) stands for the Kolmogorov-Smirnov distribution.
Remark. $\mathcal{K}$ is the distribution of the supremum of the absolute value of the Brownian bridge. For all $x > 0$,

$$
\mathbb{P}(\mathcal{K} \leq x) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k \exp(-2k^2x^2).
$$

Remark. One can observe that the CLT also holds if $(S)$ is replaced by the assumption that for some $p > 0$

$$
\sup_{n \geq 1} \mathbb{E}[|X_n|^{2+p}] < \infty,
$$

as soon as

$$
r_n^3 = o(n^{p/(2+p)}).
$$
Relative entropy

We are interested in the **large deviation principle** for the random empirical measure

\[ \mu_n = \frac{1}{r_n} \sum_{k=1}^{r_n} \delta_{S^n_k}. \]

The **relative entropy** with respect to the standard normal law with density \( \phi \) is given, for all \( \nu \in M_1(\mathbb{R}) \), by

\[ I(\nu) = \int_{\mathbb{R}} \log \frac{f(x)}{\phi(x)} f(x) dx \]

if \( \nu \) is absolutely continuous with density \( f \) and the integral exists and \( I(\nu) = +\infty \) otherwise.
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Theorem (Bercu-Bryc, 2007)

Assume that \((X_n)\) is a sequence of independent random variables satisfying \((S)\). Suppose that \(\log n = o(r_n), \ r_n^4 = o(n)\). Then, \((\mu_n)\) satisfies an LDP with speed \((r_n)\) and good rate function \(I\),

- **Upper bound:** for any closed set \(F \subset \mathcal{M}_1(\mathbb{R})\),

\[
\limsup_{n \to \infty} \frac{1}{r_n} \log \mathbb{P}(\mu_n \in F) \leq - \inf_{\nu \in F} I(\nu).
\]

- **Lower bound:** for any open set \(G \subset \mathcal{M}_1(\mathbb{R})\),

\[
\liminf_{n \to \infty} \frac{1}{r_n} \log \mathbb{P}(\mu_n \in G) \geq - \inf_{\nu \in G} I(\nu).
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