Estimation and Control for Stochastic Regression Models

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Plan

1. Introduction
   - Goals
   - Weighted least squares algorithm
   - Adaptative tracking control
   - Optimization
2. Strong law of large numbers
3. Linear regression models with adaptive control
4. Almost sure central limit theorem
5. Functional regression models with adaptive control

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Consider the stochastic regression model

$$X_{n+1} = \theta^t \Phi_n + U_n + \varepsilon_{n+1}$$

- $X_n \longrightarrow$ the system output,
- $\Phi_n \longrightarrow$ the regression vector,
- $U_n \longrightarrow$ the adaptive control that can be chosen,
- $\varepsilon_n \longrightarrow$ the dirven noise.

We have two goals

- Estimate the unknown parameter θ,
- Control the dynamic of the process (X_n).

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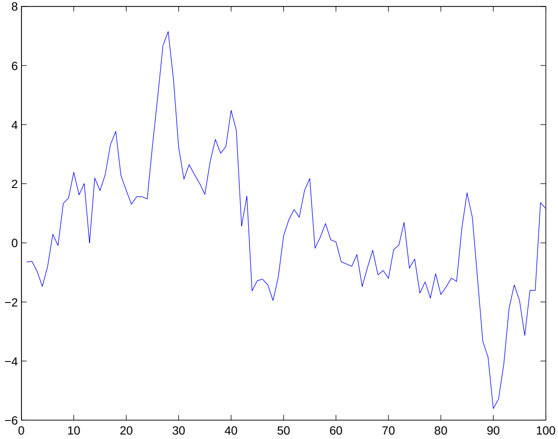
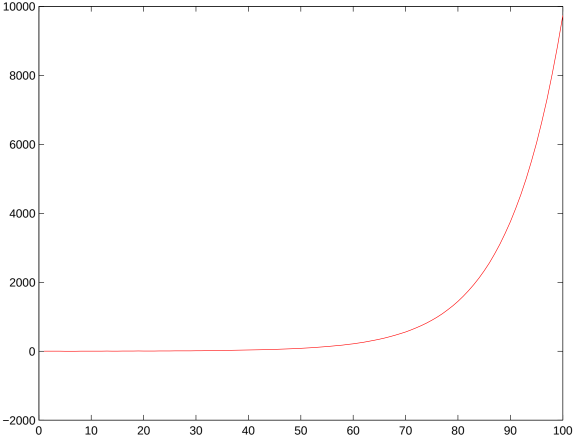
Simulation of stable autoregressive process $|\theta| < 1$



$$X_{n+1} = \theta X_n + \varepsilon_{n+1}$$

Simulation of explosive autoregressive process $|\theta| > 1$



$$X_{n+1} = \theta X_n + \varepsilon_{n+1}$$

The weighted least squares estimator $\widehat{\theta}_n$ of θ minimises

$$\Delta_n(\theta) = \frac{1}{2}\sum_{k=0}^{n-1} a_k(X_{k+1} - U_k - \theta^t\Phi_k)^2.$$

Consequently,

$$\widehat{\theta}_n = S_{n-1}^{-1}(a)\sum_{k=0}^{n-1} a_k\Phi_k(X_{k+1} - U_k),$$

$$S_n(a) = \sum_{k=0}^{n} a_k\Phi_k\Phi_k^t.$$

The **standard least squares estimator** is given by

$$a_n = 1.$$

Weighted least squares estimator

The **weighted least squares estimator** is given for $\gamma > 0$ by

$$a_n = \left(\frac{1}{\log s_n}\right)^{1+\gamma} \quad \text{where} \quad s_n = \sum_{k=0}^{n} \| \Phi_k \|^2 .$$

We always have the decomposition

$$\widehat{\theta}_n - \theta = S_{n-1}^{-1}(a) M_n(a)$$

$$M_n(a) = \sum_{k=0}^{n-1} a_k \Phi_k \varepsilon_{k+1}.$$

Adaptive tracking control

We wish to track, step by step, a given reference trajectory (x_n).
We make use of the **adaptive tracking control**

$$U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n.$$

Hence, the closed-loop system is given by

$$X_{n+1} - x_{n+1} = \pi_n + \varepsilon_{n+1}$$

where

$$\pi_n = (\theta - \widehat{\theta}_n)^t \Phi_n.$$

Optimization

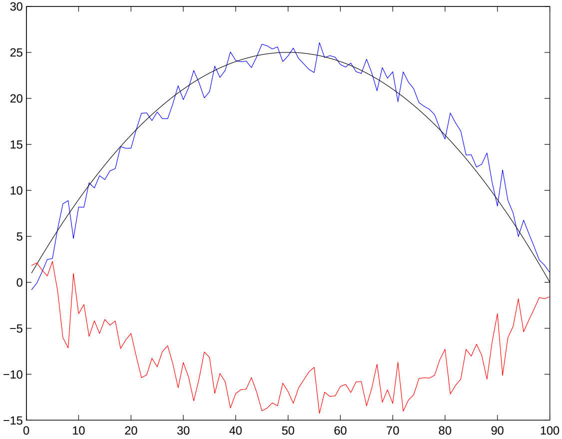
We assume that (ε_n) satisfies the **law of large numbers**

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k^2 = \sigma^2 \qquad \text{a.s.}$$

where $\sigma^2 > 0$. We shall say that the **tracking is optimal** if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (X_k - x_k)^2 = \sigma^2 \qquad \text{a.s.}$$

Simulation of controlled autoregressive process



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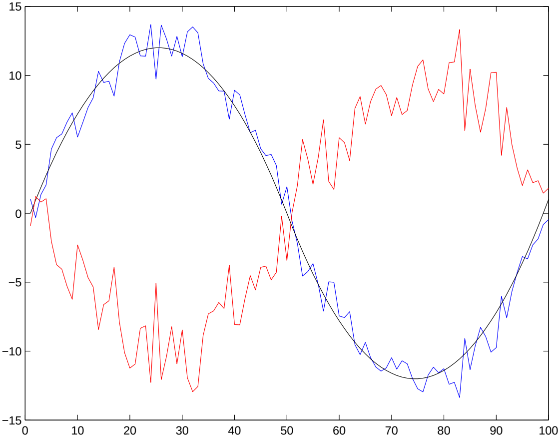

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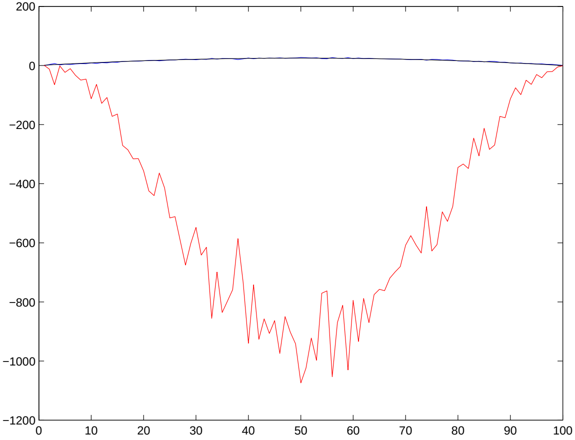

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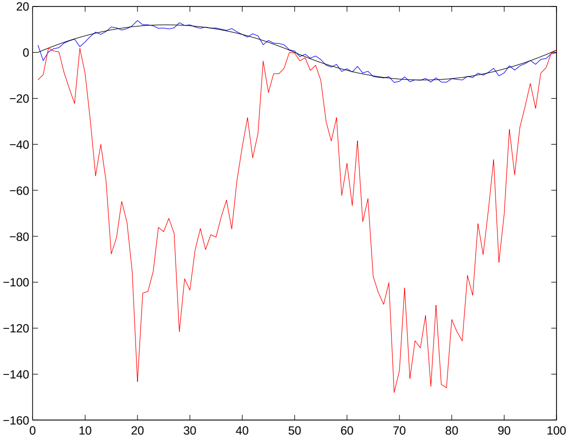

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Let (ε_n) be a sequence adapted to a filtration $\mathbb{F} = (\mathcal{F}_n)$ with

$$\mathbb{E}[\varepsilon_{n+1}|\mathcal{F}_n] = 0 \qquad \text{and} \qquad \mathbb{E}[\varepsilon_{n+1}^2|\mathcal{F}_n] = \sigma^2 > 0.$$

For a scalar sequence (Φ_n) adapted to $\mathbb{F}$, we investigate the asymptotic behavior of the **martingale transform**

$$M_n = \sum_{k=1}^{n} \Phi_{k-1}\varepsilon_k.$$

The **explosion coefficient** associated to (Φ_n) is given by

$$f_n = \frac{\Phi_n^2}{s_n} \qquad \text{where} \qquad s_n = \sum_{k=0}^{n} \Phi_k^2.$$

First law of large numbers

In all the sequel, we assume that

$$\lim_{n \to \infty} s_n = +\infty \qquad \text{a.s.}$$

Theorem (First LLN)

We have

$$\text{(LLN)} \qquad \lim_{n \to \infty} \frac{M_n}{s_{n-1}} = 0 \qquad \text{a.s.}$$

Remark. If (s_n) converges, then (M_n) also converges a.s.

Second law of large numbers

Theorem (Second LLN)

For $a > 2$, assume that

$$(A_1) \qquad \sup_{n \geqslant 0} \mathbb{E}\left[\,|\varepsilon_{n+1}|^a | \mathcal{F}_n\right] < \infty \qquad a.s.$$

Then, we have

$$\left(\frac{M_n^2}{s_{n-1}}\right) = \mathcal{O}(\log s_n) \qquad a.s.$$

$$\sum_{k=1}^{n} f_k \left(\frac{M_k^2}{s_{k-1}}\right) = \mathcal{O}(\log s_n) \qquad a.s.$$

Quadratic strong law

Theorem (Quadratic strong law)

If (A_1) holds and the explosion coefficient $f_n \longrightarrow 0$ a.s., we have

$$\text{(QSL)} \qquad \lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \left(\frac{M_k^2}{s_{k-1}} \right) = \sigma^2 \qquad \text{a.s.}$$

Remark. The **QSL** is exactly the convergence of the moment of order 2 in the **ASCLT** for (M_n).

Example

Let (ξ_n) be a sequence of **iid** random variables with $\mathbb{E}[\xi_n] = m$ and $\mathbb{V}\mathrm{ar}(\xi_n^2) = \sigma^2$. If $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, we have

(LLN) $$\lim_{n \to \infty} \frac{S_n}{n} = m \qquad \text{a.s.}$$

(QSL) $$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{S_k - km}{k} \Big)^2 = \sigma^2 \qquad \text{a.s.}$$

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Consider the autoregressive process with adaptive control

$$\boldsymbol{X_{n+1} = \theta^t \Phi_n + U_n + \varepsilon_{n+1},}$$

$$\boldsymbol{\Phi_n = (X_n, \cdots, X_{n-p+1})^t.}$$

We assume that the reference trajectory (x_n) satisfies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k^2 = \tau^2 \qquad \text{a.s.}$$

where $\tau \geqslant 0$. For all $n \geqslant 0$, let

$$S_n = \sum_{k=0}^{n} \Phi_k \Phi_k^t.$$

Lemma (Bercu)

Assume that (A_1) *holds. If* $\ell = \sigma^2 + \tau^2$, *we have*

$$\lim_{n\to\infty} \frac{S_n}{n} = \ell I_p \qquad a.s.$$

Theorem (Bercu)

If (A_1) *holds,* $\widehat{\theta}_n$ **converges almost surely** *to* θ

$$\| \widehat{\theta}_n - \theta \|^2 = \mathcal{O}\left(\frac{\log n}{n}\right) \qquad a.s.$$

In addition, the **tracking is optimal**

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - x_k)^2 = \sigma^2 \qquad a.s.$$

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In addition, the **tracking is optimal**

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k - x_k)^2 = \sigma^2 \qquad a.s.$$

Theorem (Bercu)

If (A_1) holds, we have

(CLT) $$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{\sigma^2}{\ell} I_p\Big),$$

(LIL) $$\limsup_{n\to\infty} \Big(\frac{n}{2\log\log n}\Big) \| \widehat{\theta}_n - \theta \|^2 = \frac{\sigma^2}{\ell} \qquad a.s.$$

(QSL) $$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \| \widehat{\theta}_k - \theta \|^2 = \frac{\sigma^2}{\ell} \qquad a.s.$$

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Central limit theorem

Let (ξ_n) be a sequence of **iid** random variables with $\mathbb{E}[\xi_n] = m$ and $\mathbb{V}\mathrm{ar}(\xi_n^2) = \sigma^2$. If $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, we have

$$\text{(CLT)} \qquad \frac{S_n - nm}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2).$$

In other words, for any function ***h* bounded continuous,**

$$\lim_{n\to\infty} \mathbb{E}\Big[h\Big(\frac{S_n - nm}{\sqrt{n}}\Big)\Big] = \int_{\mathbb{R}} h(x)dG(x)$$

where G stands for the Gaussian measure $\mathcal{N}(0, \sigma^2)$.

Almost sure central limit theorem

We also have

$$\text{(ASCLT)} \qquad \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\left(\frac{S_k - km}{\sqrt{k}}\right)} \Longrightarrow G \qquad \text{a.s.}$$

In other words, for any function ***h* bounded continuous,**

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We have already seen the **LLN** for the **martingale transform**

$$M_n = \sum_{k=1}^{n} \Phi_{k-1}\varepsilon_k.$$

The **explosion coefficient** associated with (Φ_n) is given by

$$f_n = \frac{\Phi_n^2}{s_n} \qquad \text{avec} \qquad s_n = \sum_{k=0}^{n} \Phi_k^2.$$

Theorem (Brown, Chaabane, Lifshits)

If (A_1) holds and the explosion coefficient $f_n \longrightarrow 0$ a.s., we have

(CLT)
$$\frac{M_n}{\sqrt{s_{n-1}}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2).$$

In addition, if
$$\sum_{n=1}^{\infty} f_n^{\gamma} < \infty \qquad a.s.$$

for some $\gamma > 0$, then we also have

(ASCLT)
$$\frac{1}{\log s_n} \sum_{k=1}^{n} f_k \, \delta_{\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)} \Longrightarrow G \qquad a.s.$$

Powers of martingales

For any function ***h* bounded continuous,** we have

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k h\Big(\frac{M_k}{\sqrt{s_{k-1}}}\Big) = \int_{\mathbb{R}} h(x)dG(x) \qquad \text{a.s.}$$

Definition. We shall say that (M_n) satisfies a **PASCLT** if this convergence holds for all **polynomial function *h*.**

Goal. Establish a **PASCLT** in order to study the stability of controlled functional regression models.

Let

$$v_n(p) = \frac{s_n^p - s_{n-1}^p}{s_n^p}.$$

Theorem (Bercu)

For some $p \geqslant 1$ and $a > 2p$, assume that

(A_p)
$$\sup_{n \geqslant 0} \mathbb{E}\left[\,|\varepsilon_{n+1}|^a | \mathcal{F}_n\right] < \infty \qquad a.s.$$

Then, we have

$$\left(\frac{M_n^2}{s_{n-1}}\right)^p = \mathcal{O}(\log s_n) \qquad a.s.$$

$$\sum_{k=1}^{n} v_k(p)\left(\frac{M_k^2}{s_{k-1}}\right)^p = \mathcal{O}(\log s_n) \qquad a.s.$$

Theorem (Bercu)

If (A_p) holds and the explosion coefficient $f_n \longrightarrow 0$ a.s., we have

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \Big(\frac{M_k^2}{s_{k-1}}\Big)^p = \frac{\sigma^{2p}(2p)!}{2^p p!} \qquad a.s.$$

Theorem (Bercu-Fort)

Assume that (A_p) holds for all $p \geqslant 1$ and $f_n \longrightarrow 0$ a.s. Then, (M_n) satisfies the **PASCLT**

$$\frac{1}{\log s_n} \sum_{k=1}^{n} f_k \, \delta_{\Big(\frac{M_k}{\sqrt{s_{k-1}}}\Big)} \Longrightarrow G \qquad a.s.$$

Explosive martingales

For all $p \geqslant 1$, let

$$\sigma_n(p) = \mathbb{E}[\varepsilon_{n+1}^p | \mathcal{F}_n] \qquad \text{a.s.}$$

Theorem (Bercu)

Assume that (A_p) holds and, for all $2 \leqslant q \leqslant 2p$, $\boldsymbol{\sigma_n(q) \to \sigma(q)}$ a.s. where $\sigma(q) = 0$ if q is odd. Also assume that $\boldsymbol{f_n \to f}$ a.s. where $\boldsymbol{0 < f < 1}$. Then, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{M_k^2}{s_{k-1}} \right)^p = l(p, f) \qquad a.s.$$

Gaussian limit distribution

The limit $I(p, f)$ is given by

$$I(p,\mathbf{f}) = \frac{1}{1-(1-\mathbf{f})^p} \sum_{k=1}^{p} C_{2p}^{2k} \mathbf{f}^k (1-\mathbf{f})^{p-k} \sigma(2k) I(p-k,\mathbf{f}).$$

This expression does not depend on f iff, for all $1 \leqslant k \leqslant p$,

$$\sigma(2k) = \frac{\sigma^{2k}(2k)!}{2^k k!}.$$

In that particular case, we have

$$I(p, f) = \frac{\sigma^{2p}(2p)!}{2^p p!} = I(p).$$

Explosive martingales

Theorem (Bercu-Fort)

Assume that (A_p) *holds for all* $p \geqslant 1$ *and* $\boldsymbol{f_n \to f}$ *a.s. where* $\boldsymbol{0 < f < 1}$. *For all* $p \geqslant 1$, *if* $\boldsymbol{l(p, f) = l(p)}$, *then* (M_n) *satisfies the* **PASCLT**

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)} \Longrightarrow G \qquad a.s.$$

Stable autoregressive process $|\theta| < 1$

Consider the stable autoregressive process

$$\boldsymbol{X_{n+1} = \theta X_n + \varepsilon_{n+1}}.$$

If (A_1) holds, we have $\boldsymbol{f_n \longrightarrow 0}$,

$$\lim_{n\to\infty} \frac{s_n}{n} = \frac{\sigma^2}{(1-\theta^2)} \qquad \text{a.s.}$$

In addition, $\boldsymbol{\widehat{\theta}_n \longrightarrow \theta}$ a.s. and

$$\boldsymbol{\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1-\theta^2)}.$$

If (A_p) holds for all $p \geqslant 1$, we have the **PASCLT**

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\sqrt{k}(\widehat{\theta}_k-\theta)} \Longrightarrow \mathcal{N}(0, 1-\theta^2) \qquad \text{a.s.}$$

In particular, for all $p \geqslant 1$, we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}k^{p-1}(\widehat{\theta}_k-\theta)^{2p} = \frac{(1-\theta^2)^p(2p)!}{2^p p!} \qquad \text{a.s.}$$

Explosive autoregressive process $|\theta| > 1$

If (H_1) holds, $\theta^{-n} X_n$ converges a.s. to the random variable

$$Y = X_0 + \sum_{k=1}^{\infty} \theta^{-k} \varepsilon_k.$$

In addition, $\boldsymbol{f_n \longrightarrow (\theta^2 - 1)/\theta^2}$,

$$\lim_{n\to\infty} \frac{s_n}{\theta^{2n}} = \frac{\theta^2 Y^2}{(\theta^2 - 1)} \qquad \text{a.s.}$$

Consequently, $\boldsymbol{\widehat{\theta}_n \longrightarrow \theta}$ a.s. Moreover, if (ε_n) is **gaussian** and $\mathcal{C}$ stands for the Cauchy distribution

$$\boldsymbol{|\theta|^n (\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{C}.}$$

If (ε_n) is **gaussian**, we have the **PASCLT**

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{|\theta|^k(\widehat{\theta}_k-\theta)} \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2(\theta^2-1)}{\gamma^2}\right) \qquad \text{a.s.}$$

In particular, for all $p \geqslant 1$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(|\theta|^k(\widehat{\theta}_k-\theta))^{2p} = \frac{\sigma^{2p}(\theta^2-1)^p(2p)!}{\gamma^{2p}2^p p!} \qquad \text{a.s.}$$

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Consider the functional autoregressive model of order $d \geqslant 1$

$$X_{n+1} = \theta f(X_n, \cdots, X_{n-d+1}) + U_n + \varepsilon_{n+1}.$$

We estimate θ by the **standard least squares estimator**

$$\widehat{\theta}_n - \theta = \frac{M_n}{s_{n-1}} \qquad \text{with} \qquad M_n = \sum_{k=1}^{n} \Phi_{k-1}\varepsilon_k.$$

We choose the **adaptive tracking control**

$$U_n = x_{n+1} - \widehat{\theta}_n \Phi_n$$

where $\Phi_n = f(X_n, \cdots, X_{n-d+1})$.

The functional class $\mathcal{C}(a, b)$

Let $\boldsymbol{\mathcal{C}(a, b)}$ with $a, b \in \mathbb{N}$ and $a \geqslant 1$ be the class of functions f from $\mathbb{R}^d$ to $\mathbb{R}$ such that, for all $x \in \mathbb{R}^d$,

$$c_1 + c_2 \parallel x \parallel^b \leqslant |f(x)| \leqslant c_3 + c_4 \parallel x \parallel^a$$

where $b \geqslant 1$ if $c_1 = 0$ and $b \geqslant 0$ otherwise.

Corollary (Bercu-Portier)

Assume that (H_a) *holds and* $\boldsymbol{f} \in \mathcal{C}(\boldsymbol{a}, \boldsymbol{b})$ *with* $\boldsymbol{a} < \boldsymbol{4}$. *Then, we have* $\widehat{\boldsymbol{\theta}}_{\boldsymbol{n}} \longrightarrow \theta$ *a.s. and*

$$(\widehat{\boldsymbol{\theta}}_{\boldsymbol{n}} - \theta)^2 = \mathcal{O}\left(\frac{\log n}{n}\right) \qquad a.s.$$

For all $1 \leqslant p \leqslant a$, *the* **tracking is stable of order** $\boldsymbol{p}$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k - x_k)^{2p} < \infty \qquad a.s.$$

If $\boldsymbol{\sigma_n(2p)} \longrightarrow \boldsymbol{\sigma(2p)}$ *a.s., the* **tracking is optimal of order** $\boldsymbol{p}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k - x_k)^{2p} = \sigma(2p) \qquad a.s.$$

The natural hypothesis (H_a)

Denote by $\mathcal{P}(a)$ the polynomial algebra with d variables and **total degree** $\leqslant a$ with $a \geqslant 1$. We assume that $f^2 \in \mathcal{P}(2a)$ together with

$$(H_a) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=d}^{n} f^2(\varepsilon_k + x_k, \ldots, \varepsilon_{k-d+1} + x_{k-d+1}) = \ell \qquad \text{a.s.}$$

where $\ell > 0$. Under (A_a) and (H_a) with $a < 4$, we can prove

$$\lim_{n\to\infty} \frac{s_n}{n} = \ell \qquad \text{a.s.}$$

Corollary (Bercu-Portier)

Under (A_a) *and* (H_a) *with* $a < 4$, *we have* $\widehat{\theta}_n \longrightarrow \theta$ *a.s. and*

(CLT)
$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{\sigma^2}{\ell}\Big),$$

(LIL)
$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)(\widehat{\theta}_n - \theta)^2 = \frac{\sigma^2}{\ell} \qquad a.s.$$

Moreover, for all $1 \leqslant p \leqslant a$, *we also have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{p-1}(\widehat{\theta}_k - \theta)^{2p} = \frac{\sigma^{2p}(2p)!}{\ell^p\, 2^p\, p!} \qquad a.s.$$

Polynomial autoregressive processes of order 2

Assume that $x_n \to 0$ and $\sigma_n(p) \to \sigma(p)$ a.s. for all $1 \leqslant p \leqslant 4$.
Consider the polynomial autoregressive processes

(1) $$X_{n+1} = \theta X_n^2 + U_n + \varepsilon_{n+1},$$

(2) $$X_{n+1} = \theta X_n(1 - X_n) + U_n + \varepsilon_{n+1},$$

(3) $$X_{n+1} = \theta X_n X_{n-1} + U_n + \varepsilon_{n+1}.$$

Then, the corollary holds with $\ell(1) = \sigma(4)$,

$$\ell(2) = \sigma(4) - 2\sigma(3) + \sigma(2), \qquad \ell(3) = \sigma(2)^2.$$

Polynomial autoregressive processes of order 3

Assume that $x_n \to 0$ and $\sigma_n(p) \to \sigma(p)$ a.s. for all $1 \leqslant p \leqslant 6$.
Consider the polynomial autoregressive processes

$$(4) \qquad X_{n+1} = \theta X_n^3 + U_n + \varepsilon_{n+1},$$

$$(5) \qquad X_{n+1} = \theta X_n^2(1 - X_n) + U_n + \varepsilon_{n+1},$$

$$(6) \qquad X_{n+1} = \theta X_n^2 X_{n-1} + U_n + \varepsilon_{n+1}.$$

Then, the corollary holds with $\ell(4) = \sigma(6)$,

$$\ell(5) = \sigma(6) - 2\sigma(5) + \sigma(4), \qquad \ell(6) = \sigma(4)\sigma(2).$$

Simulation of controlled autoregressive process

$$X_{n+1} = \theta X_n^2 + U_n + \varepsilon_{n+1}$$