Sharp large deviations for the fractional Ornstein-Uhlenbeck process

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Outline

The Gaussian example

Let (X_n) be a sequence of iid $\mathcal{N}(0, \sigma^2)$ random variables. If

$$S_n = \sum_{k=1}^{n} X_k$$

we clearly have $S_n \sim \mathcal{N}(0, \sigma^2 n)$. Consequently, for all $c > 0$

$$\mathbb{P}(S_n \geqslant nc) = \frac{\sigma}{c\sqrt{2\pi n}} \exp\Big(-\frac{c^2 n}{2\sigma^2}\Big)\Big[1 + o(1)\Big].$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant nc) = -\frac{c^2}{2\sigma^2}.$$

Question. What about the non-Gaussian case ?

Outline

Let (X_n) be a sequence of iid random variables with mean m. The **Fenchel-Legendre** dual of the log-Laplace L of (X_n) is

$$I(c) = \sup_{t \in \mathbb{R}} \{ct - L(t)\}.$$

Theorem (Cramer-Chernov)

The sequence (S_n/n) satisfies an **LDP** *with rate function I*

- **Upper bound:** *for any closed set $F \subset \mathbb{R}$*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in F\Big) \leqslant - \inf_F I,$$

- **Lower bound:** *for any open set $G \subset \mathbb{R}$*

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in G\Big) \geqslant - \inf_G I.$$

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On the Cramer-Chernov theorem

The rate function I is convex with $I(m) = 0$. For all $c > m$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant nc) = -I(c).$$

- **Gaussian:** If $X_n \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 > 0$,

$$I(c) = \frac{c^2}{2\sigma^2}.$$

- **Exponential:** If $X_n \sim \mathcal{E}(\lambda)$ with $\lambda > 0$,

$$I(c) = \begin{cases} \lambda c - 1 - \log(\lambda c) & \text{if } c > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Outline

On the Bahadur-Rao theorem

Theorem (Bahadur-Rao)

Assume that L is finite on all $\mathbb{R}$ and that the law of (X_n) is absolutely continuous. Then, for all $c > m$,

$$\mathbb{P}(S_n \geqslant nc) = \frac{\exp(-nI(c))}{\sigma_c t_c \sqrt{2\pi n}} \Big[1 + o(1) \Big]$$

where t_c is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$.

Remark. The core of the proof is the Berry-Esséen theorem.

Theorem (Bahadur-Rao)

(S_n/n) satisfies an **SLDP** *associated with L. For all $c > m$, it exists $(d_{c,k})$ such that for any $p \geqslant 1$ and n large enough*

$$\mathbb{P}(S_n \geqslant nc) = \frac{\exp(-nI(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right)\right].$$

Remark. The coefficients $(d_{c,k})$ may be explicitly calculated as functions of the derivatives $l_k = L^{(k)}(t_c)$. For example,

$$d_{c,1} = \frac{1}{\sigma_c^2}\left(\frac{l_4}{8\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} - \frac{l_3}{2t_c\sigma_c^2} - \frac{1}{t_c^2}\right).$$

Outline

Gaussian quadratic forms

Let (X_n) be a centered stationary real Gaussian process with spectral density $g \in \mathbb{L}^\infty(\mathbb{T})$

$$\mathbb{E}[X_n X_k] = \frac{1}{2\pi} \int_{\mathbb{T}} \exp(i(n-k)x) g(x)\, dx.$$

We are interested in the behavior of

$$\boldsymbol{W_n = \frac{1}{n} X^{(n)t} M_n X^{(n)}}$$

where (M_n) is a sequence of Hermitian matrices of order n and

$$X^{(n)} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Toeplitz and Cochran

Let $T_n(g)$ be the covariance matrix of $X^{(n)}$. Via Cochran,

$$W_n = \frac{1}{n}\sum_{k=1}^{n} \lambda_k^n Z_k^n$$

- $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of $T_n(g)^{1/2} M_n T_n(g)^{1/2}$
- $Z_1^n, \ldots, Z_n^n$ are iid with $\chi^2(1)$ distribution

The normalized cumulant generating function of W_n is given by

$$L_n(t) = \frac{1}{n}\log \mathbb{E}\Big[\exp(ntW_n)\Big] = -\frac{1}{2n}\sum_{k=1}^{n}\log(1-2t\lambda_k^n)$$

as soon as t belongs to $\Delta_n = \{t \in \mathbb{R} \ / \ 2t\lambda_k^n < 1\}$.

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Outline

LDP assumption. There exists $\varphi \in \mathbb{L}^\infty(\mathbb{T})$ not identically zero such that, if $m_\varphi = \text{essinf}\, \varphi$ and $M_\varphi = \text{esssup}\, \varphi$,

$$(H_1) \qquad m_\varphi \leqslant \lambda_k^n \leqslant M_\varphi$$

and, for all $h \in \mathbb{C}([m_\varphi, M_\varphi])$,

$$(H_1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} h(\lambda_k^n) = \frac{1}{2\pi} \int_{\mathbb{T}} h(\varphi(x))dx.$$

Under (H_1), the asymptotic cumulant generating function is

$$L(t) = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2t\varphi(x))dx$$

where t belongs to $\Delta = \{t \in \mathbb{R} \,/\, 2\max(m_\varphi t, M_\varphi t) < 1\}$.

Large deviation principle

The **Fenchel-Legendre** dual of L is given by

$$I(c) = \sup_{t \in \mathbb{R}} \left\{ ct + \frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2t\varphi(x))dx \right\}.$$

Theorem (Bercu-Gamboa-Lavielle)

If (H_1) holds, the sequence (W_n) satisfies an **LDP** *with rate function I. In particular, for all $c > \mu$*

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(W_n \geqslant c) = -I(c).$$

with $\mu = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{T}} \varphi(x)dx.$

Outline

Sharp large deviation results

SLDP assumption. There exists H such that, for all $t \in \Delta$

(H_2)
$$L_n(t) = L(t) + \frac{1}{n}H(t) + o\left(\frac{1}{n}\right)$$

where the remainder is uniform in t.

Theorem (Bercu-Gamboa-Lavielle)

Assume that (H_1) and (H_2) hold. Then, for all $c > \mu$

$$\mathbb{P}(W_n \geqslant c) = \frac{\exp(-nI(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}}\left[1 + o(1)\right]$$

where t_c is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$.

Sharp large deviation results

SLDP assumption. For $p \geqslant 1$, there exists $H \in \mathcal{C}^{2p+3}(\mathbb{R})$ such that, for all $t \in \Delta$ and for any $0 \leqslant k \leqslant 2p+3$

$$(H_2(p)) \qquad L_n^{(k)}(t) = L^{(k)}(t) + \frac{1}{n} H^{(k)}(t) + \mathcal{O}\left(\frac{1}{n^{p+2}}\right)$$

where the remainder is uniform in t.

Remark. Assumption $(H_2(p))$ is not really restrictive. It is fulfilled in many statistical applications.

Theorem (Bercu-Gamboa-Lavielle)

For $p \geqslant 1$, assume that (H_1) and $(H_2(p))$ hold. Then, (W_n) satisfies an **SLDP** *of order p associated with L and H. For all $c > \mu$, it exists $(d_{c,k})$ such that for n large enough*

$$\mathbb{P}(W_n \geqslant c)$$

$$= \frac{\exp(-nI(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right].$$

Remark. The coefficients $(d_{c,k})$ may be given as functions of the derivatives $l_k = L^{(k)}(t_c)$ and $h_k = H^{(k)}(t_c)$. For example,

$$d_{c,1} = \frac{1}{\sigma_c^2}\left(-\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_3 h_1}{2\sigma_c^2} - \frac{5 l_3^2}{24\sigma_c^4} + \frac{h_1}{t_c} - \frac{l_3}{2 t_c \sigma_c^2} - \frac{1}{t_c^2} \right).$$

Outline

Stable autoregressive process

Consider the stable autoregressive process

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \qquad |\theta| < 1$$

where (ε_n) is iid $\mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$. If X_0 is independent of (ε_n) with $\mathcal{N}(0, \sigma^2/(1-\theta^2))$ distribution, (X_n) is a centered stationary Gaussian process with spectral density given, for all $x \in \mathbb{T}$, by

$$g(x) = \frac{\sigma^2}{1 + \theta^2 - 2\theta \cos x}.$$

Let $\widehat{\theta}_n$ be the **least squares** estimator of the parameter θ

$$\widehat{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\sum_{k=1}^{n} X_{k-1}^2}.$$

We have $\widehat{\theta}_n \longrightarrow \theta$ a.s. and $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1-\theta^2)$. One can also estimate θ by the **Yule-Walker** estimator

$$\widetilde{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\sum_{k=0}^{n} X_k^2}.$$

$$a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \qquad \text{and} \qquad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}.$$

Theorem (Bercu-Gamboa-Rouault)

- $(\widehat{\theta}_n)$ *satisfies an* **LDP** *with rate function*

$$J(c) = \begin{cases} \frac{1}{2}\log\left(\frac{1+\theta^2-2\theta c}{1-c^2}\right) & \text{if } c \in [a,b], \\ \log|\theta - 2c| & \text{otherwise.} \end{cases}$$

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$$I(c) = \begin{cases} \frac{1}{2}\log\left(\frac{1+\theta^2-2\theta c}{1-c^2}\right) & \text{if } c \in]-1,1[, \\ +\infty & \text{otherwise.} \end{cases}$$

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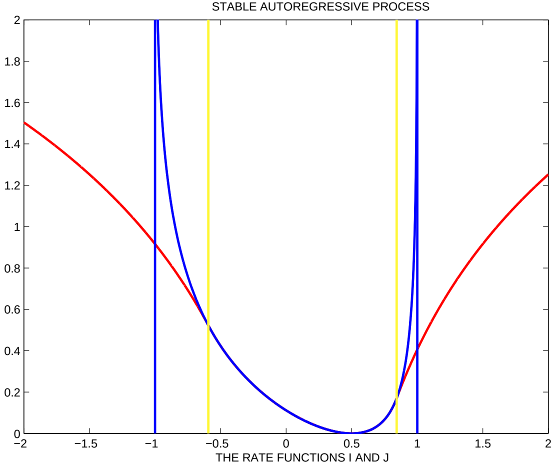
Least squares and Yule-Walker



Yule-Walker

Theorem (Bercu-Gamboa-Lavielle)

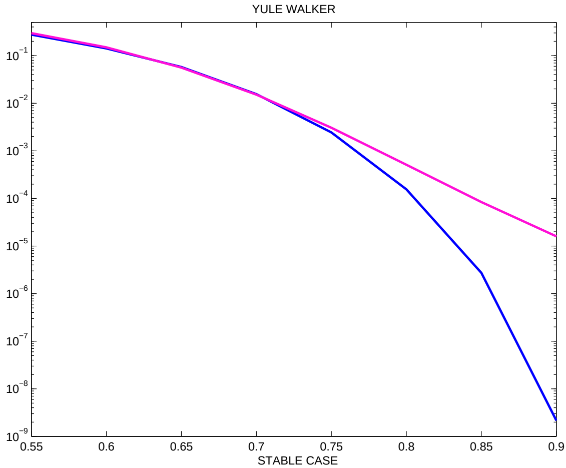
The sequence $(\widetilde{\theta}_n)$ satisfies an **SLDP**. *For all $c \in \mathbb{R}$ with $c > \theta$ and $|c| < 1$, it exists a sequence $(d_{c,k})$ such that for any $p \geqslant 1$ and n large enough*

$$\mathbb{P}(\widetilde{\theta}_n \geqslant c) = \frac{\exp(-nI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}} \right) \right]$$

$$t_c = \frac{c(1+\theta^2) - \theta(1-c^2)}{1-c^2}, \qquad \sigma_c^2 = \frac{1-c^2}{(1+\theta^2-2\theta c)^2},$$

$$H(c) = -\frac{1}{2} \log \left(\frac{(1-c\theta)^4}{(1-\theta)^2(1+\theta^2-2\theta c)(1-c^2)^2} \right).$$

Yule-Walker


YULE WALKER
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0.55
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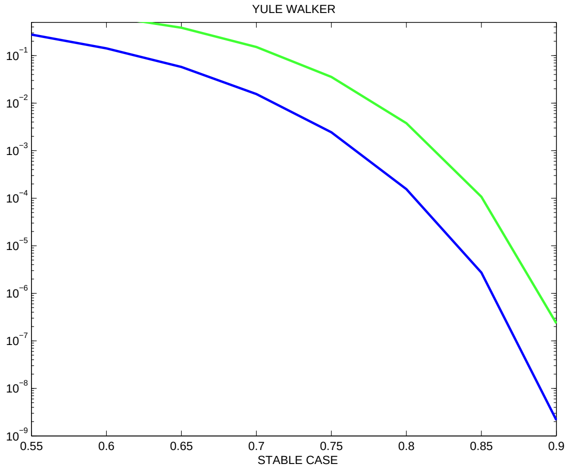
0.65
0.7
0.75
0.8
0.85
0.9
STABLE CASE

Yule-Walker


YULE WALKER
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STABLE CASE

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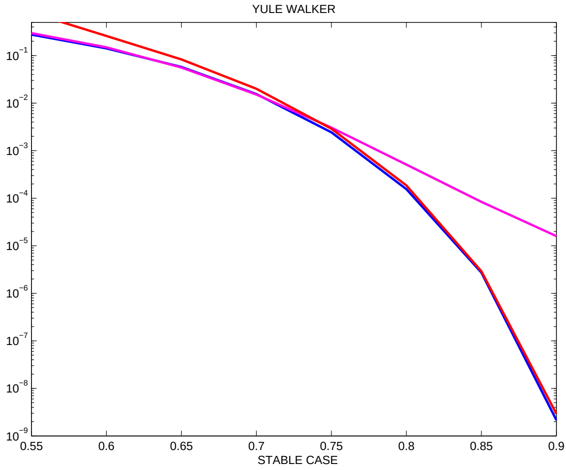
YULE WALKER
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0.65
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0.75
0.8
0.85
0.9
STABLE CASE

Explosive autoregressive process

Consider the explosive autoregressive process

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \qquad |\theta| > 1$$

where (ε_n) is iid $\mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$. The **Yule-Walker** estimator satisfies $\widetilde{\theta}_n \longrightarrow 1/\theta$ a.s. together with

$$|\theta|^n \Big(\widetilde{\theta}_n - \frac{1}{\theta} \Big) \xrightarrow{\mathcal{L}} \frac{(\theta^2 - 1)}{\theta^2} \, \mathcal{C}$$

where $\mathcal{C}$ stands for the Cauchy distribution.

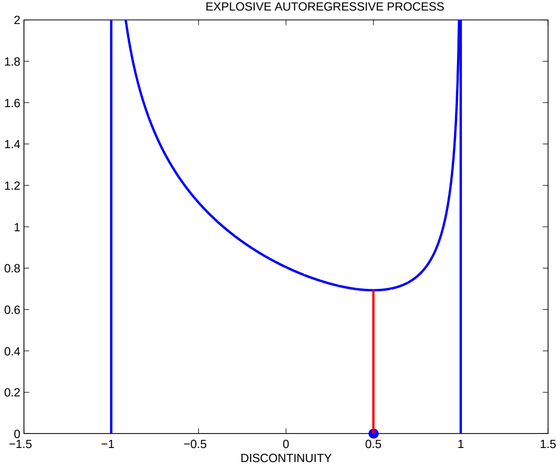
Explosive autoregressive process

Theorem (Bercu)

The sequence $(\widetilde{\theta}_n)$ *satisfies an* **LDP** *with rate function*

$$I(c) = \begin{cases} \dfrac{1}{2}\log\left(\dfrac{1+\theta^2-2\theta c}{1-c^2}\right) & \text{if } c \in]-1,1[, \ c \neq 1/\theta, \\ 0 & \text{if } c = 1/\theta, \\ +\infty & \text{otherwise.} \end{cases}$$

Discontinuity



Yule-Walker

Theorem (Bercu)

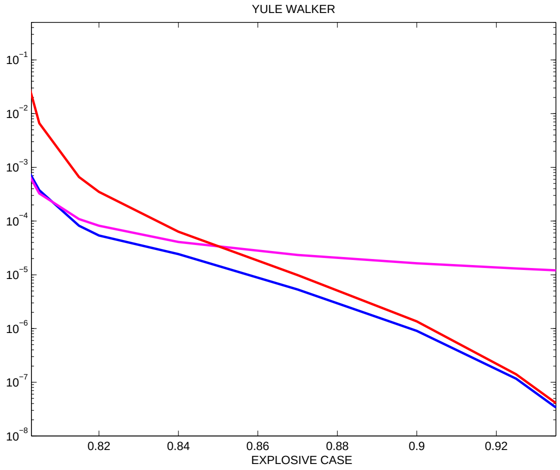
The sequence $(\widetilde{\theta}_n)$ satisfies an **SLDP**. *For all $c \in \mathbb{R}$ with $c > 1/\theta$ and $|c| < 1$, it exists a sequence $(d_{c,k})$ such that for any $p \geqslant 1$ and n large enough*

$$\mathbb{P}(\widetilde{\theta}_n \geqslant c) = \frac{\exp(-nI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}} \right) \right]$$

$$t_c = \frac{(\theta c - 1)(\theta - c)}{1 - c^2}, \qquad \sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},$$

$$H(c) = -\frac{1}{2} \log \left(\frac{(\theta c - 1)^2}{(1 + \theta^2 - 2\theta c)(1 - c^2)} \right).$$

Yule-Walker


YULE WALKER
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0.82
0.84
0.86
0.88
0.9
0.92
EXPLOSIVE CASE

Outline

Stable Ornstein-Uhlenbeck process

Consider the stable Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dB_t, \qquad \theta < 0$$

with initial state $X_0 = 0$, where (B_t) is a standard Brownian motion. We are interested in **SLDP** for **the energy**

$$S_T = \int_0^T X_t^2 \, dt$$

and **the maximum likelihood** estimator of θ

$$\widehat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt} = \frac{X_T^2 - T}{2 \int_0^T X_t^2 \, dt}.$$

Strong laws and Central Limit Theorems

Theorem

We have the **SLLN** $S_T/T \longrightarrow -1/2\theta$ *a.s. Moreover, we have the* **CLT**

$$\frac{1}{\sqrt{T}}\left(S_T + \frac{T}{2\theta}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, -\frac{1}{2\theta^3}\right).$$

Theorem

We have the **SLLN** $\widehat{\theta}_T \longrightarrow \theta$ *a.s. Moreover, we have the* **CLT**

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, -2\theta\right).$$

Stable Ornstein-Uhlenbeck process

Theorem (Bryc-Dembo)

The sequence (S_T/T) *satisfies an* **LDP** *with rate function*

$$I(c) = \begin{cases} \dfrac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem (Florens-Pham)

The sequence $(\widehat{\theta}_T)$ *satisfies an* **LDP** *with rate function*

$$I(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c < \theta/3, \\ 2c - \theta & \text{otherwise.} \end{cases}$$

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

The sequence (S_T/T) *satisfies an* **SLDP**. *For all* $c > -1/2\theta$, *it exists a sequence* $(b_{c,k})$ *such that, for any* $p \geqslant 1$ *and* T *large enough*

$$\mathbb{P}(S_T \geqslant cT) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

$$t_c = \frac{4\theta^2 c^2 - 1}{8c^2}, \qquad H(c) = -\frac{1}{2}\log\left(\frac{1}{2}(1 - 2\theta c)\right)$$

$\sigma_c^2 = 4c^3$.

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

The sequence $(\widehat{\theta}_T)$ *satisfies an* **SLDP**. *For all* $\theta < c < \theta/3$, *it exists a sequence* $(d_{c,k})$ *such that, for any* $p \geqslant 1$ *and* T *large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}} \right) \right]$$

$$t_c = \frac{c^2 - \theta^2}{2c}, \qquad H(c) = -\frac{1}{2} \log \left(\frac{(c+\theta)(3c-\theta)}{4c^2} \right)$$

$\sigma_c^2 = -1/2c$. *Similar expansion holds for* $c > \theta/3$ *with* $c \neq 0$.

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

- *For $c = 0$, it exists a sequence (b_k) such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant 0) = \frac{\exp(\theta T)}{\sqrt{\pi T}\sqrt{-\theta}}\left[1 + \sum_{k=1}^{p} \frac{b_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right].$$

- *For $c = \theta/3$, it exists a sequence (d_k) such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c))}{4\pi T^{1/4}\tau_\theta}\left[1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]$$

where $\tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4)$.

Outline

Fractional Ornstein-Uhlenbeck process

Consider the fractional Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dB_t^H, \qquad \theta < 0$$

where (B_t^H) is a **fractional Brownian motion** with **Hurst parameter** $0 < H < 1$, (B_t^H) is a Gaussian process with continuous paths such that $B_0^H = 0$, $\mathbb{E}[B_t^H] = 0$ and

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).$$

The weighting function

$$w(t,s) = w_H^{-1} s^{-H+1/2}(t-s)^{-H+1/2}$$

plays a crucial role for stochastic calculus associated with (B_t^H).

A Gaussian martingale

For all $t > 0$ and $H > 1/2$, let

$$M_t = \int_0^t w(t,s)\, dB_s^H.$$

Then, (M_t) is a Gaussian martingale with quadratic variation

$$< M >_t = \frac{t^{2-2H}}{\lambda_H}$$

$$\lambda_H = \frac{8H(1-H)\Gamma(1-2H)\Gamma(H+1/2)}{\Gamma(1/2-H)}$$

where Γ stands for the classical gamma function.

For all $t > 0$, let

$$Y_t = \int_0^t w(t,s)\, dX_s$$

$$Q_t = \frac{\ell_H}{2}\left(t^{2H-1} Y_t + \int_0^t s^{2H-1}\, dY_s \right)$$

where $\ell_H = \lambda_H/(2(1-H))$. **The energy** is given by

$$S_T = \int_0^T Q_t^2 \, d\!<M>_t$$

while **the maximum likelihood** estimator of θ is

$$\widehat{\theta}_T = \frac{\int_0^T Q_t \, dY_t}{\int_0^T Q_t^2 \, d\!<M>_t}.$$

Strong laws and Central Limit Theorems

Theorem

We have the **SLLN** $S_T/T \longrightarrow -1/2\theta$ *a.s. Moreover, we have the* **CLT**

$$\frac{1}{\sqrt{T}}\left(S_T + \frac{T}{2\theta}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, -\frac{1}{2\theta^3}\right).$$

Theorem

We have the **SLLN** $\widehat{\theta}_T \longrightarrow \theta$ *a.s. Moreover, we have the* **CLT**

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, -2\theta\right).$$

The energy

Theorem (Bercu-Coutin-Savy)

The sequence (S_T/T) *satisfies an* **LDP** *with rate function*

$$I(c) = \begin{cases} \dfrac{(2\theta c+1)^2}{8c} & \text{if } 0 < c \leqslant -\dfrac{1}{2\theta\delta_H}, \\ \dfrac{c\theta^2}{2}(1-\delta_H^2) + \dfrac{\theta}{2}(1-\delta_H) & \text{if } c \geqslant -\dfrac{1}{2\theta\delta_H}, \\ +\infty & \text{otherwise.} \end{cases}$$

where $\delta_H = (1 - \sin(\pi H))/(1 + sin(\pi H))$.

Remark. In the particular case $H = 1/2$, $\delta_H = 0$ and the **LDP** for (S_T/T) is exactly the one established by Bryc and Dembo.

Theorem (Bercu-Coutin-Savy)

The sequence (S_T/T) *satisfies an* **SLDP**. *For all* $c > -1/(2\theta)$ *with* $c < -1/(2\theta\delta_H)$, *it exists a sequence* $(b^H_{c,k})$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbb{P}(S_T \geqslant cT) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{b^H_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where $\sigma_c^2 = 4c^3$,

$$t_c = \frac{4\theta^2 c^2 - 1}{8c^2}, \qquad J(c) = -\frac{1}{2}\log\left(\frac{1-2\theta c}{2}\right),$$

$$K_H(c) = -\frac{1}{2}\log\left(\frac{(1+\sin(\pi H))(1+2\theta c\delta_H)}{2\sin(\pi H)}\right).$$

Theorem (Bercu-Coutin-Savy)

For all $c > -1/(2\theta\delta_H)$*, it exists a sequence* $(d_{c,k}^H)$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbb{P}(S_T \geqslant cT) = \frac{\exp(-TI(c) + P_H(c) + Q_H(c))}{\sigma_H t_H \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where $\sigma_H^2 = -1/2\theta^3\delta_H^3$,

$$t_H = \frac{\theta^2(1-\delta_H^2)}{2}, \qquad P_H(c) = -\frac{1}{2}\log\left(\frac{-(1+2\theta c\delta_H)}{4\delta_H \sin(\pi H)}\right),$$

$$Q_H(c) = \frac{(2H-1)^2 \sin(\pi H)(1+2\theta c\delta_H)}{2(1-(\sin(\pi H))^2)}.$$

Theorem (Bercu-Coutin-Savy)

For $c = -1/(2\theta\delta_H)$, it exists a sequence (d_k^H) such that, for any $p > 0$ and T large enough

$$\mathbb{P}(S_T \geqslant cT) = \frac{\exp(-TI(c) + K_H)\Gamma(1/4)}{2\pi\sigma_H t_H T^{1/4}} \left[1 + \sum_{k=1}^{2p} \frac{d_k^H}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]$$

where

$$K_H = \frac{1}{2}\log(\delta_H \sin(\pi H)) + \frac{1}{4}\log(-\theta\delta_H).$$

the maximum likelihood estimator

Theorem (Bercu-Coutin-Savy)

The sequence $(\widehat{\theta}_T)$ satisfies an **LDP** *with rate function*

$$I(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c < \theta/3, \\ 2c - \theta & \text{otherwise.} \end{cases}$$

Remark. One can observe that $(\widehat{\theta}_T)$ shares the same LDP than the one established by Florens-Landais and Pham for $H = 1/2$.

Theorem (Bercu-Coutin-Savy)

The sequence $(\widehat{\theta}_T)$ *satisfies an* **SLDP**. *For all* $\theta < c < \theta/3$, *it exists a sequence* $(b^H_{c,k})$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c t_c \sqrt{2\pi T}}$$

$$\left[1 + \sum_{k=1}^{p} \frac{b^H_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where $\sigma_c^2 = -1/2c$, $p_H = (1 - \sin(\pi H))/sin(\pi H)$,

$$t_c = \frac{c^2 - \theta^2}{2c}, \qquad J(c) = -\frac{1}{2}\log\left(\frac{(c+\theta)(3c-\theta)}{4c^2}\right),$$

$$K_H(c) = -\frac{1}{2}\log\left(1 + p_H\frac{(c-\theta)^2}{4c^2}\right).$$

Theorem (Bercu-Coutin-Savy)

For all $c > \theta/3$ with $c \neq 0$, it exists a sequence $(d_{c,k}^H)$ such that, for any $p > 0$ and T large enough,

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c) + P(c))\sqrt{\sin(\pi H)}}{\sigma^c t^c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

where

$$t^c = 2(c-\theta), \qquad (\sigma^c)^2 = \frac{c^2}{2(2c-\theta)^3},$$

$$P(c) = -\frac{1}{2}\log\left(\frac{(c-\theta)(3c-\theta)}{4c^2}\right)$$

Theorem (Bercu-Coutin-Savy)

- *For $c = 0$, it exists a sequence (b_k^H) such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant 0) = \frac{\exp(\theta T)\sqrt{\sin(\pi H)}}{\sqrt{\pi T}\sqrt{-\theta}}\left[1 + \sum_{k=1}^{p} \frac{b_k^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right].$$

- *For $c = \theta/3$, it exists a sequence (d_k^H) such that, for any $p \geqslant 1$ and T large enough, and $\tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4)$,*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c))\sqrt{\sin(\pi H)}}{4\pi T^{1/4}\tau_\theta}$$
$$\left[1 + \sum_{k=1}^{2p} \frac{d_k^H}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right].$$