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Limiting distribution for subcritical controlled branching processes with random control function[☆]

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Abstract

This paper concerns the controlled branching process with random control function introduced by Yanev (Theor. Prob. Appl. 20 (1976) 421). Some relationships between its probability generating functions are established and the convergence in distribution of the population size to a nondegenerate and finite random variable is investigated.

Keywords: Controlled branching processes; Subcritical branching processes; Probability generating functions; Limiting distribution

1. Introduction

Introduced by Yanev (1976), the controlled branching process (CBP) with random control function is a stochastic model defined in the form

$$Z_0 = N, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj}, \quad n = 0, 1, \ldots , \tag{1}$$

where the empty sum is considered to be 0, N is a positive integer and $\{X_{nj} : n=0,1,\ldots;\ j=1,2,\ldots\}$, $\{\phi_n(k) : n, k = 0, 1, \ldots\}$ are independent sets of nonnegative integer-valued random variables defined on the same probability space. The variables X_{nj} are independent and identically distributed with

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common probability law $p_k := P(X_{01} = k)$, $k = 0, 1, \ldots$, called offspring probability distribution. For $n=0, 1, \ldots$, $\{\phi_n(k)\}_{k=0}^{\infty}$ are independent stochastic processes with identical one dimensional probability distributions. For simplicity we shall denote, when necessary, by $P_k(i) := P(\phi_0(k)=i)$, $k, i = 0, 1, \ldots$.

Intuitively, X_{nj} is the number of descendants originated by the jth individual of the generation n and Z_{n+1} represents the total number of individuals in the $(n+1)$th generation. The individuals originate descendants independently with the same offspring probability distribution for each generation, but when in a certain generation n there are k individuals, the random variable $\phi_n(k)$ produces a control in the process in such a way that, if $\phi_n(k) = j$, then j progenitors will take part in the reproduction process that will determine Z_{n+1}. This branching model could reasonably describe the probabilistic evolution of populations in which, for various reasons of an environmental, social or other nature, a random mechanism establishes the number of progenitors who participate in each generation.

From (1), it can be easily verified that $\{Z_n\}_{n=0}^{\infty}$ is a homogeneous Markov chain on the nonnegative integers. To avoid trivialities, we will assume that $p_0 < 1$. Next result provides a necessary and sufficient condition for 0 to be and absorbing state.

Proposition 1.1. *Let $\{Z_n\}_{n=0}^{\infty}$ be a CBP with a random control function. Then, 0 is an absorbing state if and only if $P(\phi_0(0) = 0) = 1$.*

Proof. Suppose that $P(\phi_0(0) = 0) = 1$ then, taking into account the process definition, we deduce that $P(Z_{n+r} = 0 \mid Z_n = 0) = 1$, $r = 1, 2, \ldots$, and therefore 0 is an absorbing state.

On the other hand, $P(\phi_0(0)=0) < 1$ implies the existence of $t > 0$ such that $P_0(t) > 0$. Moreover, since $p_0 < 1$, there exists $k \geqslant 1$ such that $p_k > 0$. Consequently, we get that

$$P(Z_{n+1} = kt \mid Z_n = 0) \geqslant P\left(\sum_{j=1}^{t} X_{nj} = kt\right) P_0(t) \geqslant (p_k)^t P_0(t) > 0$$

and we derive that 0 is not an absorbing state. □

Remark 1.1. Considering that $P(\phi_0(0)=0)=1$, Yanev (1976) proved that if $p_0 > 0$ or $P(\phi_0(k) = 0) > 0$, $k = 1, 2, \ldots$ then, the positive integers are transient states, so from well-known results on Markov chains (e.g., see Feller, 1968) it is verified the following classical duality extinction–explosion in branching process theory:

$$P(\lim_{n\to\infty} Z_n = 0) + P(\lim_{n\to\infty} Z_n = \infty) = 1.$$

Assuming that $\phi_n(k) = \alpha_n k(1 + o(1))$ almost surely, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of independent, identically distributed and nonnegative random variables, sufficient conditions for the extinction or nonextinction of the process were also determined by Yanev (1976). With similar requirements about the control variables $\phi_n(k)$ but omitting the assumption of independence in the sequence $\{\alpha_n\}_{n=0}^{\infty}$, sufficient conditions for the extinction were provided by Bruss (1980). Imposing certain hypotheses to the α-th absolute moments $(1 < \alpha \leqslant 2)$ of the offspring probability distribution and the control variables, a limiting result was established by Nakagawa (1994). Recently, González et al. (2002), under a more general context, have determined necessary and sufficient conditions for the almost sure extinction and have investigated the probabilistic evolution of Z_n suitably normed. Finally, concerning with its inferential theory, Dion and Essebbar (1995) obtained estimators for the main parameters of a CBP with some particular control variables.

This work aims to continue the research about this controlled branching model. In Section 2, some relationships between the probability generating functions involved in the process are determined. Section 3 is devoted to investigating the convergence in distribution of Z_n, as $n \to \infty$, to a positive, nondegenerate and finite random variable. In order to study this question, which has not previously been considered in the literature about this stochastic model, some transition properties will be established.

2. Probability generating functions and moments

Let $\{Z_n\}_{n=0}^{\infty}$ be a CBP with random control function. Let us consider the probability generating functions:

$$f(s) := E[s^{X_{01}}],\ g_k(s) := E[s^{\phi_0(k)}],\ F_n(s) := E[s^{Z_n}],\ h_k(s) := E[s^{Z_{n+1}} \mid Z_n = k],$$

where $0 \leqslant s \leqslant 1$, $n = 0, 1, \ldots$; $k = 1, 2, \ldots$.

The relationships between the functions $F_n(\cdot)$, $h_k(\cdot)$, $g_k(\cdot)$ and $f(\cdot)$ provided in the next result can be easily verified.

Proposition 2.1. *For* $0 \leqslant s \leqslant 1$,

(i) $F_n(s) = E[h_{Z_{n-1}}(s)], \quad n = 1, 2, \ldots,$
(ii) $h_k(s) = g_k(f(s)), \quad k = 0, 1, \ldots.$

To determine some conditional and unconditional moments of Z_{n+1}, it will be useful to introduce the functions $\varepsilon(\cdot)$ and $\sigma^2(\cdot)$ on the nonnegative integers:

$$\varepsilon(k) := E[\phi_n(k)], \quad \sigma^2(k) := \mathrm{Var}[\phi_n(k)], \quad k = 0, 1, \ldots .$$

We assume that $\sigma^2(k) < \infty$, $k = 0, 1, \ldots$ and that the offspring probability distribution is such that $\tau^2 := \sum_{k=0}^{\infty} (k - m)^2 p_k < \infty$, where $m := \sum_{k=0}^{\infty} k p_k$.

From Proposition 2.1, it is matter of some straightforward calculation to prove:

Proposition 2.2. *For* $n = 0, 1, \ldots$,

(i) $E[Z_{n+1} \mid \mathscr{F}_n] = m\varepsilon(Z_n) \quad$ *a.s.*,
(ii) $\mathrm{Var}[Z_{n+1} \mid \mathscr{F}_n] = m^2\sigma^2(Z_n) + \tau^2\varepsilon(Z_n) \quad$ *a.s.*

where $\mathscr{F}_n$ *is the* σ*-algebra generated by the variables* $Z_0, \ldots, Z_n$.

Notice that, as consequence of Proposition 2.2, we have that

$$E[Z_{n+1}] = mE[\varepsilon(Z_n)], \quad \mathrm{Var}[Z_{n+1}] = \tau^2 E[\varepsilon(Z_n)] + m^2(E[\sigma^2(Z_n)] + \mathrm{Var}[\varepsilon(Z_n)]).$$

3. Limiting distribution in the subcritical case

We now consider a CBP with a random control function $\{Z_n\}_{n=0}^{\infty}$ verifying the following assumptions about the offspring probability distribution and the control variables:

$$\text{A1}: \quad p_0 > 0, \quad p_0 + p_1 < 1$$

$$\text{A2}: \quad P(\phi_0(i) > i) > 0, \quad i = 0, 1, \ldots .$$

In this section, the limiting behaviour of $\{Z_n\}_{n=0}^{\infty}$ will be investigated. Under A1 and A2, assuming certain limiting condition on the sequence $\{k^{-1}\varepsilon(k)\}_{k=1}^{\infty}$, we will deduce analogous results to those obtained for subcritical branching processes with immigration (see Jagers, 1975). Note that under A2, according to Proposition 1.1, 0 is not an absorbing state. Firstly, it will be necessary to verify that $\{Z_n\}_{n=0}^{\infty}$ is an irreducible and aperiodic Markov chain.

Let $S := \{i \geqslant 0 : P(Z_n = i) > 0 \quad \text{for some } n \geqslant 0\}$ be the state space of $\{Z_n\}_{n=0}^{\infty}$. For simplicity, we will write $p_{ij}^{(n)} := P(Z_{m+n} = j \mid Z_m = i)$, $n = 1, 2, \ldots$ where $i, j \in S$ and $P(Z_m = i) > 0$ for some $m \geqslant 0$. When $n = 1$ such a probability will be written simply p_{ij}. Finally, given $i \in S$, $C_i := \{j \in S : p_{ij}^{(n)} > 0 \quad \text{for some } n \geqslant 1\}$ will denote the set of the states which i leads to.

Proposition 3.1. *Let $\{Z_n\}_{n=0}^{\infty}$ be a CBP with a random control function verifying* A1 *and* A2. *Then,*

(i) *Given a nonnegative integer v there exists $j > v$ such that $j \in C_0$,*
(ii) $C_i = C_0$ for all $i \in S$,
(iii) *$\{Z_n\}_{n=0}^{\infty}$ is an irreducible and aperiodic Markov chain.*

Proof. (i) From A1, there exists $k > 1$ such that $p_k > 0$. Moreover $P(\phi_n(0) > 0) > 0$ guarantees the existence of $d_0 > 0$ such that $P_0(d_0) > 0$. Consequently,

$$p_{0k} = P\left(\sum_{l=1}^{\phi_n(0)} X_{nl} = k\right) \geqslant p_k\, p_0^{d_0 - 1} P_0(d_0) > 0. \tag{2}$$

Now, taking into account A2 and considering an inductive procedure on t, it is obtained that

$$P(Z_{n+t} = k^{t+1} \mid Z_n = k) > 0, \quad t = 1, 2, \ldots . \tag{3}$$

Since $k^t \to \infty$ as $t \to \infty$, given a nonnegative integer v, there exists $t \geqslant 1$ such that $k^t > v$ hence, by (2) and (3), we have that $j = k^t \in C_0$.

(ii) Firstly, let us prove that for all $i \in S$ it is verified that $p_{i0} > 0$, so $0 \in C_i$. In fact, if $P(Z_n = i) > 0$ for some n then,

$$p_{i0} = P\left(\sum_{j=1}^{\phi_n(i)} X_{nj} = 0\right) = \sum_{k=0}^{\infty} P\left(\sum_{j=1}^{k} X_{nj} = 0\right) P_i(k) = \sum_{k=0}^{\infty} p_0^k P_i(k) = E[p_0^{\phi_n(i)}] > 0.$$

If $j \in C_0$, there exists a positive integer n such that $p_{0j}^{(n)} > 0$. Therefore, given $i \in S$ we deduce that $p_{ij}^{(n+1)} \geqslant p_{i0}\, p_{0j}^{(n)} > 0$. Thus $j \in C_i$ and we get that $C_0 \subseteq C_i$.

Reciprocally, if $j \in C_i$, there exists a positive integer t such that $p_{ij}^{(t)} > 0$. Since $0 \in C_i$, for all $i \in S$, it can be assumed that $j \neq 0$. Let us prove that $j \in C_0$.

If $t = 1$, then it is verified, for a certain n, that

$$p_{ij} = P\left(\sum_{l=1}^{\phi_n(i)} X_{nl} = j\right) = \sum_{v=1}^{\infty} P\left(\sum_{l=1}^{v} X_{nl} = j\right) P_i(v) > 0.$$

Hence, there exists $v > 0$ such that $P_i(v) > 0$ and $P\left(\sum_{l=1}^{v} X_{nl} = j\right) > 0$.

By (i), it is derived the existence $h > v$ such that $h \in C_0$. Now, using A2, it is deduced that $P_h(d_h) > 0$ for some $d_h > h$ and it follows that,

$$p_{hj} = P\left(\sum_{l=1}^{\phi_n(h)} X_{nl} = j\right) \geqslant P\left(\sum_{l=1}^{v} X_{nl} = j\right) p_0^{d_h - v} P_h(d_h) > 0$$

so $j \in C_h$ and, since $h \in C_0$, we have that $j \in C_0$.

If $t > 1$ then, for a certain n, we obtain that

$$p_{ij}^{(t)} = \sum_{h=0}^{\infty} p_i(h) p_{hj}^{(t-1)} > 0$$

and therefore, there must exist $h \geqslant 0$ such that

$$p_{hj}^{(t-1)} > 0 \quad \text{and} \quad p_{ih} > 0.$$

If $h=0$, then $p_{0j}^{(t-1)} > 0$ and we get that $j \in C_0$, Otherwise, since $p_{ih} > 0$, using a similar reasoning as the previously considered for the case $t = 1$, it follows that $h \in C_0$. Therefore, since $p_{hj}^{(t-1)} > 0$, we have that $j \in C_0$.

(iii) From (ii), it is derived that if one state leads to another state then 0 also leads to it, so the state space is formed by only one essential class, the class of 0. Consequently, any state in C_0 communicates to those states that leads to and moreover any positive state leads to 0 in one step. Let us prove that the states are aperiodic. In fact, since $p_0 + p_1 < 1$, there exists $k > 1$ such that $p_k > 0$. By A2, we have guaranteed the existence of $d_k > k$ such that $P_k(d_k) > 0$, and by A1, $p_0 > 0$. Then,

$$p_{kk} \geqslant P\left(\sum_{v=1}^{d_k} X_{nv} = k\right) P_k(d_k) \geqslant p_k p_0^{d_k - 1} P_k(d_k) > 0.$$

Thus $\{Z_n\}_{n=0}^{\infty}$ is an irreducible and aperiodic Markov chain. □

Next result establishes, for the subcritical case, namely when $\limsup_{k\to\infty} k^{-1}\varepsilon(k) < m^{-1}$, the convergence in distribution of $\{Z_n\}_{n=0}^{\infty}$ to a positive, finite and nondegenerate random variable.

Theorem 3.1. *Let $\{Z_n\}_{n=0}^{\infty}$ be a CBP with a random control function verifying* A1 *and* A2. *Then, if* $\limsup_{k\to\infty} k^{-1}\varepsilon(k) < m^{-1}$, *it is verified that $\{Z_n\}_{n=0}^{\infty}$ converges in distribution to a positive, finite and nondegenerate random variable Z.*

Proof. From Proposition 3.1, we deduce that $\{Z_n\}_{n=0}^{\infty}$ is an irreducible Markov chain formed by only one essential aperiodic class. Let us prove that $\{Z_n\}_{n=0}^{\infty}$ is positive recurrent and therefore, by

Markov chains theory, we will deduce that $\{Z_n\}_{n=0}^{\infty}$ converges in distribution to a positive, finite and nondegenerate random variable whose probability law will be the stationary probability distribution corresponding to the Markov chain (see Chung (1967) for details).

Firstly, we shall prove that

$$\limsup_{n\to\infty} E[Z_n] < \infty. \tag{4}$$

Since $\limsup_{k\to\infty} k^{-1}\varepsilon(k) < m^{-1}$, it is guaranteed the existence of $k_0 > 0$ and $t < 1$ such that

$$mk^{-1}\varepsilon(k) < t, \quad k = k_0 + 1,\ k_0 + 2, \ldots$$

and we deduce that

$$m\varepsilon(k) \leqslant kt + mC, \quad k = 0, 1, \ldots\ , \tag{5}$$

where $C := \max_{0\leqslant k\leqslant k_0} \varepsilon(k)$.

By (5) and Proposition 2.2(i), we obtain that

$$E[Z_{n+1}] = E[E[Z_{n+1} \mid Z_n]] \leqslant tE[Z_n] + mC, \quad n = 0, 1, \ldots \tag{6}$$

so, by an iterative procedure in (6) and using the fact that $Z_0 = N$, we get

$$E[Z_{n+1}] \leqslant M \sum_{k=0}^{n+1} t^k, \quad n = 0, 1, \ldots\ ,$$

where $M := \max\{N, mC\}$. Now, since $t < 1$,

$$E[Z_{n+1}] \leqslant M(1 - t)^{-1} < \infty$$

and (4) holds.

To conclude, it remains to verify that $\{Z_n\}_{n=0}^{\infty}$ is positive recurrent. Suppose that it is not a positive recurrent Markov chain, then $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$. Therefore, considering $Z_0 = N$, it is followed that:

$$\limsup_{n\to\infty} E[Z_n] \geqslant \limsup_{n\to\infty} \sum_{i=n_0}^{\infty} i\, p_{Ni}^{(n)} \geqslant n_0 \limsup_{n\to\infty} \left(1 - \sum_{i=1}^{n_0-1} p_{Ni}^{(n)}\right) = n_0, \quad n_0 = 2, 3, \ldots$$

and contradiction with (4) is obtained. □

Corollary 3.1. *Under conditions in Theorem* 3.1, *it is verified that*

$$A(s) = E[g_Z(f(s))], \quad 0 \leqslant s \leqslant 1, \tag{7}$$

where $A(s) := E[s^Z]$, *begin* Z *the limit variable of* $\{Z_n\}_{n=0}^{\infty}$ *and recall that* $g_k(s) = E[s^{\phi_0(k)}]$ *and* $f(s) = E[s^{X_{01}}]$.

Proof. By Theorem 3.1 we have that $\lim_{n\to\infty} P(Z_n = k) = P(Z = k)$, $k = 0, 1, \ldots$ and therefore, given $0 \leqslant s \leqslant 1$, we deduce that $\lim_{n\to\infty} F_n(s) = A(s)$. Now, by Proposition 2.1, we know that

$$F_{n+1}(s) = \sum_{k=0}^{\infty} g_k(f(s))P(Z_n = k), \quad 0 \leqslant s \leqslant 1$$

so, making $n \to \infty$, we deduce

$$A(s) = \sum_{k=0}^{\infty} g_k(f(s))P(Z = k) = E[g_Z(f(s))], \quad 0 \leqslant s \leqslant 1$$

and (7) holds.

From (7), differentiating and evaluating at $s = 1$, it is derived that

$$E[Z] = m \sum_{k=0}^{\infty} \varepsilon(k)P(Z = k) = mE[\varepsilon(Z)]. \qquad \square \tag{8}$$

Let now consider some particular cases:

(i) If $\phi_0(k) := k + Y$, $k = 0, 1, \ldots$, where Y is a nonnegative, integer-valued random variable, with $0 < E[Y] < \infty$, and independent of $\{X_{ni} : n = 0, 1, \ldots; \ i = 1, 2, \ldots\}$, then, $\{Z_n\}_{n=0}^{\infty}$ is a Galton–Watson process with immigration. It is clear that $\varepsilon(k) = k + \lambda$, $k = 0, 1, \ldots$, where $\lambda := E[Y]$, consequently if the offspring probability distribution is such that $m < 1$, it follows that

$$\lim_{k \to \infty} k^{-1}\varepsilon(k) = 1 < m^{-1}.$$

Using the fact that $g_k(f(s)) = f(s)^k g(f(s))$, $0 \leqslant s \leqslant 1$, where $g(s) := E[s^Y]$, by (7), it is derived that

$$A(s) = E[f(s)^Z g(f(s))], \quad 0 \leqslant s \leqslant 1$$

which leads to the equation

$$A(s) = g(f(s))A(f(s)), \quad 0 \leqslant s \leqslant 1 \tag{9}$$

Note that (9) is the classical functional equation for the limiting probability generating function of a subcritical Galton–Watson process with immigration (see Jagers, 1975). Moreover, by (8), it is obtain that $E[Z] = mE[Z + \lambda]$ which implies that $E[Z] = \lambda m(1 - m)^{-1}$.

(ii) Yanev and Mitov (1980, 1984) considered branching processes with random migration components (emigration and immigration), where the control random variables have, for $k = 0, 1, \ldots$, the following distribution:

$$P(\phi_0(k) = \max\{k - 1, 0\}) = p, \quad P(\phi_0(k) = k) = q, \quad P(\phi_0(k) = k + 1) = r$$

with $p + q + r = 1$, $(r > 0)$.

Notice that

$$\varepsilon(0) = r, \quad \varepsilon(k) = k + r - p, \quad k = 1, 2, \ldots$$

and therefore, if $m < 1$, we deduce

$$\lim_{k \to \infty} k^{-1}\varepsilon(k) = 1 < m^{-1}.$$

For $0 \leqslant s \leqslant 1$, we have that

$$g_0(s) = E[s^{\phi_0(0)}] = 1 - r + rs, \quad g_k(s) = E[s^{\phi_0(k)}] = s^{k-1}(p + qs + rs^2), \quad k = 1, 2, \ldots$$

and, by (7), we get

$$A(s)=P(Z=0)(1-r+rf(s))+(p+qf(s)+rf(s)^2)\sum_{k=1}^{\infty}P(Z=k)f(s)^{k-1}.$$

Now, it is not difficult to derive the following functional equation:

$$A(s)f(s)=(p+qf(s)+rf(s)^2)A(f(s))-pA(0)(1-f(s)),\quad 0\leqslant s\leqslant 1. \tag{10}$$

For the case of pure immigration, namely when $r=1$, it is easily checked that equation (9), with $g(s)=s$, $0\leqslant s\leqslant 1$, follows on from (10).

On the other hand, taking into account that

$$E[\varepsilon(Z)]=\sum_{k=0}^{\infty}\varepsilon(k)P(Z=k)=E[Z]+r-p+pP(Z=0)$$

it is obtained, by (8), that

$$E[Z]=m(1-m)^{-1}(r-p+pP(Z=0))$$

where $P(Z=0)=A(0)$ is determined by Eq. (7) putting $s=0$.

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