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The best constant in the Topchii–Vatutin inequality for martingales

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Abstract

Consider the class of even convex functions $\phi : \mathbb{R} \to [0,\infty)$ with $\phi(0) = 0$ and concave derivative on $(0,\infty)$. Given any ϕ-integrable martingale $(M_n)_{n\geqslant 0}$ with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \geqslant 1$, the Topchii–Vatutin inequality (Theory Probab. Appl. 42 (1997) 17) asserts that

$$E\phi(M_n) - E\phi(M_0) \leqslant C\sum_{k=1}^{n} E\phi(D_k)$$

with $C = 4$. It is proved here that the best constant in this inequality is $C = 2$ for general ϕ-integrable martingales $(M_n)_{n\geqslant 0}$, and $C = 1$ if $(M_n)_{n\geqslant 0}$ is further nonnegative or having symmetric conditional increment distributions.

MSC: 60G42; 60E15

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1. Introduction and result

Let $(M_n)_{n\geqslant 0}$ be a martingale with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \geqslant 1$, and associated absolute maxima $M_n^* \stackrel{\text{def}}{=} \max_{0\leqslant k\leqslant n} |M_k|$, $n \geqslant 0$. Let further $\mathscr{G}_0$ be the class of even convex functions ϕ: $\mathbb{R} \to [0,\infty)$ with $\phi(0) = 0$ and $\mathscr{G}_1$ its subclass of $\phi \in \mathscr{G}_0$ with a concave derivative on $(0,\infty)$.

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Note that the latter class comprises the functions $\phi(x)=|x|^p$ for $p\in[1,2]$ as well as $\phi(x)=(|x|+a)^p\log^r(|x|+a)-a^p\log^r a$ for $p\in[1,2)$, $r>0$ and $a>0$ sufficiently large. The following convex function inequality is due to Topchii and Vatutin (1997): There exists a finite positive constant C such that for all $\phi\in\mathscr{G}_1$, all martingales $(M_n)_{n\geqslant 0}$ and all $n\geqslant 1$

$$E\phi(M_n)-E\phi(M_0)\ \leqslant\ C\sum_{k=1}^n E\phi(D_k). \tag{1.1}$$

More precisely, they showed (1.1) be true with $C=4$ and $M_0=0$. If $\phi(x)=|x|$ or $\phi(x)=x^2$, then it is well-known that (1.1) holds true with $C=1$ and that this value cannot be improved. We shall prove in this note that the best constant for general $\phi\in\mathscr{G}_1$ and general ϕ-integrable martingales is $C=2$, but that $C=1$ is optimal when imposing certain additional restrictions on the class of considered martingales. The result is stated as the following theorem.

Theorem 1. *If $0\not\equiv\phi\in\mathscr{G}_1$ and $M=(M_k)_{0\leqslant k\leqslant n}$ is a ϕ-integrable martingale, then*

$$E\phi(M_n)-E\phi(M_0)<2\sum_{k=1}^n E\phi(D_k). \tag{1.2}$$

The constant 2 is sharp in the sense that, for each $\varepsilon\in(0,1)$, there exists a bounded martingale M and some $\phi\in\mathscr{G}_1$ such that

$$E\phi(M_n)-E\phi(M_0)\geqslant(2-\varepsilon)\sum_{k=1}^n E\phi(D_k). \tag{1.3}$$

If M is nonnegative or having symmetric conditional increment distributions, then inequality (1.1) holds true with $C=1$.

An analogue of (1.1) for the maximum M_n^* can be quite easily inferred from the following Burkholder–Davis–Gundy inequality (see e.g. Chow and Teicher, 1997 Theorem 1, p. 425): Let $v>0$ and $\mathscr{G}_0^{(v)}$ be the class of all $\phi\in\mathscr{G}_0$ satisfying $\phi(2x)\leqslant v\phi(x)$ for all x. Then there exists a constant $C_v^*\in(0,\infty)$ such that for all $\phi\in\mathscr{G}_0^{(v)}$ and all martingales $(M_n)_{n\geqslant 0}$ having $M_0=0$

$$E\phi(M_n^*)\leqslant C_v^* E\phi\left(\left(\sum_{k=1}^n D_k^2\right)^{1/2}\right). \tag{1.4}$$

This inequality applies to class $\mathscr{G}_1$ because $\mathscr{G}_1\subset\mathscr{G}_0^{(4)}$ as will be shown in Lemma 2 at the end of Section 2. Defining $\psi(t)\stackrel{\text{def}}{=}\phi(t^{1/2})$, the same lemma will further show that ψ is concave and subadditive on $[0,\infty)$, that is $\psi(\sum_{k=1}^n x_k)\leqslant\sum_{k=1}^n\psi(x_k)$ for all $x_1,\ldots,x_n\geqslant 0$ and $n\in\mathbb{N}$. Utilizing this last fact on the right-hand side in (1.4), we obtain

$$E\phi(M_n^*)\leqslant C_4^*\sum_{k=1}^n E\phi(D_n). \tag{1.5}$$

Let us finally mention that sharp inequalities similar to those considered here were derived in a recent paper by de la Peña et al. (2002) for infinite degree order statistics.

2. Proof of Theorem 1

The proof of Theorem 1 and in particular of the sharpness of the constant $C=2$ in (1.1) are heavily based on several reductions, the main one being that it suffices to consider only certain extremal elements $\phi \in \mathscr{G}_1$. This was also used by Alsmeyer (1996) and Rösler (1995) for the study of odd functional moments of positive random variables with a decreasing density. The general background is that the class of increasing convex (or concave) functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0)=0$ as well as many important subclasses like $\mathscr{G}_1$ form a convex cone for which Choquet theory tells us that each element ϕ can be written as an integral of its extremal elements with respect to some measure on $[0,\infty]$ (depending on ϕ). For the given classes these integral representations are obtained by simple partial integration. The following lemma provides the result for the class $\mathscr{G}_1$ and exemplifies the general procedure.

Lemma 1. *For each $\phi \in \mathscr{G}_1$, there exists a unique finite measure Q_ϕ on $[0,\infty]$ such that*

$$\phi(x) = \int_{[0,\infty]} \phi_t(x) Q_\phi(\mathrm{d}t), \quad x \geqslant 0, \tag{2.1}$$

where $\phi_0(x)=|x|$, $\phi_\infty(x)=x^2$, and

$$\phi_t(x) \overset{\text{def}}{=} \begin{cases} x^2 & \text{if } |x| \leqslant t \\ 2xt - t^2 & \text{if } |x| > t \end{cases} \tag{2.2}$$

for $t \in (0,\infty)$.

Note that the functions ϕ_t also arise in problems of robust estimation and are known in statistics as Huber functions or Huber's ρ-functions, see e.g. Huber (1964, 1973).

Proof. Each $\phi \in \mathscr{G}_1$ has a concave derivative ϕ' with $\phi'_+(0) \overset{\text{def}}{=} \lim_{x\to+0} \phi'(x) \geqslant 0$ and thus also a nonincreasing second right derivative ϕ''_+ with asymptotic value $\phi''_+(\infty) \overset{\text{def}}{=} \lim_{x\to\infty} \phi''_+(x) \geqslant 0$. Therefore $\Lambda_{\phi'}((x,\infty)) \overset{\text{def}}{=} \phi''_+(x) - \phi''_+(\infty)$ for $x \geqslant 0$ defines a measure on $(0,\infty)$. Put

$$\mathscr{G}_1^* \overset{\text{def}}{=} \{\phi \in \mathscr{G}_1 : \phi'_+(0)=0, \phi''_+(\infty)=0\}.$$

and $\phi^*(x) \overset{\text{def}}{=} \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ which is an element of $\mathscr{G}_1^*$. Partial integration now gives for $x>0$

$$\begin{aligned} \phi'(x) - \phi'_+(0) - \phi''_+(\infty)x &= \int_0^x (\phi''_+(y) - \phi''_+(\infty))\,\mathrm{d}y \\ &= \int_0^x \int_{(y,\infty)} \Lambda_{\phi'}(\mathrm{d}t)\,\mathrm{d}y \\ &= \int_{(0,\infty)} (x \wedge t)\Lambda_{\phi'}(\mathrm{d}t) \end{aligned}$$

and also

$$\phi^*(x) = \int_0^x (\phi'(y) - \phi'_+(0) - \phi''_+(\infty)y)\,\mathrm{d}y$$
$$= \int_{(0,\infty)} \int_0^x (y \wedge t)\,\mathrm{d}y\Lambda_{\phi'}(\mathrm{d}t)$$
$$= \int_{(0,\infty)} \phi_t(x) Q_{\phi^*}(\mathrm{d}t),$$

where $Q_{\phi^*} \stackrel{\text{def}}{=} \Lambda_{\phi'}/2$. We conclude (2.1) with $Q_\phi \stackrel{\text{def}}{=} \phi'_+(0)\delta_0 + \frac{1}{2}\phi''_+(\infty)\delta_\infty + Q_{\phi^*}$. □

Proof of Theorem 1. The following reduction arguments will show that it suffices to prove

$$E\phi_1(s+D) \leqslant \phi_1(s) + C\,E\phi_1(D) \tag{2.3}$$

for all $s \geqslant 0$ and all centered random variables D having a two point distribution, where $C=2$ in the general case, while $C=1$ if D is symmetric or $s+D \geqslant 0$. Of course, ϕ_1 is the function defined by (2.2). Note that in terms of the martingales under consideration the former means nothing but a reduction to martingales of the form $(M_0, M_1) = (s, s+D)$.

First reduction: As noted above, for each $\phi \in \mathscr{G}_1$ the even function $\phi^*(x) = \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ is an element of $\mathscr{G}_1^*$. Since

$$E\phi(M_n) = E\phi^*(M_n) + \phi'_+(0)E|M_n| + \frac{\phi''_+(\infty)}{2} EM_n^2$$
$$\leqslant E\phi^*(M_n) + \phi'_+(0)\left(E|M_0| + \sum_{k=1}^n E|D_k|\right) + \frac{\phi''_+(\infty)}{2}\left(EM_0^2 + \sum_{k=1}^n ED_k^2\right),$$

it suffices to prove Theorem 1 for functions $\phi \in \mathscr{G}_1^*$.

Second reduction: Using (2.2), $\phi_t(x) = t^2\phi_1(x/t)$ for all $t \in (0,\infty)$ and $Q_\phi(\{0,\infty\}) = 0$ if $\phi'_+(0) = 0$ and $\phi''_+(\infty) = 0$ (see at the end of the proof of Lemma 1), we infer for each $\phi \in \mathscr{G}_1^*$

$$E\phi(M_n) = \int_{(0,\infty)} E\phi_t(M_n) Q_\phi(\mathrm{d}t) = \int_{(0,\infty)} t^2 E\phi_1(M_n/t) Q_\phi(\mathrm{d}t).$$

Since $(M_k/t)_{0 \leqslant k \leqslant n}$ is still a martingale, it suffices to prove Theorem 1 with $\phi = \phi_1$.

Third reduction: By conditioning

$$E\phi_1(M_n) - E\phi_1(M_{n-1}) - C\,E\phi_1(D_n)$$
$$= \int (E(\phi_1(s+D_n)|M_{n-1}=s) - \phi_1(s) - C\,E(\phi_1(D_n)|M_{n-1}=s))P(M_{n-1} \in \mathrm{d}s),$$

where, given $M_{n-1} = s$, D_n has conditonal mean 0. This reduces the proof to that of (2.3) for any centered random variable D and any $s \in \mathbb{R}$. We may further restrict to $s \geqslant 0$ because $E\phi_1(s+D) = E\phi_1(-s-D)$ and $-D$ is also centered.

Fourth reduction: Finally, since every centered distribution is a mixture of centered two point distributions, we conclude that it is indeed enough to prove (2.3) for all $s \geqslant 0$ and all centered D taking only two values, see e.g. Hoeffding (1955).

In the following, we simply write f' and always mean f'_+ in those cases where left and right derivatives are different.

Proof of (2.3) *with* $C=1$ *for symmetric* D. Suppose D has distribution $(\delta_{-a}+\delta_a)/2$ for some $a \geqslant 0$ and let

$$\Delta(s) \stackrel{\text{def}}{=} E\phi_1(s+D) - E\phi_1(D) - \phi_1(s), \quad s \geqslant 0.$$

Then

$$\Delta(s) = \frac{\phi_1(s+a)+\phi_1(s-a)}{2} - \phi_1(a) - \phi_1(s),$$

$$\Delta'(s) = \frac{\phi_1'(s+a)+\phi_1'(s-a)}{2} - \phi_1'(s),$$

$$\Delta''(s) = \frac{\phi_1''(s+a)+\phi_1''(s-a)}{2} - \phi_1''(s)$$

for $s \geqslant 0$. In particular $\Delta(0)=\Delta'(0)=0$ and $\Delta''(0) \leqslant 0$. Note that

$$\phi_1'(x) \stackrel{\text{def}}{=} \begin{cases} 2x & \text{if } |x| \leqslant 1 \\ 2\,\mathrm{sign}(x) & \text{if } |x| \geqslant 1 \end{cases} \quad \text{and} \quad \phi_1''(x) = 2 1_{[-1,1]}(x)\lambda - \text{a.e},$$

where λ denotes Lebesgue measure on $\mathbb{R}$ and 1_B the indicator function of a set B. Hence, if $a \in [0,1]$, then λ-a.e.

$$\Delta''(s) = \begin{cases} 0 & \text{if } 0 \leqslant s \leqslant 1-a \text{ or } s > a+1, \\ -1 & \text{if } 1-a < s \leqslant 1, \\ 1 & \text{if } 1 < s \leqslant a+1 \end{cases}$$

while in case $a \in (1,2]$

$$\Delta''(s) = \begin{cases} -2 & \text{if } 0 \leqslant s \leqslant a-1, \\ -1 & \text{if } a-1 < s \leqslant 1, \\ 1 & \text{if } 1 < s \leqslant a+1, \\ 0 & \text{if } s > a+1 \end{cases}$$

and in case $a > 2$

$$\Delta''(s) = \begin{cases} 0 & \text{if } 1 < s \leqslant a-1 \text{ or } s > a+1, \\ -2 & \text{if } 0 \leqslant s \leqslant 1, \\ 1 & \text{if } a-1 < s \leqslant a+1. \end{cases}$$

We also have that $\Delta(s)$ and $\Delta'(s)$ vanish at $s=0$ and (by linearity of ϕ_1 on $(1,\infty)$) for sufficiently large s. From this we see that Δ' is everywhere nonpositive and unimodal which in turn yields $\Delta(s) \leqslant 0$ for all $s \geqslant 0$ and thus (2.3) with $C=1$.

Proof of (2.3) *with* $C=1$ *for nonnegative* $s+D$. Let D be a centered random variable with distribution $p\delta_{-a}+q\delta_b$ for $a,b \geqslant 0$, hence $p+q=1$ and $qb-pa=0$. The function Δ now takes the form

$$\Delta(s) = p\phi_1(s-a) + q\phi_1(s+b) - p\phi_1(-a) - q\phi_1(b) - \phi_1(s)$$

and has derivative $\Delta'(s) = p\phi_1'(s-a) + q\phi_1'(s+b) - \phi_1'(s)$. By concavity of ϕ_1' on $[0,\infty)$,

$$\Delta'(s) \leqslant \phi_1'(s - pa + qb) - \phi_1'(s) = 0$$

for all $s \geqslant a$. Consequently, $E\phi_1(s+D) \leqslant E\phi_1(D) + \phi_1(s)$ follows for all $s > a$ if this is true for $s = a$.

If $s = a \leqslant 1$, then $\phi_1(s) = s^2$ whence $\phi_1(s+x) - \phi_1(x) \leq (s+x)^2 - x^2 = s(2x+s)$ for all $x \geqslant -s$ implies the asserted inequality, namely

$$E\phi_1(s+D) - E\phi_1(D) \leqslant sE(2D+s) = s^2.$$

Now fix $s = a \geqslant 1$, note that $ED = 0$ implies $p = b/(s+b)$, and look at $\Delta(s)$ as a function $G(b)$, say, of b. We obtain

$$\begin{aligned} G(b) &= q\phi_1(s+b) - p\phi_1(-s) - q\phi_1(b) - \phi_1(s) \\ &= q\phi_1(s+b) - q\phi_1(b) - (1+p)\phi_1(s) \\ &= \frac{s\phi_1(s+b) - s\phi_1(b) - (s+2b)\phi_1(s)}{s+b}. \end{aligned}$$

This implies in case $b \geqslant 1$

$$G(b) = \frac{s(2(s+b)-1) - s(2b-1) - (s+2b)(2s-1)}{s+b} = \frac{s+2b-4sb}{s+b} \leqslant 0,$$

and in case $0 < b < 1$

$$G(b) = \frac{s(2(s+b)-1) - sb^2 - (s+2b)(2s-1)}{s+b} = \frac{-2b(s-1) - sb^2}{s+b} \leqslant 0.$$

So we have again shown that (2.3) holds with $C = 1$.

Proof of (2.3) *with* $C = 2$ *for general* D. The assertion to prove may be rephrased in terms of $\Delta(s)$ as

$$\Delta(s) \leqslant E\phi_1(D) = p\phi_1(s-a) + q\phi_1(s+b)$$

for all $s \geqslant 0$. Since $s = 0$ is trivial, fix an arbitrary $s > 0$, let D have distribution $p\delta_{-a} + q\delta_b$ and suppose $\theta \stackrel{\text{def}}{=} a - s \geqslant 0$ (only this case needs to be considered after the previous part of the proof). Note that $ED = 0$ implies $b = (p/q)a$ and thus $D \stackrel{d}{=} p\delta_{-s-\theta} + q\delta_{(p/q)(s+\theta)}$. In order to prove (2.3) with $C = 2$, fix any $p \in (0,1)$ and consider

$$\begin{aligned} H(\theta) &\stackrel{\text{def}}{=} E\phi_1(s+D) - 2\,E\phi_1(D) - \phi_1(s) \\ &= p(\phi_1(\theta) - 2\phi_1(s+\theta)) + q(\phi_1(s+(p/q)(s+\theta)) - 2\phi_1((p/q)(s+\theta))) - \phi_1(s) \end{aligned}$$

for $\theta \geqslant 0$. Since $s + D \geqslant 0$ if $\theta = 0$, we infer $H(0) \leqslant -E\phi_1(D) < 0$ from the previous part of the proof. Differentiation with respect to θ gives

$$\begin{aligned} H'(\theta) &= p(\phi_1'(\theta) - 2\phi_1'(s+\theta)) + p(\phi_1'(s+(p/q)(s+\theta)) - 2\phi_1'((p/q)(s+\theta))) \\ &= p((\phi_1'(s+(p/q)(s+\theta)) - \phi_1'((p/q)(s+\theta))) - (\phi_1'(s+\theta) - \phi_1'(\theta)) \\ &\quad -(\phi_1'(s+\theta) + \phi_1'((p/q)(s+\theta)))). \end{aligned} \tag{2.4}$$

The function ϕ_1' is monotone and is subadditive as a nonnegative concave function on $[0,\infty)$. It follows that

$$\phi_1'(s+\theta)+\phi_1'((p/q)(s+\theta)) \;\geq\; \phi_1'(s+\theta+(p/q)(s+\theta)) \geqslant \phi_1'(s+(p/q)(s+\theta))$$

and thereby in (2.4)

$$H'(\theta) \leqslant -\,p(\phi_1'((p/q)(s+\theta))+(\phi_1'(s+\theta)-\phi_1'(\theta))) \leqslant 0.$$

Consequently, H is nonincreasing on $[0,\infty)$ with $H(0)<0$ and therefore everywhere negative. This proves (2.3) with $C=2$ and strict inequality.

Attaining the bound in (2.3) *with* $C=2$. We finally have to provide examples showing that the bound $C=2$ is sharp. Let $s\geqslant 1$ and D be distributed as $[b/(a+b)]\delta_{-a}+[a/(a+b)]\delta_b$ for some $a\geqslant 1+s$ and $b\in[0,1]$. Then

$$\begin{aligned}&E\phi_1(s+D)-\phi_1(s)-(2-\varepsilon)E\phi_1(D)\\ &\quad=\frac{1}{a+b}(b(2a-2s-1)+a(2s+2b-1)-(a+b)(2s-1)-(2-\varepsilon)(b(2a-1)+ab^2))\\ &\quad=\frac{b}{a+b}(2-2ab-4s+\varepsilon(2a-1+ab)).\end{aligned}$$

Now it is easily seen that, for any $\varepsilon>0$, a positive value is obtained when choosing $b=1/a$ and a sufficiently large. The proof of Theorem 1 is herewith complete. □

Recall that $\mathscr{G}_0^{(v)}$ denotes the class of all $\phi\in\mathscr{G}_0$ satisfying $\phi(2x)\leq v\phi(x)$ for all x. We claimed in the Introduction that $\mathscr{G}_1\subset\mathscr{G}_0^{(4)}$ as well as $\mathscr{G}_1\subset\mathscr{G}_2$, where $\mathscr{G}_2$ denotes the subclass of $\mathscr{G}_0$ containing those ϕ for which $\psi(x)\stackrel{\text{def}}{=}\phi(x^{1/2})$ is concave on $[0,\infty)$. These claims are finally confirmed in the subsequent lemma.

Lemma 2. $\mathscr{G}_1\subset\mathscr{G}_2$ *and* $\mathscr{G}_1\subset\mathscr{G}_0^{(4)}$.

Proof. Note that each nonnegative concave function f on $[0,\infty)$ is subadditive and that $f(x)/x$ is nonincreasing (because $-f$ is evidently star-shaped, see Marshall and Olkin, 1979, p. 453). Given any $\phi\in\mathscr{G}_1$, use this for $f=\phi'$ to see that the pertinent ψ is indeed concave because $\psi'(x)=[\phi'(x^{1/2})]/2x^{1/2}$. Moreover, the subadditivity of ϕ' on $[0,\infty)$ implies $\phi'(2x)\leqslant 2\phi'(x)$ and thus

$$\phi(2x)=\int_0^{2x}\phi'(t)\,\mathrm{d}t=\int_0^x 2\phi'(2t)\,\mathrm{d}t\leqslant\int_0^x 4\phi'(t)\,\mathrm{d}t=4\phi(x)$$

for all $x\geqslant 0$. □

Note added in Proof

In a recent paper, Li [7, Theorem 2.1] proved the following large deviation inequality for a martingale $(M_n)_{n\geqslant 0}$ with $M_0=0$: If $1<p\leqslant 2$ and $K\stackrel{\text{def}}{=}\sup_{n\geqslant 1}E|M_n|^p<\infty$, then

$$P\left(\max_{1\leqslant i\leqslant n}|M_n|>nx\right)\leqslant CKn^{1-p}x^{-p}$$

for all $x > 0$, $n \geqslant 1$ and $C = (18\,pq^{1/2})^p$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. A combination of Doob's maximal inequality (see [2, p. 255]) with our Theorem 1 for $\phi(x) = |x|^p$ immediately shows that Li's inequality actually holds true with the considerably smaller constant $C = 2$.

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References

Alsmeyer, G., 1996. Nonnegativity of odd functional moments of positive random variables with decreasing density. Statist. Probab. Lett. 26, 75–82.

Chow, Y.S., Teicher, H., 1997. Probability Theory: Independence, Interchangeability, Martingales, 3rd Edition. Springer, New York.

de la Peña, V.H., Ibragimov, R., Sharakhmetov, S., 2002. On sharp Burkholder–Rosenthal type inequalities for infinite degree order statistics. Ann. Inst. H. Poincaré 38, 973–990.

Hoeffding, W., 1955. The extrema of the expected value of a function of independent random variables. Ann. Math. Statist. 26, 268–275.

Huber, P., 1964. Robust estimation of a location parameter. Ann. Math. Statist. 35, 73–101.

Huber, P., 1973. Robust regression: asymptotics, conjectures and Monte Carlo. Ann. Statist. 1, 799–821.

Marshall, A.W., Olkin, I., 1979. Inequalities: theory of majorization and its applications. Mathematics in Science and Engineering, Vol. 143. Academic Press, New York.

Rösler, U., 1995. Distributions slanted to the right. Statist. Neerlandica 49, 83–93.

Topchii, V.A., Vatutin, V.A., 1997. Maximum of the critical Galton–Watson processes and left continuous random walks. Theory Probab. Appl. 42, 17–27.