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Statistics & Probability Letters 62 (2003) 79–86

STATISTICS & PROBABILITY LETTERS

www.elsevier.com/locate/stapro

A limit theorem for partial sums of random variables and its applications

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Received September 2001; received in revised form November 2002

Abstract

In this paper, we establish a new limit theorem for partial sums of random variables. As corollaries, we generalize the extended Borel–Cantelli lemma, and obtain some strong laws of large numbers for Markov chains as well as a generalized strong ergodic theorem for irreducible and positive recurrent Markov chains.

Keywords: Extended Borel–Cantelli lemma; Strong laws of large numbers

1. Introduction and the main results

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space, and let $(\mathscr{F}_n, n \geqslant 0)$ be an increasing sequence of sub-σ-algebras of $\mathscr{F}$. The main purpose of this paper is to establish a general limit theorem for partial sums of random variables. As corollaries, we generalize the extended Borel–Cantelli lemma (see Theorem 2), and obtain some strong laws of large numbers for Markov chains (see Theorem 3, Corollary 2 and Theorem 4) and a strong ergodic theorem for irreducible and positive recurrent Markov chains (see Theorem 5).

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[1] His work was supported by the Ministry of Science and Technology, the 973 Project on Mathematics, and the Knowledge Innovation Program of the Chinese Academy of Sciences.

0167-7152/03/$ - see front matter
doi:10.1016/S0167-7152(02)00428-5

The main result is the following:

Theorem 1. *Let* $(\xi_n,\ n \geqslant 1)$ *be an* $(\mathscr{F}_n)$*-adapted sequence of r.v.'s, such that there is a positive finite* K *for which* $|\xi_n| \leqslant K$ *for every* $n \geqslant 1$. *Let* (a_n) *be a sequence of non-negative r.v.'s defined on* $(\Omega, \mathscr{F}, \mathbf{P})$. *We denote* $\mathbf{E}[\xi_n|\mathscr{F}_{n-1}]$ *by* $\widetilde{\xi}_n$, *and put*

$$\Omega_0 = \left\{ \lim_{n\to\infty} a_n = \infty, \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \mathbf{E}[|\xi_i|\,|\mathscr{F}_{i-1}] < \infty \right\}. \tag{1}$$

Then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} (\xi_i - \widetilde{\xi}_i) = 0, \quad a.s.\ on\ \Omega_0. \tag{2}$$

In order to prove Theorem 1, we prepare two lemmas.

Lemma 1. *For* $0 \leqslant x \leqslant 1$ *and* $0 < \lambda < \infty$, *we have*

$$x \log \lambda \leqslant \log(1 + (\lambda - 1)x). \tag{3}$$

For $1 \leqslant \lambda < \infty$ *and* $x \geqslant 0$, *or for* $0 < \lambda < 1$ *and* $0 \leqslant x < 1/(1-\lambda)$, *we have*

$$\log(1 + (\lambda - 1)x) \leqslant x\lambda \log \lambda. \tag{4}$$

Proof. For a given $0 \leqslant x \leqslant 1$, we put

$$f(\lambda) = \log(1 + (\lambda - 1)x) - x \log \lambda, \quad 0 < \lambda < \infty,$$

Then

$$f'(\lambda) = \frac{x}{1 + (\lambda - 1)x} - \frac{x}{\lambda} = \frac{(\lambda - 1)(1 - x)x}{\lambda(1 + (\lambda - 1)x)}.$$

Thus, $f'(\lambda) \geqslant 0$ for $\lambda > 1$ and $f'(\lambda) \leqslant 0$ for $0 < \lambda \leqslant 1$. Since $f(1) = 0$, we must have $f(\lambda) \geqslant 0$. (3) is proved. □

We are now beginning the proof of (4). We only need to consider the $\lambda \neq 1$ case. Since $a/(1+a) \leqslant \log(1 + a) \leqslant a$ for all $a > -1$, we have

$$0 \leqslant \log(1 + (\lambda - 1)x) \leqslant (\lambda - 1)x, \quad \log \lambda \geqslant (\lambda - 1)/\lambda > 0 \quad \forall \lambda > 1,\ x \geqslant 0,$$

$$\log(1 + (\lambda - 1)x) \leqslant (\lambda - 1)x \leqslant 0, \quad 0 > \log \lambda \geqslant (\lambda - 1)/\lambda \quad \forall 0 < \lambda < 1,\ 0 \leqslant x < \frac{1}{1-\lambda}.$$

Consequently, we obtain that

$$\frac{\log(1+(\lambda-1)x)}{\log\lambda} \leqslant \frac{(\lambda-1)x}{(\lambda-1)/\lambda} = x\lambda \quad \forall \lambda > 1,\ x \geqslant 0,$$

$$\frac{\log(1+(\lambda-1)x)}{\log\lambda} \geqslant \frac{(\lambda-1)x}{(\lambda-1)/\lambda} = x\lambda \quad \forall 0 < \lambda < 1,\ 0 \leqslant x < \frac{1}{1-\lambda}.$$

This proves (4). □

Lemma 2. *Let* $(\xi_n,\ n \geqslant 1)$ *be an* $(\mathscr{F}_n)$*-adapted sequence of non-negative r.v.'s such that* $\xi_n \leqslant 1$ *for all* $n \geqslant 1$. *We denote* $\mathbf{E}[\xi_i|\mathscr{F}_{i-1}]$ *by* $\widetilde{\xi}_i$. *Let* $\lambda > 0$. *We set* $M_0(\lambda) = 1$ *and put*

$$M_n(\lambda) = \frac{\lambda^{\sum_{i=1}^n \xi_i}}{\prod_{i=1}^n (1+(\lambda-1)\widetilde{\xi}_i)}, \quad n \geqslant 1.$$

Then $(M_n(\lambda),\ n \geqslant 0)$ *is a non-negative supermartingale w.r.t.* $(\mathscr{F}_n)$.

Proof. The nonnegativity is obvious; we need only establish the supermartingale claim. By inequality (3) we get

$$\lambda^{\xi_n} \leqslant 1 + (\lambda-1)\xi_n,$$

so that

$$\mathbf{E}[\lambda^{\xi_n}|\mathscr{F}_{n-1}] \leqslant 1 + (\lambda-1)\widetilde{\xi}_n.$$

Consequently,

$$\mathbf{E}[M_n(\lambda)|\mathscr{F}_{n-1}] = M_{n-1}(\lambda)\mathbf{E}\left[\frac{\lambda^{\xi_n}}{1+(\lambda-1)\widetilde{\xi}_n}|\mathscr{F}_{n-1}\right] \leqslant M_{n-1}(\lambda).$$

This shows that $(M_n(\lambda),\ n \geqslant 0)$ is a supermartingale w.r.t. $(\mathscr{F}_n)$. □

The Proof of Theorem 1. It is obvious that we may assume $K = 1$, since otherwise the theorem applies to the random variables ξ_n/K, and the set Ω_0 and (2) remain unchanged. Instead of considering two sequences (ξ_n^+) and (ξ_n^-), we can further assume that all ξ_n are non-negative. Here ξ_n^+ and ξ_n^- stand for the positive parts and negative parts of ξ_n. In the following proof we follow an approach proposed by Liu and Yang (1995) and Yang and Liu (2000). We use notations found in Lemma 2. By Doob's Martingale Convergence Theorem, $M_n(\lambda)$ a.s. converges to a finite non-negative r.v. $M_\infty(\lambda)$. In particular, for $\lambda > 1$ and $0 < \mu < 1$, we have on the set $\{\lim_{n\to\infty} a_n = \infty\}$

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^n \left(\xi_i - \frac{\log(1+(\lambda-1)\widetilde{\xi}_i)}{\log\lambda}\right) = \limsup_{n\to\infty} \frac{1}{a_n} \frac{\log M_n(\lambda)}{\log\lambda} \leqslant 0 \quad \text{a.s.,} \tag{5}$$

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^n \left(\xi_i - \frac{\log(1+(\mu-1)\widetilde{\xi}_i)}{\log\mu}\right) = \liminf_{n\to\infty} \frac{1}{a_n} \frac{\log M_n(\mu)}{\log\mu} \geqslant 0 \quad \text{a.s..} \tag{6}$$

On the other hand, for $\lambda > 1$ and $0 < \mu < 1$, by inequality (4) we have

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \left(\frac{\log(1+(\lambda-1)\widetilde{\xi}_i)}{\log \lambda} - \widetilde{\xi}_i \right) \leqslant (\lambda - 1) \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \widetilde{\xi}_i, \tag{7}$$

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \left(\frac{\log(1+(\mu-1)\widetilde{\xi}_i)}{\log \mu} - \widetilde{\xi}_i \right) \geqslant (\mu - 1) \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \widetilde{\xi}_i. \tag{8}$$

Noting that

$$\limsup_{n\to\infty} (\alpha_n + \beta_n) \leqslant \limsup_{n\to\infty} \alpha_n + \limsup_{n\to\infty} \beta_n,$$

$$\liminf_{n\to\infty} (\alpha_n + \beta_n) \geqslant \liminf_{n\to\infty} \alpha_n + \liminf_{n\to\infty} \beta_n,$$

from (5)–(8) we have on the set $\{\lim_{n\to\infty} a_n = \infty\}$

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} (\xi_i - \widetilde{\xi}_i) \leqslant (\lambda - 1) \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \widetilde{\xi}_i \quad \text{a.s.,} \tag{7$'$}$$

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} (\xi - \widetilde{\xi}_i) \geqslant (\mu - 1) \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \widetilde{\xi}_i \quad \text{a.s..} \tag{8$'$}$$

Letting $\lambda \downarrow 1$ and $\mu \uparrow 1$ in $(7)'$ and $(8)'$, we obtain that

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} (\xi_i - \widetilde{\xi}_i) \leqslant 0 \leqslant \liminf_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} (\xi_i - \widetilde{\xi}_i) \quad \text{a.s. on } \Omega_0,$$

which implies (2). □

As a corollary of Theorem 1 we obtain the following result which generalizes the extended Borel–Cantelli lemma.

Theorem 2. *Let* $(\xi_n,\ n \geqslant 1)$ *be an* $(\mathscr{F}_n)$*-adapted sequence of non-negative r.v.'s such that there is a positive finite* K *for which* $\xi_n \leqslant K$ *for all* $n \geqslant 1$. *We denote* $\mathbf{E}[\xi_n|\mathscr{F}_{n-1}]$ *by* $\widetilde{\xi}_n$. *Put*

$$A = \left\{ \sum_{i=1}^{\infty} \xi_i = \infty \right\}; \quad B = \left\{ \sum_{i=1}^{\infty} \widetilde{\xi}_i = \infty \right\}.$$

Then we have $A = B$ *a.s. (i.e.* $\mathbf{P}(A \Delta B) = 0$*), and*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \xi_i}{\sum_{i=1}^{n} \widetilde{\xi}_i} = 1 \quad \textit{a.s. on } B. \tag{9}$$

Proof. If we let $a_n = \sum_{i=1}^n \widetilde{\xi}_i$, then from (1) we obtain (9), and (9) implies $B \subset A$ a.s.. If we let $a_n = \sum_{i=1}^n \xi_i$, then from (2) we see that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \widetilde{\xi}_i}{\sum_{i=1}^n \xi_i} = 1 \quad \text{a.s. on } A \cap B^c,$$

which implies $A \cap B^c = \emptyset$ a.s., i.e., $A \subset B$, a.s.. Thus we must have $A = B$ a.s.. □

Corollary 1. *Let $(\mathscr{F}_n,\ n \geqslant 0)$ be an increasing sequence of σ-algebras and let $B_n \in \mathscr{F}_n$. Put*

$$A = \left\{ \sum_{i=1}^{\infty} I_{B_i} = \infty \right\}; \quad B = \left\{ \sum_{i=1}^{\infty} \mathbf{P}(B_i|\mathscr{F}_{i-1}) = \infty \right\}.$$

Then we have $A = B$ a.s., and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n I_{B_i}}{\sum_{i=1}^n \mathbf{P}(B_i|\mathscr{F}_{i-1})} = 1 \quad \textit{a.s. on } B.$$

Proof. Letting $\xi_n = I_{B_n}$ in Theorem 2 immediately gives the result.

The first part of this corollary is the Extended Borel–Cantelli Lemma (see Chow and Teicher, 1988, p. 249), and the second part of this corollary is the sharper form of the Borel–Cantelli lemma (see Dubins and Freedman, 1965). □

2. The strong law of large numbers and strong ergodic theorem for Markov chains

Let $\{Y_n,\ n \geqslant 0\}$ be a non-homogeneous Markov chain with state space $S = \{1, 2, \cdots\}$. We let $p_n(i,j) = \mathbf{P}(Y_n = j | Y_{n-1} = i)$, $i, j \in S$. As an application of Theorem 1, we obtain the following strong law of large numbers on the ordered couples of states for Markov chain $\{Y_n,\ n \geqslant 0\}$.

Theorem 3. *Let $S_n(k,l)$ be the number of occurrences of the couple (k,l) in the sequence of ordered couples $(Y_0, Y_1), (Y_1, Y_2), \ldots, (Y_{n-1}, Y_n)$, and let (a_n) be a sequence of positive random variables. Let $\delta_j(i) = \delta_{i,j}$ be the Kronecker δ-function. Set*

$$A = \left\{ \lim_{n\to\infty} a_n = \infty, \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n} \delta_k(Y_{m-1}) p_m(k,l) < \infty \right\}. \tag{10}$$

Then,

$$\lim_{n\to\infty} \frac{1}{a_n} \left\{ S_n(k,l) - \sum_{m=1}^{n} \delta_k(Y_{m-1}) p_m(k,l) \right\} = 0 \quad \textit{a.s. on } A. \tag{11}$$

If we set

$$D(k,l)=\left\{\lim_{n\to\infty}\sum_{m=1}^{n}\delta_k(Y_{m-1})p_m(k,l)=\infty\right\},$$

then

$$\lim_{n\to\infty}\frac{S_n(k,l)}{\sum_{m=1}^{n}\delta_k(Y_{m-1})p_m(k,l)}=1 \quad a.s.\ on\ D(k,l). \tag{12}$$

Proof. Let $\xi_n=\delta_k(Y_{n-1})\delta_l(Y_n)$ and $\mathscr{F}_n=\sigma(Y_0,\dots,Y_n)$ in Theorem 1. Then we have $S_n(k,l)=\sum_{m=1}^{n}\xi_m$, and

$$\mathbf{E}[\xi_n|\mathscr{F}_{n-1}]=\mathbf{E}[\delta_k(Y_{n-1})\delta_l(Y_n)|Y_{n-1}]=\delta_k(Y_{n-1})p_n(k,l).$$

Thus, from (2) we obtain (11). Letting $a_n=\sum_{m=1}^{n}\delta_k(Y_{m-1})p_m(k,l)$ in (11) gives (12). □

The following corollary gives a strong law of large numbers on the frequencies of occurrences of the states for Markov chain $\{Y_n,\ n\geqslant 0\}$.

Corollary 2. *Let $S_n(k,l)$ be defined as above and $S_n(k)$ be the number of occurrences of k in the sequence $\{Y_0,Y_1,\dots,Y_{n-1}\}$. Set*

$$D(k)=\{\lim_{n\to\infty}S_n(k)=\infty\}.$$

Then

$$\lim_{n\to\infty}\frac{S_n(k,l)-\sum_{m=1}^{n}\delta_k(Y_{m-1})p_m(k,l)}{S_n(k)}=0 \quad a.s.\ on\ D(k).$$

If moreover,

$$\lim_{n\to\infty}p_n(k,l)=p(k,l),$$

then

$$\lim_{n\to\infty}\frac{S_n(k,l)}{S_n(k)}=p(k,l) \quad a.s.\ on\ D(k).$$

Proof. Letting $a_n=S_n(k)$ in (11), and noting that

$$S_n(k)=\sum_{m=1}^{n}\delta_k(Y_{m-1})\geqslant\sum_{m=1}^{n}\delta_k(Y_{m-1})p_m(k,l),$$

we immediately get the result. □

As an application of Theorem 1, we obtain another strong law of large numbers on the frequencies of occurrences of the states for Markov chain $\{Y_n,\ n\geqslant 0\}$.

Theorem 4. *Let (a_n) be a sequence of positive random variables. Set*

$$A = \left\{ \lim_{n\to\infty} a_n = \infty, \ \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n} p_m(Y_{m-1}, k) < \infty \right\}.$$

Then

$$\lim_{n\to\infty} \frac{1}{a_n} \left\{ S_n(k) - \sum_{m=1}^{n} p_m(Y_{m-1}, k) \right\} = 0 \quad a.s.\ on\ A. \tag{13}$$

In particular, we have

$$\lim_{n\to\infty} \frac{1}{n} \left\{ S_n(k) - \sum_{m=1}^{n} p_m(Y_{m-1}, k) \right\} = 0 \quad a.s.$$

If we set

$$H(k) = \left\{ \lim_{n\to\infty} \sum_{m=1}^{n} p_m(Y_{m-1}, k) = \infty \right\},$$

then

$$\lim_{n\to\infty} \frac{S_n(k)}{\sum_{m=1}^{n} p_m(Y_{m-1}, k)} = 1 \quad a.s.\ on\ H(k). \tag{14}$$

Proof. Let $\xi_n = \delta_k(Y_n)$ and $\mathscr{F}_n = \sigma(Y_0, Y_1, \ldots, Y_n)$ in Theorem 1. Then

$$\sum_{m=1}^{n} \xi_m = S_n(k) + \delta_k(Y_n) - \delta_k(Y_0),$$

and

$$\mathbf{E}[\xi_n | \mathscr{F}_{n-1}] = \mathbf{E}[\delta_k(Y_n) | Y_{n-1}] = p_n(Y_{n-1}, k).$$

Thus, from (2) we have that (13) follows. Letting $a_n = \sum_{m=1}^{n} p_m(Y_{m-1}, k)$ in (13) gives (14). $\square$

As another application of Theorem 1, we obtain the following generalized strong ergodic theorem for irreducible and positive recurrent Markov chains.

Theorem 5. *Let $(X_n,\ n \geqslant 0)$ be an irreducible and positive recurrent Markov chain with state space $S = \{1, 2, \ldots\}$, transition probability matrix $\{p(i,j)\}$, and invariant probability distribution $\{\pi(i)\}$. Let $F(x_0, x_1, \ldots, x_N)$ be a bounded function on S^{N+1}. Then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(X_k, \ldots, X_{k+N}) = \sum_{j_0=1}^{\infty} \cdots \sum_{j_N=1}^{\infty} \pi(j_0) F(j_0, \ldots, j_N) \prod_{l=1}^{N} p(j_l, j_{l+1}). \tag{15}$$

Proof. First we prove (15) for $N=1$. Let $\xi_k=F(X_k,X_{k+1})$, and $\mathscr{F}_k=\sigma(X_0,\dots,X_{k+1})$. Then by the Markov property,

$$\mathbf{E}[\xi_k|\mathscr{F}_{k-1}]=\mathbf{E}[F(X_k,X_{k+1})|X_k]=\sum_j F(X_k,j)p(X_k,j).$$

By Theorem 1 we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[F(X_k,X_{k+1})-\sum_j F(X_k,j)p(X_k,j)\right]=0 \quad \text{a.s..} \tag{16}$$

However, by the Strong Ergodic Theorem for Markov chain (X_n) (see Freedman, 1983, p. 75), we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}F(X_k,j)p(X_k,j)=\sum_{i=1}^{\infty}\pi(i)F(i,j)p(i,j), \quad \text{a.s..}$$

This together with (16) implies that (15) holds for the case $N=1$.

Now we are going to prove (15) for general N by induction. Assume (15) holds for $N=m$. Let $\xi_k=F(X_k,\dots,X_{k+m+1})$ and $\mathscr{F}_k=\sigma(X_0,\dots,X_{k+m+1})$. Then by the Markov property of $\{X_n\}$ we have

$$\begin{aligned}\mathbf{E}[\xi_k|\mathscr{F}_{k-1}]&=\mathbf{E}[F(x_k,\dots,x_{k+m},X_{k+m+1})|\mathscr{F}_{k-1}]|_{x_k=X_k,\dots,x_{k+m}=X_{k+m}}\\&=\mathbf{E}[F(x_k,\dots,x_{k+m},X_{k+m+1})|X_{k+m}]|_{x_k=X_k,\dots,x_{k+m}=X_{k+m}}\\&=\sum_j F(X_k,\dots,X_{k+m},j)p(X_{k+m},j).\end{aligned}$$

Consequently, by Theorem 1 and the induction assumption that (15) holds for $N=m$, we see that (15) holds for $N=m+1$. This completes the proof of the theorem. □

Acknowledgements

The authors would like to thank an anonymous referee for helpful suggestions.

References

Chow, Y.S., Teicher, H., 1988. Probability Theory, 2nd Edition. Springer, New York.

Dubins, L.E., Freedman, D.A., 1965. A sharper form of the Borel–Cantelli lemma and the strong law. Ann. Math. Statist. 36, 800–807.

Freedman, D., 1983. Markov Chains. Springer, New York.

Liu, W., Yang, W.G., 1995. A limit theorem for the entropy density of nonhomogeneous Markov information source, Statist. Probab. Lett. 22, 295–301.

Yang, W.G., Liu, W., 2000. Strong law of large numbers for Markov chain fields on a Bethe tree. Statist. Probab. Lett. 49, 245–250.