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SHARP LARGE DEVIATIONS FOR THE FRACTIONAL ORNSTEIN–UHLENBECK PROCESS*

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Abstract. We investigate the sharp large deviation properties of the energy and the maximum likelihood estimator for the Ornstein–Uhlenbeck process driven by a fractional Brownian motion with Hurst index greater than one half.

Key words. large deviations, Ornstein–Uhlenbeck process, fractional Brownian motion

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1. Introduction. Since the pioneering works of Kolmogorov, Hurst, and Mandelbrot, a wide range of literature has been available on the statistical properties of fractional Brownian motion. On the other hand, one can realize that its large deviations properties were not deeply investigated. Our purpose is to establish sharp large deviations for functionals associated with the Ornstein–Uhlenbeck process driven by a fractional Brownian motion

$$dX_t = \theta X_t\, dt + dW_t^H, \tag{1.1}$$

where the initial state X_0 is 0 and the drift parameter θ is strictly negative. The process (W_t^H) is a fractional Brownian motion with the Hurst parameter $0 < H < 1$, which means that (W_t^H) is a Gaussian process with continuous paths such that $W_0^H = 0$, $\mathbf{E}\,[W_t^H] = 0$ and

$$\mathbf{E}\,[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The weighting function w defined, for all $0 < s < t$, by $w(t,s) = w_H^{-1} s^{-H+\frac{1}{2}}(t-s)^{-H+\frac{1}{2}}$, where w_H is a normalizing positive constant, plays a fundamental role for stochastic calculus associated with (W_t^H). In particular, for $t > 0$, let

$$M_t = \int_0^t w(t,s)\, dW_s^H. \tag{1.2}$$

It was proved in [14, p. 578] that (M_t) is a Gaussian martingale with quadratic variation $\langle M\rangle_t = \lambda_H^{-1} t^{2-2H}$, where

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$$

and Γ stands for the classical gamma function. Moreover, it follows from Jost's formula [11] on the transformation of the process (W_t^H) into (W_t^{1-H}) that

$$W_t^H = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}} \int_0^t (t-s)^{2H-1}\, dW_s^{1-H}.$$

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Consequently, we assume in what follows that $\frac{1}{2} < H < 1$, since our investigation also holds without any problem in the case $0 < H < \frac{1}{2}$. It is also more convenient to study the behavior of

$$Y_t = \int_0^t w(t,s)\,dX_s = \theta \int_0^t w(t,s) X_s\,ds + M_t. \tag{1.3}$$

It was shown in [12] that relation (1.3) can be rewritten as

$$Y_t = \theta \int_0^t Q_s\,d\langle M\rangle_s + M_t, \tag{1.4}$$

where the process (Q_t) satisfies for all $t > 0$

$$Q_t = \frac{l_H}{2}\left(t^{2H-1} Y_t + \int_0^t s^{2H-1}\,dY_s \right)$$

with $l_H = \lambda_H/(2(1-H))$. It follows from (1.4) that the score function, which is the derivative of the log-likelihood function from observations over the interval $[0, T]$, is given by

$$\Sigma_T(\theta) = \int_0^T Q_t\,dY_t - \theta \int_0^T Q_t^2\,d\langle M\rangle_t.$$

Via an approach similar to the one of [3] for the Ornstein–Uhlenbeck process, we shall investigate the large deviation properties for random variables associated with (1.1) such as the energy

$$S_T = \int_0^T Q_t^2\,d\langle M\rangle_t \tag{1.5}$$

as well as the maximum likelihood estimator of θ, solution of $\Sigma_T(\theta) = 0$, given by

$$\hat{\theta}_T = \frac{\int_0^T Q_t\,dY_t}{\int_0^T Q_t^2\,d\langle M\rangle_t}. \tag{1.6}$$

We also wish to mention the recent work [4] concerning the large deviation properties of the log-likelihood ratio

$$(\theta_0 - \theta_1) \int_0^T Q_t\,dY_t - \frac{\theta_0^2 - \theta_1^2}{2} \int_0^T Q_t^2\,d\langle M\rangle_t.$$

As usual, we say that a family of real random variables (Z_T) satisfies a Large Deviation Principle (LDP) with a rate function I if I is a lower semicontinuous function from $\mathbf{R}$ to $[0, +\infty]$ such that, for any closed set $F \subset \mathbf{R}$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(Z_T \in F) \leqq -\inf_{x\in F} I(x),$$

while for any open set $G \subset \mathbf{R}$,

$$-\inf_{x\in G} I(x) \leqq \liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(Z_T \in G).$$

Moreover, I is a good rate function if its level sets are compact subsets of $\mathbf{R}$. We refer the reader to the excellent book [7] on the theory of large deviations; see also [2], [9]. A classical tool for proving an LDP for S_T and $\hat{\theta}_T$ is the normalized cumulant generating function

$$\mathcal{L}_T(a,b) = \frac{1}{T}\log \mathbf{E}\Big[\exp(\mathcal{Z}_T(a,b))\Big], \tag{1.7}$$

where, for any $(a,b) \in \mathbf{R}^2$,

$$\mathcal{Z}_T(a,b) = a\int_0^T Q_t\,dY_t + b\int_0^T Q_t^2\,d\langle M\rangle_t. \tag{1.8}$$

The random variable $\mathcal{Z}_T(a,b)$ allows for a unified presentation of our results. In order to establish an LDP for S_T and $\hat{\theta}_T$, it is enough to prove an LDP for $\mathcal{Z}_T(0,a)$ and $\mathcal{Z}_T(a,-ac)$, respectively. As a matter of fact, we have for all $a, c \geqq 0$,

$$\mathbf{P}(S_T \geqq cT) = \mathbf{P}(Z_T(0,a) \geqq acT), \quad \mathbf{P}(\hat{\theta}_T \geqq c) = \mathbf{P}(Z_T(a,-ac) \geqq 0).$$

The following lemma was partially established by formula (5.12) in [12]. It provides an asymptotic expansion for $\mathcal{L}_T$ which enlightens the role of the limit $\mathcal{L}$ of $\mathcal{L}_T$ for the LDP, as well as the first order terms $\mathcal{H}$ and $\mathcal{K}_T$ for the sharp LDP (in what follows, abbreviated SLDP). One can observe that it is the keystone of all our results.

LEMMA 1. *Let Δ_H be the effective domain of the limit $\mathcal{L}$ of $\mathcal{L}_T$:*

$$\Delta_H = \Big\{(a,b) \in \mathbf{R}^2 \colon\ \theta^2 - 2b > 0 \text{ and } \sqrt{\theta^2 - 2b} > \max(a+\theta; -\delta_H(a+\theta))\Big\},$$

where $\delta_H = (1-\sin(\pi H))/(1+\sin(\pi H))$. Then, for any (a,b) in the interior of Δ_H, if $\varphi(b) = \sqrt{\theta^2 - 2b}$, $\tau(a,b) = \varphi(b) - (a+\theta)$, and $r_T(b) = r_H(\varphi(b)T/2)\exp(-T\varphi(b)) - 1$, we have the decomposition

$$\mathcal{L}_T(a,b) = \mathcal{L}(a,b) + \frac{1}{T}\,\mathcal{H}(a,b) + \frac{1}{T}\,\mathcal{K}_T(a,b) + \frac{1}{T}\,\mathcal{R}_T(a,b), \tag{1.9}$$

where

$$\mathcal{L}(a,b) = -\frac{1}{2}(a + \theta + \varphi(b)), \tag{1.10}$$

$$\mathcal{H}(a,b) = -\frac{1}{2}\,\log\frac{\tau(a,b)}{2\varphi(b)}, \tag{1.11}$$

$$\mathcal{K}_T(a,b) = -\frac{1}{2}\,\log\left(1 + \frac{2\varphi(b) - \tau(a,b)}{2\varphi(b)} r_T(b)\right), \tag{1.12}$$

with the function r_H defined for all $z \in \mathbf{C}$, with $|\arg z| < \pi$, by

$$r_H(z) = \frac{\pi z}{\sin(\pi H)}\,(I_H(z)I_{1-H}(z) + I_{-H}(z)I_{H-1}(z)),$$

where I_H is the modified Bessel function of the first kind [13]. *Finally, the remainder is*

$$\mathcal{R}_T(a,b) = -\frac{1}{2}\,\log\left(1 + \frac{(2\varphi(b) - \tau(a,b))^2}{\tau(a,b)(2\varphi(b) + r_T(b)(2\varphi(b) - \tau(a,b)))}\,e^{-2T\varphi(b)}\right). \tag{1.13}$$

The proof of Lemma 1 is given in Appendix A.

Remark 1. By use of the duplication formula for the gamma function [13], one can realize that if $H = \frac{1}{2}$, then $r_H(z) = e^{2z} + e^{-2z}$, which immediately leads to $r_T(b) = e^{-2T\varphi(b)}$. Consequently, in that particular case, $\mathcal{K}_T(a,b)$ as well as $\mathcal{R}_T(a,b)$ go exponentially fast to zero, and we find again Lemma 2.1 of [3] which is the keystone for all results in [3].

2. The energy. First of all, we shall focus our attention on the energy (S_T). One can observe that the strong law of large numbers as well as the central limit theorem (CLT) for the sequence (S_T) were not previously established in the literature.

PROPOSITION 1. *We have*

$$\lim_{T\to\infty} \frac{S_T}{T} = -\frac{1}{2\theta} \quad a.s. \tag{2.1}$$

Moreover, we also have the CLT

$$\frac{1}{\sqrt{T}}\left(S_T + \frac{T}{2\theta}\right) \xrightarrow{\text{Law}} \mathcal{N}\left(0, -\frac{1}{2\theta^3}\right). \tag{2.2}$$

Proof. The almost sure convergence (2.1) clearly follows from Theorem 2 below together with the Borel–Cantelli lemma. As a matter of fact, the sequence (S_T/T) satisfies an SLDP with good rate function I given by (2.11) and $I(c) = 0$ if and only if $c = -1/(2\theta)$. In order to prove the CLT given by (2.2), denote

$$V_T = \frac{1}{\sqrt{T}}\left(S_T + \frac{T}{2\theta}\right).$$

For all $a \in \mathbf{R}$, let $\Lambda_T(a) = \mathbf{E}\left[\exp(aV_T)\right]$. We clearly have

$$\Lambda_T(a) = \exp\left(\frac{a\sqrt{T}}{2\theta}\right)\mathbf{E}\left[\exp\left(\frac{aS_T}{\sqrt{T}}\right)\right].$$

Hence, we deduce from the decomposition (1.9) with $a = 0$ and $b = a$ that

$$\begin{aligned}\Lambda_T(a) = \exp\Bigg(&\frac{a\sqrt{T}}{2\theta} + T\mathcal{L}\left(0, \frac{a}{\sqrt{T}}\right) + \mathcal{H}\left(0, \frac{a}{\sqrt{T}}\right)\\ &+ \mathcal{K}_T\left(0, \frac{a}{\sqrt{T}}\right) + \mathcal{R}_T\left(0, \frac{a}{\sqrt{T}}\right)\Bigg).\end{aligned} \tag{2.3}$$

On the one hand,

$$\mathcal{L}\left(0, \frac{a}{\sqrt{T}}\right) = -\frac{1}{2}(\theta + \varphi_T),$$

where

$$\varphi_T = \sqrt{\theta^2 - \frac{2a}{\sqrt{T}}} = -\theta\sqrt{1 - \frac{2a}{\theta^2\sqrt{T}}}.$$

Consequently, since

$$\varphi_T = -\theta + \frac{a}{\theta\sqrt{T}} + \frac{a^2}{2\theta^3 T} + o\left(\frac{1}{T}\right),$$

we obtain that

$$\lim_{T\to\infty} \frac{a\sqrt{T}}{2\theta} + T\mathcal{L}\Big(0, \frac{a}{\sqrt{T}}\Big) = -\frac{a^2}{4\theta^3}. \tag{2.4}$$

On the other hand, since (φ_T) converges to $-\theta$, one can check that

$$\lim_{T\to\infty} \left(\mathcal{H}\Big(0, \frac{a}{\sqrt{T}}\Big) + \mathcal{K}_T\Big(0, \frac{a}{\sqrt{T}}\Big) + \mathcal{R}_T\Big(0, \frac{a}{\sqrt{T}}\Big)\right) = 0. \tag{2.5}$$

Therefore, we infer from (2.3)–(2.5) that

$$\lim_{T\to\infty} \Lambda_T(a) = \exp\Big(-\frac{a^2}{4\theta^3}\Big). \tag{2.6}$$

Convergence (2.6) holds for all a in a neighborhood of the origin which leads to (2.2) and completes the proof of Proposition 1.

In order to obtain the large deviation properties for (S_T), we shall make use of Lemma 1 with $a = 0$ and $b = a$. On the one hand, let

$$D_H = \Big\{a \in \mathbf{R}\colon\ \theta^2 - 2a > 0 \text{ and } \sqrt{\theta^2 - 2a} > -\delta_H\theta\Big\}.$$

It is not hard to see that $D_H = \,]-\infty, a_H[\,$, where

$$a_H = \frac{\theta^2}{2}(1 - \delta_H^2). \tag{2.7}$$

Consequently, since $|\delta_H| < 1$, one can observe that the origin always belongs to the interior of D_H. On the other hand, for all $a \in D_H$, let $\varphi(a) = \sqrt{\theta^2 - 2a}$,

$$L(a) = \mathcal{L}(0, a) = -\frac{1}{2}(\theta + \sqrt{\theta^2 - 2a}), \tag{2.8}$$

$$H(a) = \mathcal{H}(0, a) = -\frac{1}{2}\log\frac{\varphi(a) - \theta}{2\varphi(a)}, \tag{2.9}$$

$$K_T(a) = \mathcal{K}_T(0, a) = -\frac{1}{2}\log\Big(1 + \frac{\varphi(a) + \theta}{2\varphi(a)}\, r_T(a)\Big). \tag{2.10}$$

The main difficulty compared to [3] is that the function L is not steep. Actually,

$$L'(a) = \frac{1}{2\sqrt{\theta^2 - 2a}},$$

which implies that $L'(a_H) = -1/(2\theta\delta_H)$. Moreover, for all $c > 0$, $L'(a) = c$ if and only if $a = a_c$ with $a_c = (4\theta^2c^2 - 1)/(8c^2)$. Hence, $a_c < a_H$ whenever $0 < c < -1/(2\theta\delta_H)$. Denote by I the Fenchel–Legendre transform of the function L. Our first result on the large deviation properties of (S_T/T) is as follows.

THEOREM 1. *The sequence (S_T/T) satisfies an LDP with good rate function*

$$I(c) = \begin{cases} \dfrac{(2\theta c + 1)^2}{8c} & \text{if } 0 < c \leqq -\dfrac{1}{2\theta\delta_H}, \\ \dfrac{c\theta^2}{2}(1 - \delta_H^2) + \dfrac{\theta}{2}(1 - \delta_H) & \text{if } c \geqq -\dfrac{1}{2\theta\delta_H}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}$$

Remark 2. If $H = \frac{1}{2}$, then $\delta_H = 0$ and the LDP for (S_T/T) is exactly the one established in [6] for general centered Gaussian processes.

We are now going to improve this result by an SLDP for (S_T/T) inspired by the well-known Bahadur–Rao theorem [1] on the sample mean.

THEOREM 2. *The sequence (S_T/T) satisfies an SLDP associated with L, H, and K_T given by* (2.8), (2.9), *and* (2.10), *respectively.*

(a) *For all $-1/(2\theta) < c < -1/(2\theta\delta_H)$, there exists a sequence $(b_{c,k}^H)$ such that, for any $p > 0$ and T large enough,*

$$\mathbf{P}(S_T \geqq cT) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{a_c\sigma_c\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{2.12}$$

while, for $0 < c < -1/(2\theta)$,

$$\mathbf{P}(S_T \leqq cT) = -\frac{\exp(-TI(c) + J(c) + K_H(c))}{a_c\sigma_c\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{2.13}$$

where

$$J(c) = -\frac{1}{2}\log\frac{1-2\theta c}{2}, \quad K_H(c) = -\frac{1}{2}\log\frac{(1+\sin(\pi H))(1+2\theta c\delta_H)}{2\sin(\pi H)}, \tag{2.14}$$

with

$$a_c = \frac{4\theta^2c^2 - 1}{8c^2}, \quad \sigma_c^2 = 4c^3. \tag{2.15}$$

Moreover, the coefficients $b_{c,1}^H, \ldots, b_{c,p}^H$ can be calculated explicitly as functions of the derivatives of L and H evaluated at point a_c. They also depend on the Taylor expansion of K_T and its derivatives at a_c.

(b) *For all $c > -1/(2\theta\delta_H)$, there exists a sequence $(d_{c,k}^H)$ such that, for any $p > 0$ and T large enough,*

$$\mathbf{P}(S_T \geqq cT) = \frac{\exp(-TI(c) + P_H(c) + Q_H(c))}{a_H\sigma_H\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{2.16}$$

where

$$P_H(c) = -\frac{1}{2}\log\frac{-(1+2\theta c\delta_H)}{4\delta_H\sin(\pi H)}, \quad Q_H(c) = \frac{(2H-1)^2\sin(\pi H)(1+2\theta c\delta_H)}{2(1-(\sin(\pi H))^2)} \tag{2.17}$$

with

$$a_H = \frac{\theta^2(1-\delta_H^2)}{2}, \quad \sigma_H^2 = -\frac{1}{2\theta^3\delta_H^3}. \tag{2.18}$$

Moreover, the coefficients $d_{c,1}^H, \ldots, d_{c,p}^H$ can be calculated explicitly as functions of the derivatives of L and H evaluated at point a_H. They also depend on the Taylor expansion of K_T and its derivatives at a_H.

Remark 3. For example, the first coefficient $b_{c,1}^H$ is given by

$$b_{c,1}^H = \frac{1}{\sigma_c^2}\left(\frac{s_1}{a_c} - \frac{s_1^2}{2} - \frac{s_2}{2} - \frac{s_3}{2a_c\sigma_c^2} + \frac{s_1\ell_3}{2\sigma_c^2} - \frac{5\ell_3^2}{24\sigma_c^4} + \frac{\ell_4}{8\sigma_c^2} - \frac{1}{a_c^2}\right) + k_{c,1}^H,$$

where $\ell_q = L^{(q)}(a_c)$, $h_q = H^{(q)}(a_c)$, $s_q = h_q + k_q$, with

$$k_1 = \lim_{T\to\infty} K'_T(a_c) = \frac{-4\theta p_H c^3}{2 + p_H(1+2\theta c)},$$

$$k_2 = \lim_{T\to\infty} K''_T(a_c) = \frac{16\theta p_H c^5(6 + p_H(3+2\theta c))}{(2 + p_H(1+2\theta c))^2},$$

$$k_{c,1}^H = \lim_{T\to\infty} T(K_T(a_c) - K_H(c)) = \frac{c(1+2\theta c)(2H-1)^2}{2\sin(\pi H)(2 + p_H(1+2\theta c))},$$

$p_H = (1-\sin(\pi H))/\sin(\pi H)$. In addition, one can also observe that $\sigma_c^2 = L''(a_c)$ and $\sigma_H^2 = L''(a_H)$.

THEOREM 3. *For $c = -1/(2\theta\delta_H)$, there exists a sequence (d_k^H) such that, for any $p > 0$ and T large enough,*

$$\mathbf{P}(S_T \geqq cT) = \frac{\exp(-TI(c) + K_H)\Gamma(1/4)}{2\pi a_H \sigma_H T^{1/4}} \left[1 + \sum_{k=1}^{2p} \frac{d_k^H}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right], \tag{2.19}$$

where a_H and σ_H^2 are given by (2.18) *and*

$$K_H = \frac{1}{2}\log(\delta_H \sin(\pi H)) + \frac{1}{4}\log(-\theta\delta_H).$$

As before, the coefficients $d_1^H, \dots, d_{2p}^H$ may be calculated explicitly.

The proofs of Theorems 2 and 3 are given in section 4.

3. The maximum likelihood estimator. Our purpose is now to establish similar results for the maximum likelihood estimator $\hat{\theta}_T$ of θ. An alternative estimator of θ may be found in [10]. The almost sure convergence of $\hat{\theta}_T$ towards θ was already proved in [12]; see also [15], [16] for related results. We recently learned that the CLT was established via a different approach in [5].

PROPOSITION 2. *We have*

$$\lim_{T\to\infty} \hat{\theta}_T = \theta \quad a.s. \tag{3.1}$$

Moreover, we also have the CLT

$$\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{\text{Law}} \mathcal{N}(0, -2\theta). \tag{3.2}$$

Proof. The almost sure convergence (3.1) is given in [12, Proposition 2.2]. For all $c \in \mathbf{R}$, denote

$$V_T(c) = \frac{1}{\sqrt{T}} \int_0^T Q_t\, dY_t - \left(\frac{c}{\sqrt{T}} + \theta\right) \frac{S_T}{\sqrt{T}}.$$

One can easily check that

$$\mathbf{P}(\sqrt{T}(\hat{\theta}_T - \theta) \leqq c) = \mathbf{P}(V_T(c) \leqq 0). \tag{3.3}$$

Consequently, in order to prove the CLT given by (3.2), it is necessary only to establish the asymptotic behavior of the sequence $(V_T(c))$, where the threshold c can be seen as a parameter. For all $(a, c) \in \mathbf{R}^2$, let $\Lambda_T(a, c) = \mathbf{E}\left[\exp(aV_T(c))\right]$. We clearly have

$$\Lambda_T(a, c) = \exp\left(TL_T\left(\frac{a}{\sqrt{T}}, c_T\right)\right), \quad \text{where } c_T = -\frac{a}{\sqrt{T}}\left(\frac{c}{\sqrt{T}} + \theta\right).$$

Thus, it follows from the decomposition (1.9) that

$$\Lambda_T(a,c) = \exp\left(T\mathcal{L}\left(\frac{a}{\sqrt{T}}, c_T\right) + \mathcal{H}\left(\frac{a}{\sqrt{T}}, c_T\right) + \mathcal{K}_T\left(\frac{a}{\sqrt{T}}, c_T\right) + \mathcal{R}_T\left(\frac{a}{\sqrt{T}}, c_T\right)\right). \tag{3.4}$$

On the one hand,

$$\mathcal{L}\left(\frac{a}{\sqrt{T}}, c_T\right) = -\frac{1}{2}\left(\frac{a}{\sqrt{T}} + \theta + \varphi_T\right),$$

where $\varphi_T = \sqrt{\theta^2 - 2c_T} = -\theta\sqrt{1 - 2c_T/\theta^2}$. Hence, since

$$\varphi_T = -\theta - a/\sqrt{T} + (a^2 - 2ac)/(2\theta T) + o(1/T),$$

we deduce that

$$\lim_{T\to\infty} T\mathcal{L}\left(\frac{a}{\sqrt{T}}, c_T\right) = -\frac{a^2 - 2ac}{4\theta}. \tag{3.5}$$

On the other hand, since (φ_T) converges to $-\theta$, it is not hard to see that

$$\lim_{T\to\infty}\left(\mathcal{H}\left(\frac{a}{\sqrt{T}}, c_T\right) + \mathcal{K}_T\left(\frac{a}{\sqrt{T}}, c_T\right) + \mathcal{R}_T\left(\frac{a}{\sqrt{T}}, c_T\right)\right) = 0. \tag{3.6}$$

The conjunction of (3.4), (3.5), and (3.6) leads to

$$\lim_{T\to\infty} \Lambda_T(a,c) = \exp\left(-\frac{a^2 - 2ac}{4\theta}\right). \tag{3.7}$$

This convergence holds for all a in a neighborhood of the origin. Consequently,

$$V_T(c) \xrightarrow{\text{Law}} \mathcal{N}\left(\frac{c}{2\theta}, \frac{-1}{2\theta}\right). \tag{3.8}$$

Denote by $V(c)$ the limiting distribution of $(V_T(c))$. It follows from a standard Gaussian calculation that

$$\mathbf{P}(V(c) \leqq 0) = \frac{1}{-4\pi\theta}\int_{-\infty}^{c} \exp\left(\frac{x^2}{2\theta}\right) dx. \tag{3.9}$$

Finally, (3.3) and (3.9) imply (3.2), which completes the proof of Proposition 2.

In order to establish the large deviation properties of $(\hat{\theta}_T)$, we shall make use of the auxiliary random variable defined for all $c \in \mathbf{R}$ by

$$Z_T(c) = \int_0^T Q_t\, dY_t - c\int_0^T Q_t^2\, d\langle M\rangle_t, \tag{3.10}$$

where we recall that $\mathbf{P}(\hat{\theta}_T \geqq c) = \mathbf{P}(Z_T(c) \geqq 0)$. Let

$$D_H = \left\{a \in \mathbf{R}\colon\ \theta^2 + 2ac > 0 \text{ and } \sqrt{\theta^2 + 2ac} > \max(a + \theta; -\delta_H(a + \theta))\right\}.$$

After some straightforward calculations, it is not hard to see that

$$D_H = \begin{cases}]a_1^H, a_2^H[& \text{if } c \leqq \dfrac{\theta}{2}, \\ ]a_1^H, a^c[& \text{if } c > \dfrac{\theta}{2}, \end{cases}$$

where $a^c = 2(c-\theta)$ and

$$a_1^H = \frac{c - \theta\mu_H - \sqrt{c^2 - 2\theta c\mu_H + \theta^2\mu_H}}{\mu_H},$$

$$a_2^H = \frac{c - \theta\mu_H + \sqrt{c^2 - 2\theta c\mu_H + \theta^2\mu_H}}{\mu_H},$$

with $\mu_H = \delta_H^2$. In addition, for all $a \in D_H$, let $\varphi(a) = \sqrt{\theta^2 + 2ac}$ and

$$L(a) = \mathcal{L}(a, -ca) = -\frac{1}{2}\left(a + \theta + \sqrt{\theta^2 + 2ac}\right), \tag{3.11}$$

$$H(a) = \mathcal{H}(a, -ca) = -\frac{1}{2}\log\frac{\varphi(a) - a - \theta}{2\varphi(a)}, \tag{3.12}$$

$$K_T(a) = \mathcal{K}_T(a, -ca) = -\frac{1}{2}\log\left(1 + \frac{\varphi(a) + a + \theta}{2\varphi(a)} r_T(-ac)\right). \tag{3.13}$$

The function L is not steep, since the derivative of L is finite at the boundary of D_H. Moreover, $L'(a) = 0$ if and only if $a = a_c$ with $a_c = (c^2 - \theta^2)/(2c)$ and $a_c \in D_H$ whenever $c < \theta/3$.

THEOREM 4. *The maximum likelihood estimator* $(\hat{\theta}_T)$ *satisfies an LDP with good rate function*

$$I(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c < \dfrac{\theta}{3}, \\ 2c - \theta & \text{if } c \geqq \dfrac{\theta}{3}. \end{cases} \tag{3.14}$$

Remark 4. One can observe that the rate function I is totally free of the parameter H. Hence, $(\hat{\theta}_T)$ shares the same LDP as the one established in [8] for the standard Ornstein–Uhlenbeck process with $H = \frac{1}{2}$.

THEOREM 5. *The maximum likelihood estimator* $(\hat{\theta}_T)$ *satisfies an SLDP associated with* L, H, *and* K_T *given by* (3.11), (3.12), *and* (3.13), *respectively.*

(a) *For all* $\theta < c < \theta/3$, *there exists a sequence* $(b_{c,k}^H)$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbf{P}(\hat{\theta}_T \geqq c) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c a_c\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{3.15}$$

while, for $c < \theta$,

$$\mathbf{P}(\hat{\theta}_T \leqq c) = -\frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c a_c\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{b_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{3.16}$$

where

$$J(c) = -\frac{1}{2}\log\frac{(c+\theta)(3c-\theta)}{4c^2}, \quad K_H(c) = -\frac{1}{2}\log\left(1 + p_H\,\frac{(c-\theta)^2}{4c^2}\right) \tag{3.17}$$

with $p_H = (1-\sin(\pi H))/\sin(\pi H)$*,*

$$a_c = \frac{c^2-\theta^2}{2c}, \quad \sigma_c^2 = -\frac{1}{2c}. \tag{3.18}$$

Moreover, the coefficients $b_{c,1}^H, \ldots, b_{c,p}^H$ *can be calculated explicitly as in Theorem* 2*.*

(b) *For all* $c > \theta/3$ *with* $c \neq 0$*, there exists a sequence* $(d_{c,k}^H)$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbf{P}(\hat{\theta}_T \geqq c) = \frac{\exp(-TI(c)+P(c))\sqrt{\sin(\pi H)}}{\sigma^c a^c\sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p}\frac{d_{c,k}^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{3.19}$$

where

$$P(c) = -\frac{1}{2}\log\frac{(c-\theta)(3c-\theta)}{4c^2}, \tag{3.20}$$

$$a^c = 2(c-\theta), \quad (\sigma^c)^2 = \frac{c^2}{2(2c-\theta)^3}. \tag{3.21}$$

In addition, the coefficients $d_{c,1}^H, \ldots, d_{c,p}^H$ *can be calculated explicitly as in Theorem* 2*.*

(c) *Finally, for* $c = 0$*, for any* $p > 0$ *and for* T *large enough,*

$$\mathbf{P}(\hat{\theta}_T \geqq 0) = 2\,\frac{\exp(-TI(c))\sqrt{\sin(\pi H)}}{\sqrt{2\pi T}\sqrt{-2\theta}}\left[1 + \sum_{k=1}^{p}\frac{d_k^H}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{3.22}$$

As before, the coefficients $d_1^H, \ldots, d_p^H$ *can be calculated explicitly.*

THEOREM 6. *For* $c = \theta/3$*, there exists a sequence* (e_k^H) *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbf{P}\left(\hat{\theta}_T \geqq \frac{\theta}{3}\right) = \frac{\exp(-TI(c))\Gamma(1/4)}{4\pi T^{1/4}\,a_\theta^{3/4}\,\sigma_\theta}\sqrt{\sin(\pi H)}\left[1 + \sum_{k=1}^{2p}\frac{e_k^H}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right], \tag{3.23}$$

where a_θ *and* σ_θ *are given by*

$$a_\theta = -\frac{4\theta}{3}, \quad \sigma_\theta^2 = -\frac{3}{2\theta}. \tag{3.24}$$

As before, the coefficients $e_1^H, \ldots, e_{2p}^H$ *can be calculated explicitly.*

The proofs are left to the reader, since they essentially follow along the same lines as those for the energy.

4. Proofs of the main results.

4.1. Proof of Theorem 2, first part. We first focus our attention on the SLDP for the energy S_T in the easy case $-1/(2\theta) < c < -1/(2\theta\delta_H)$. Let L_T be the normalized cumulant generating function of S_T. We already have seen that a_c, given by (2.15), belongs to the domain D_H whenever $c < -1/(2\theta\delta_H)$. For all $-1/(2\theta) < c < -1/(2\theta\delta_H)$, we can split $\mathbf{P}(S_T \geqq cT)$ into two terms, $\mathbf{P}(S_T \geqq cT) = A_T B_T$ with

$$A_T = \exp(T(L_T(a_c) - ca_c)), \tag{4.1}$$

$$B_T = \mathbf{E}_T\Big[\exp(-a_c(S_T - cT))\mathbf{1}_{S_T \geqq cT}\Big], \tag{4.2}$$

where $\mathbf{E}_T$ stands for the expectation after the usual change of probability

$$\frac{d\mathbf{P}_T}{d\mathbf{P}} = \exp\Big(a_c S_T - TL_T(a_c)\Big). \tag{4.3}$$

On one hand, we can deduce from (1.9) with $L(a) = \mathcal{L}(0,a)$, $H(a) = \mathcal{H}(0,a)$, $K_T(a) = \mathcal{K}_T(0,a)$, and $R_T(a) = \mathcal{R}_T(0,a)$ together with (2.11) and (2.14) that

$$A_T = \exp\Big(T(L(a_c) - ca_c) + H(a_c) + K_T(a_c) + R_T(a_c)\Big),$$

$$A_T = \exp\Big(- TI(c) + J(c) + K_T(a_c) + R_T(a_c)\Big). \tag{4.4}$$

Consequently, we infer from (2.10) and (2.14) that for any $p > 0$ and T large enough

$$K_T(a_c) = K_H(c) + \sum_{k=1}^{p} \frac{\gamma_k}{T^k} + \mathcal{O}\bigg(\frac{1}{T^{p+1}}\bigg), \tag{4.5}$$

where the coefficients (γ_k), which also depend on H, may be calculated explicitly. In addition, it is not hard to see from (1.13) that $R_T(a_c) = \mathcal{O}(\exp(-T/c))$. Therefore, we deduce from (4.4) and (4.5) that for any $p > 0$ and T large enough

$$A_T = \exp(-TI(c) + J(c) + K_H(c))\bigg[1 + \sum_{k=1}^{p} \frac{\alpha_k}{T^k} + \mathcal{O}\bigg(\frac{1}{T^{p+1}}\bigg)\bigg], \tag{4.6}$$

where, as before, the coefficients (α_k) can be calculated explicitly. For example,

$$\alpha_1 = \frac{-2c(1+2\theta c)r_1^H}{\sin(\pi H)(2 + (1+2\theta c)p_H)}.$$

It now remains to give the expansion for B_T which can be rewritten as

$$B_T = \mathbf{E}_T\Big[\exp(-a_c\sigma_c\sqrt{T}U_T)\mathbf{1}_{U_T \geqq 0}\Big], \tag{4.7}$$

where

$$U_T = \frac{S_T - cT}{\sigma_c\sqrt{T}}.$$

LEMMA 2. *For all $-1/(2\theta) < c < -1/(2\theta\delta_H)$, the distribution of U_T under $\mathbf{P}_T$ converges, as T goes to infinity, to the $\mathcal{N}(0,1)$ distribution. Moreover, there exists a sequence (β_k) such that, for any $p > 0$ and T large enough,*

$$B_T = \frac{\beta_0}{\sqrt{T}}\bigg[1 + \sum_{k=1}^{p} \frac{\beta_k}{T^k} + \mathcal{O}\bigg(\frac{1}{T^{p+1}}\bigg)\bigg]. \tag{4.8}$$

The sequence (β_k) *depends only on the derivatives of* L *and* H *evaluated at point* a_c. *They also depend on the Taylor expansion of* K_T *and its derivatives at* a_c. *In particular*,

$$\beta_0 = \frac{1}{a_c \sigma_c \sqrt{2\pi}},$$
$$\beta_1 = \frac{1}{\sigma_c^2}\left(\frac{s_1}{a_c} - \frac{s_1^2}{2} - \frac{s_2}{2} - \frac{s_3}{2a_c\sigma_c^2} + \frac{s_1\ell_3}{2\sigma_c^2} - \frac{5\ell_3^2}{24\sigma_c^4} + \frac{\ell_4}{8\sigma_c^2} - \frac{1}{a_c^2}\right),$$

where $\ell_q = L^{(q)}(a_c)$, $h_q = H^{(q)}(a_c)$, *and* $s_q = h_q + k_q$.

The proof of Lemma 2 is given in section B.1.

Proof of Theorem 2, *first part.* The expansions (2.12) and (2.13) immediately follow from the conjunction of (4.6) and (4.8).

4.2. Proof of Theorem 2, second part. We are now going to establish the SLDP for the energy S_T in the more complicated case $c > -1/(2\theta\delta_H)$. We have already seen that the point a_c, given by (2.11), belongs to the domain $D_H =]-\infty, a_H[$ whenever $c < -1/(2\theta\delta_H)$. Consequently, for $c > -1/(2\theta\delta_H)$, it is necessary to make use of a slight modification of the strategy of time varying change of probability proposed in [6].

Now, we show a modification of the decomposition (1.9) which allows us to replace $r_T(a)$ by p_H in K_T, putting the difference into the remainder term. The modified remainder will be denoted by $\check{R}_T$.

LEMMA 3. *For any* $a \in D_H$, *we have the following decomposition*:

$$L_T(a) = L(a) + \frac{1}{T}H(a) + \frac{1}{T}K(a) + \frac{1}{T}\check{R}_T(a), \tag{4.9}$$

where L *and* H *are given by* (2.8) *and* (2.9), *respectively*,

$$K(a) = -\frac{1}{2}\log\left(1 + \frac{\varphi(a)+\theta}{2\varphi(a)}\,p_H\right), \tag{4.10}$$

and the remainder term has the form

$$\check{R}_T(a) = -\frac{1}{2}\log\Bigg(1 + \frac{(\varphi(a)+\theta)(r_T(a)-p_H)}{(2+p_H)\varphi(a)+\theta\delta_H} + \frac{(\varphi(a)+\theta)^2}{(\varphi(a)-\theta)((2+p_H)\varphi(a)+\theta\delta_H)}\,e^{-2T\varphi(a)}\Bigg). \tag{4.11}$$

Proof. It follows from (1.9) that

$$\begin{aligned} L_T(a) &= L(a) + \frac{1}{T}H(a) + \frac{1}{T}K_T(a) + \frac{1}{T}R_T(a) \\ &= L(a) + \frac{1}{T}H(a) + \frac{1}{T}K(a) + \frac{1}{T}\check{R}_T(a), \end{aligned}$$

where

$$\check{R}_T(a) = K_T(a) - K(a) + R_T(a).$$

Hence, if we denote $\varphi = \sqrt{\theta^2 - 2a}$, then it is not hard to see that

$$\begin{aligned}\exp(-2\check{R}_T(a)) &= \frac{2\varphi + (\varphi+\theta) r_T(a)}{2\varphi + (\varphi+\theta)p_H} \\ &\quad \times \left(1 + \frac{(\varphi+\theta)^2}{(\varphi-\theta)(2\varphi + (\varphi+\theta)r_T(a))}\, e^{-2T\varphi}\right) \\ &= \frac{2\varphi + (\varphi+\theta) r_T(a)}{2\varphi + (\varphi+\theta)p_H} + \frac{(\varphi+\theta)^2}{(\varphi-\theta)(2\varphi + (\varphi+\theta)p_H)} e^{-2T\varphi} \\ &= 1 + \frac{(\varphi+\theta)(r_T(a) - p_H)}{2\varphi + (\varphi+\theta)p_H} + \frac{(\varphi+\theta)^2}{(\varphi-\theta)(2\varphi + (\varphi+\theta)p_H)} e^{-2T\varphi}\end{aligned}$$

which ends the proof of Lemma 3.

Denote by Λ_T the suitable approximation of the normalized cumulant generating function L_T given by

$$\Lambda_T(a) = L(a) + \frac{1}{T} H(a) + \frac{1}{T} K(a). \tag{4.12}$$

The previous lemma enable us to write

$$L_T(a) = \Lambda_T(a) + \frac{1}{T} \check{R}_T(a). \tag{4.13}$$

One can observe that Λ_T is a holomorphic function in the domain D_H. In addition, there exists a unique a_T, which belongs to the interior of D_H and converges to its border a_H, solution of the implicit equation

$$\Lambda_T'(a) = c. \tag{4.14}$$

After some tedious but straightforward calculations, we can deduce from (4.14) that there exists a sequence (a_k) such that, for any $p > 0$ and T large enough,

$$a_T = \sum_{k=0}^{p} \frac{a_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \tag{4.15}$$

with

$$a_0 = a_H, \quad a_1 = -\frac{\theta\delta_H}{1 + 2\theta c\delta_H},$$

$$a_2 = \frac{2\theta c\delta_H(4 + \sin(\pi H)) + 2 + \sin(\pi H)}{2(1 + 2\theta c\delta_H)^3}.$$

Moreover, if $\varphi_T = \varphi(a_T) = \sqrt{\theta^2 - 2a_T}$, then we also have for any $p > 0$ and T large enough,

$$\varphi_T = \sum_{k=0}^{p} \frac{\varphi_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \tag{4.16}$$

with

$$\varphi_0 = -\theta\delta_H, \quad \varphi_1 = \frac{-1}{1 + 2\theta c\delta_H},$$

$$\varphi_2 = \frac{2\theta c\delta_H(5 + \sin(\pi H)) + 3 + \sin(\pi H)}{2\theta\delta_H(1 + 2\theta c\delta_H)^3}.$$

Hereafter, we introduce the new probability measure

$$\frac{d\mathbf{P}_T}{d\mathbf{P}} = \exp\Big(a_T S_T - T L_T(a_T)\Big) \tag{4.17}$$

and we denote by $\mathbf{E}_T$ the expectation under $\mathbf{P}_T$. It clearly leads to the decomposition $\mathbf{P}(S_T \geqq cT) = A_T B_T$, where

$$A_T = \exp\Big(T L_T(a_T) - cTa_T\Big), \tag{4.18}$$

$$B_T = \mathbf{E}_T\Big[\exp(-a_T(S_T - cT))\mathbf{1}_{S_T \geqq cT}\Big]. \tag{4.19}$$

The proof now splits into two parts; the first is devoted to the expansion of A_T, while the second gives the expansion of B_T. It follows from (4.12), (4.13), and (4.18) that

$$A_T = \exp(T(L(a_T) - ca_T) + H(a_T) + K(a_T) + \check{R}_T(a_T)). \tag{4.20}$$

We can deduce from the Taylor expansions of a_T and φ_T given by (4.15) and (4.16) that

$$\begin{aligned} T(L(a_T) - ca_T) &= -\frac{T}{2}(\theta + \varphi_T + 2ca_T) \\ &= -T(ca_H - L(a_H)) - \frac{\varphi_1}{2} - ca_1 - \frac{1}{2}\sum_{k=1}^{p} \frac{\varphi_{k+1} + 2ca_{k+1}}{T^k} + \mathcal{O}\Big(\frac{1}{T^{p+1}}\Big) \\ &= -TI(c) + \frac{1}{2} - \frac{1}{2}\sum_{k=1}^{p} \frac{\varphi_{k+1} + 2ca_{k+1}}{T^k} + \mathcal{O}\Big(\frac{1}{T^{p+1}}\Big). \end{aligned}$$

Consequently, we obtain that for any $p > 0$ and T large enough,

$$\exp\Big(T(L(a_T) - ca_T)\Big) = \exp(-TI(c))\sqrt{e}\Big[1 + \sum_{k=1}^{p} \frac{\alpha_k}{T^k} + \mathcal{O}\Big(\frac{1}{T^{p+1}}\Big)\Big], \tag{4.21}$$

where the coefficients (α_k) can be calculated explicitly. For example,

$$\alpha_1 = \frac{-1}{4\theta\delta_H(1 + 2c\theta\delta_H)^2}\big(2\theta c\delta_H(4 + \sin(\pi H)) + 3 + \sin(\pi H)\big).$$

By the same way, we find that for any $p > 0$ and T large enough,

$$\exp(H(a_T)) = \sqrt{\frac{2\varphi_T}{\varphi_T - \theta}} = \sqrt{1 - \sin(\pi H)}\Big[1 + \sum_{k=1}^{p} \frac{\beta_k}{T^k} + \mathcal{O}\Big(\frac{1}{T^{p+1}}\Big)\Big], \tag{4.22}$$

where the coefficients (β_k) can be calculated explicitly. For example,

$$\beta_1 = \frac{1 + \sin(\pi H)}{4\theta\delta_H(1 + 2\theta c\delta_H)}.$$

The expansions for $K(a_T)$ and $\check{R}_T(a_T)$ are much more tricky. On the one hand,

$$\exp(K(a_T)) = \sqrt{\frac{2\varphi_T}{2\varphi_T + (\varphi_T + \theta)p_H}}.$$

One can observe that $2\varphi_0 + (\varphi_0 + \theta)p_H = 0$. Hence, multiplying the numerator and the denominator by T, we obtain that for any $p > 0$ and T large enough,

$$\exp(K(a_T)) = \sqrt{\frac{2T\varphi_T}{2T\varphi_T + T(\varphi_T + \theta)p_H}} = \sqrt{\theta T \delta_H (1-\delta_H)(1+2\theta c \delta_H)}$$

$$\times\left[1 + \sum_{k=1}^{p} \frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{4.23}$$

where, as before, the coefficients (γ_k) can be calculated explicitly. On the other hand, $\check{R}_T(a_T) = K_T(a_T) - K(a_T) + R_T(a_T)$. It is not hard to see that

$$\begin{aligned}\exp(-2\check{R}_T(a_T)) &= \frac{2\varphi_T + (\varphi_T+\theta)r_T(a_T)}{2\varphi_T + (\varphi_T+\theta)p_H} \\ &\quad + \frac{(\varphi_T+\theta)^2}{(\varphi_T-\theta)(2\varphi_T + (\varphi_T+\theta)p_H)}\exp(-2T\varphi_T), \\ &= \frac{2\varphi_T + (\varphi_T+\theta)r_T(a_T)}{2\varphi_T + (\varphi_T+\theta)p_H} + \mathcal{O}(T\exp(2\theta T\delta_H)).\end{aligned}$$

Therefore,

$$\exp(\check{R}_T(a_T)) = \sqrt{\frac{2\varphi_T + (\varphi_T+\theta)p_H}{2\varphi_T + (\varphi_T+\theta)r_T(a_T)}}\Big(1 + \mathcal{O}(T\exp(2\theta T\delta_H))\Big).$$

Recall that $r_T(a) = r_H(\varphi(a)T/2)\exp(-T\varphi(a)) - 1$. It is shown (equation (A.9) in Appendix A) that for any $p > 0$ and T large enough,

$$r_T(a_T) = p_H + \frac{1}{\sin(\pi H)}\sum_{k=1}^{p}\frac{2^k r_k^H}{\varphi_T^k T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right), \tag{4.24}$$

where the coefficients (r_k^H) can be calculated explicitly. For example,

$$r_1^H = -\frac{(2H-1)^2}{4}.$$

Consequently, we infer from (4.24) that

$$T(r_T(a_T) - p_H) = w_T(a_T) + \mathcal{O}\left(\frac{1}{T^p}\right),$$

where

$$w_T(a_T) = \frac{1}{\sin(\pi H)}\sum_{k=1}^{p}\frac{2^k r_k^H}{\varphi_T^k T^{k-1}}.$$

If $\mu_T = T(2\varphi_T + (\varphi_T+\theta)p_H)$, then we obtain that for any $p > 0$ and T large enough,

$$\begin{aligned}\exp(\check{R}_T(a_T)) &= \sqrt{\frac{\mu_T}{\mu_T + (\varphi_T+\theta)T(r_T(a_T) - p_H)}}\Big(1 + \mathcal{O}(T\exp(2\theta T\delta_H))\Big), \\ &= \sqrt{\frac{1-\sin^2(\pi H)}{1-\sin^2(\pi H) + 4r_1^H\sin(\pi H)(1+2\theta c\delta_H)}}\end{aligned}$$

$$\times\left[1 + \sum_{k=1}^{p}\frac{\delta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{4.25}$$

where, as before, the coefficients (δ_k) can be calculated explicitly. Putting together the four contributions (4.21), (4.22), (4.23), and (4.25), we find from (4.20) that for any $p > 0$ and T large enough,

$$A_T = \exp(-TI(c) + R_H(c))\delta_H \sqrt{2e\theta T \sin(\pi H)(1 + 2\theta c\delta_H)} \times \left[1 + \sum_{k=1}^{p} \frac{\alpha_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{4.26}$$

where the coefficients (α_k) can be calculated explicitly and

$$R_H(c) = -\frac{1}{2}\log\left(1 - \frac{(2H-1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}{1 - \sin^2(\pi H)}\right). \tag{4.27}$$

The rest of the proof concerns the expansion of B_T which can be rewritten as

$$B_T = \mathbf{E}_T\Big[\exp(-a_T T U_T)\mathbf{1}_{U_T \geqq 0}\Big], \tag{4.28}$$

where

$$U_T = \frac{S_T - cT}{T}.$$

LEMMA 4. *For all $c > -1/(2\theta\delta_H)$, the distribution of U_T under $\mathbf{P}_T$ converges, as T goes to infinity, to the distribution of $\nu_H N^2 - \gamma_H$, where N stands for the standard $\mathcal{N}(0,1)$ distribution,*

$$\gamma_H = c - L'(a_H) = \frac{1 + 2\theta c\delta_H}{2\theta\delta_H}, \tag{4.29}$$

$$\nu_H = \frac{(1 - \sin^2(\pi H))\gamma_H}{1 - \sin^2(\pi H) - (2H-1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}. \tag{4.30}$$

In other words, the limit of the characteristic function of U_T under $\mathbf{E}_T$ is

$$\Phi(u) = \frac{\exp(-i\gamma_H u)}{\sqrt{1 - 2i\nu_H u}}. \tag{4.31}$$

Moreover, there exists a sequence (β_k) such that, for any $p > 0$ and T large enough,

$$B_T = \sum_{k=1}^{p} \frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \tag{4.32}$$

The coefficients (β_k) depend only on the Taylor expansion of a_T at the neighborhood of a_H together with the derivatives of L and H evaluated at point a_H. They also depend on the Taylor expansion of K_T and its derivatives at a_H. In particular,

$$\beta_1 = \frac{1}{a_H \gamma_H \sqrt{2\pi e}} \exp(Q_H(c) - R_H(c)),$$

where $R_H(c)$ is given by (4.27) and

$$Q_H(c) = \frac{(2H-1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}{2(1 - \sin^2(\pi H))}.$$

The proof of Lemma 4 is given in section B.2.

Proof of Theorem 2, second part. The expansions (4.26) and (4.32) imply (2.16), which completes the proof of Theorem 2.

4.3. Proof of Theorem 3. We shall now proceed to the proof of Theorem 3 which essentially follows along the same lines as those of Theorem 2, second part. First of all, one can observe that if $c = -1/(2\theta\delta_H)$, then we have exactly $a_c = a_H$. As in the proof of Theorem 2, there exists a unique a_T, which belongs to the interior of $D_H =]-\infty, a_H[$ and converges to its border a_H, solution of the implicit equation

$$\Lambda'_T(a) = c = -\frac{1}{2\theta\delta_H}, \tag{4.33}$$

where Λ_T is given by (4.12). We deduce from (4.33) that

$$T(\varphi_T + \theta\delta_H)^2 = \frac{\theta(\varphi_T + \theta p_H)}{c\varphi_T(\varphi_T - \theta)(2 + p_H)}. \tag{4.34}$$

Consequently, we infer from (4.33) and (4.34) that there exists a sequence (a_k) such that, for any $p > 0$ and T large enough,

$$a_T = \sum_{k=0}^{2p} \frac{a_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right), \quad \varphi_T = \sum_{k=0}^{2p} \frac{\varphi_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right) \tag{4.35}$$

with

$$a_0 = a_H, \quad \varphi_0 = -\theta\delta_H, \qquad a_1 = -(-\theta\delta_H)^{\frac{3}{2}}, \quad \varphi_1 = \sqrt{-\theta\delta_H},$$

$$a_2 = -\frac{\theta\delta_H}{4}(1 + \sin(\pi H)), \quad \varphi_2 = -\frac{1}{4}(3 + \sin(\pi H)).$$

Furthermore, we have the decomposition $\mathbf{P}(S_T \geqq cT) = A_T B_T$, where A_T and B_T are given by (4.18) and (4.19), respectively. Via the same lines as in the proof of the expansion (4.26), we find that for any $p > 0$ and T large enough,

$$\begin{aligned} A_T &= \exp(-TI(c))(-\theta\delta_H eT)^{1/4}\sqrt{2\delta_H \sin(\pi H)} \\ &\quad \times \left[1 + \sum_{k=1}^{2p} \frac{\alpha_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right], \end{aligned} \tag{4.36}$$

where the coefficients (α_k) can be calculated explicitly. It still remains to give the expansion of B_T, which can be rewritten as

$$B_T = \mathbf{E}_T\left[\exp(-a_T\sqrt{T}U_T)\mathbf{1}_{U_T \geqq 0}\right], \tag{4.37}$$

where

$$U_T = \frac{S_T - cT}{\sqrt{T}}. \tag{4.38}$$

Lemma 5. *For $c = -1/(2\theta\delta_H)$, the distribution of U_T under $\mathbf{P}_T$ converges, as T goes to infinity, to the distribution of $\sigma_H N_1 + \eta_H(N_2^2 - 1)$, where N_1 and N_2 are two independent $\mathcal{N}(0,1)$ random variables and*

$$\sigma_H^2 = L''(a_H) = -\frac{1}{2(\theta\delta_H)^3}, \tag{4.39}$$

$$\eta_H = \frac{1}{2(-\theta\delta_H)^{\frac{3}{2}}}. \tag{4.40}$$

In other words, the limit of the characteristic function of U_T under $\mathbf{E}_T$ is

$$\Phi(u) = \frac{\exp(-i\eta_H u - u^2\sigma_H^2/2)}{\sqrt{1-2i\eta_H u}}. \tag{4.41}$$

Moreover, there exists a sequence (β_k) such that, for any $p > 0$ and T large enough,

$$B_T = \sum_{k=1}^{2p} \frac{\beta_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right), \tag{4.42}$$

where the sequence (β_k) can be calculated explicitly. In particular,

$$\beta_1 = \frac{1}{4\pi a_H \eta_H} \exp\left(-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right).$$

The proof of Lemma 5 is given in section B.3.

Proof of Theorem 3. The expansions (4.36) and (4.42) imply (2.19), which completes the proof of Theorem 3.

Appendix A: On the main asymptotic expansion. We shall first prove the asymptotic expansion (1.9) of the normalized cumulant generating function $\mathcal{L}_T(a,b)$. This result was partially established by formula (5.12) in [12]. By Girsanov's theorem, $\mathcal{L}_T(a,b)$ can be rewritten as

$$\begin{aligned}\mathcal{L}_T(a,b) &= \frac{1}{T}\log \mathbf{E}\left[\exp\left(a\int_0^T Q_t\,dY_t + bS_T\right)\right]\\ &= \frac{1}{T}\log \mathbf{E}_\varphi\left[\exp\left((a+\theta-\varphi)\int_0^T Q_t\,dY_t + \frac{1}{2}(2b-\theta^2+\varphi^2)S_T\right)\right]\end{aligned}$$

for all $\varphi \in \mathbf{R}$, where $\mathbf{E}_\varphi$ stands for the expectation after the usual change of probability

$$\frac{d\mathbf{P}_\varphi}{d\mathbf{P}} = \exp\left((\varphi-\theta)\int_0^T Q_t\,dY_t - \frac{1}{2}(\varphi^2-\theta^2)S_T\right).$$

If $\theta^2 - 2b > 0$, we can choose $\varphi = \sqrt{\theta^2-2b}$ and $\tau = \varphi - (a+\theta)$ which leads to

$$\mathcal{L}_T(a,b) = \frac{1}{T}\log \mathbf{E}_\varphi\left[\exp\left(-\tau\int_0^T Q_t\,dY_t\right)\right]. \tag{A.1}$$

By Itô's formula, we also have

$$\int_0^T Q_t\,dY_t = \frac{1}{2}\left(l_H Y_T \int_0^T t^{2H-1}\,dY_t - T\right).$$

Consequently, we obtain from (A.1) that

$$\mathcal{L}_T(a,b) = \frac{\tau}{2} + \frac{1}{T}\log \mathbf{E}_\varphi\left[\exp\left(-\frac{\tau l_H}{2} Y_T \int_0^T t^{2H-1}\,dY_t\right)\right]. \tag{A.2}$$

Under the new probability $\mathbf{P}_\varphi$, the pair $(Y_T, \int_0^T t^{2H-1}\,dY_t)$ is Gaussian with mean zero and covariance matrix $\Gamma_T(\varphi)$. Denote by I and J the two matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As soon as the matrix

$$M_T(a,b) = I + \frac{\tau l_H}{2}\,\Gamma_T^{\frac{1}{2}}(\varphi) J \Gamma_T^{\frac{1}{2}}(\varphi)$$

is positive definite, we deduce from (A.2) together with standard calculus on the Gaussian distribution that

$$\mathcal{L}_T(a,b) = \frac{\tau}{2} - \frac{1}{2T}\,\log\det(M_T(a,b)). \tag{A.3}$$

Furthermore, it has been already proven by relation (5.12) of [12] that, if $\tau > 0$, then

$$\det(M_T(a,b)) = \frac{1}{z_T}\left[x_T\left(1 + \frac{\tau}{\varphi}\,e^{\delta_T}\sinh\delta_T\right)^2 - y_T\left(1 - \frac{\tau}{\varphi}\,e^{\delta_T}\cosh\delta_T\right)^2\right] \tag{A.4}$$

with $\delta_T = T\varphi/2$, $x_T = I_{H-1}(\delta_T)I_{-H}(\delta_T)$, $y_T = I_{1-H}(\delta_T)I_H(\delta_T)$, and

$$z_T = x_T - y_T = \frac{4\sin(\pi H)}{\pi\varphi T},$$

where I_H is the modified Bessel function of the first kind. We refer the reader to [13, Chap. 5] for the main properties of Bessel functions. Therefore, if $p_T = (x_T + y_T)/z_T$ and $r_T = 2p_T e^{-T\varphi} - 1$, then we deduce from (A.4) after some straightforward calculations that

$$\begin{aligned}\det(M_T(a,b)) &= \frac{(2\varphi-\tau)^2}{4\varphi^2} + p_T\,\frac{\tau(2\varphi-\tau)}{2\varphi^2}\,e^{T\varphi} + \frac{\tau^2}{4\varphi^2}\,e^{2T\varphi}\\ &= \frac{\tau}{2\varphi}\,e^{2T\varphi}\left(1 + \frac{2\varphi-\tau}{2\varphi}\,r_T + \frac{(2\varphi-\tau)^2}{2\varphi\tau}\,e^{-2T\varphi}\right).\end{aligned} \tag{A.5}$$

Consequently, we infer from (A.3) and (A.5) that

$$\begin{aligned}\mathcal{L}_T(a,b) = -\frac{1}{2}\,(a+\theta+\varphi) - \frac{1}{2T}\,\log\frac{\tau}{2\varphi} - \frac{1}{2T}\,\log\left(1 + \frac{2\varphi-\tau}{2\varphi}\,r_T\right)\\ - \frac{1}{2T}\,\log\left(1 + \frac{(2\varphi-\tau)^2}{\tau(2\varphi + r_T(2\varphi-\tau))}\,e^{-2T\varphi}\right).\end{aligned}$$

In order to complete the proof of Lemma 1, it remains to show that the limiting domain Δ_H reduces to $\theta^2 - 2b > 0$ and $\sqrt{\theta^2 - 2b} > \max(a+\theta; -\delta_H(a+\theta))$. On the one hand, we have already seen that our calculation is true as soon as $\theta^2 - 2b > 0$ and $\tau > 0$, which can be rewritten as

$$\varphi > a + \theta.$$

On the other hand, we also have the second constraint

$$1 + \frac{2\varphi - \tau}{2\varphi} r_T > 0 \tag{A.6}$$

leading to

$$\sqrt{\theta^2 - 2b} > -\delta_H(a+\theta). \tag{A.7}$$

As a matter of fact, it follows from the asymptotic expansion (5.11.10) of [13] for the Bessel function I_H that for all $z \in \mathbf{C}$ with $|z|$ large enough and $|\arg(z)| \leqq \pi/2 - \delta$, where δ is an arbitrarily small positive number, and for any $p > 0$

$$r_H(z) = \frac{\exp(2z)}{\sin(\pi H)} \left[1 + \sum_{k=1}^{p} \frac{r_k^H}{z^k} + \mathcal{O}\left(\frac{1}{|z|^{p+1}} \right) \right]. \tag{A.8}$$

Moreover, the coefficients (r_k^H) can be calculated explicitly. For example, one can check that $r_1^H = -(2H-1)^2/4$ and $r_2^H = (2H-1)^2(2H+1)(2H-3)/32$. In addition, all the coefficients (r_k^H) vanish if $H = \frac{1}{2}$. Consequently,

$$r_T(a) = p_H + \frac{1}{\sin(\pi H)} \sum_{k=1}^{p} \frac{2^k r_k^H}{(\varphi(a))^k T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}} \right) \tag{A.9}$$

with $p_H = (1 - \sin(\pi H))/\sin(\pi H)$. Hence, as T tends to infinity, (A.6) reduces to $2\varphi + (\varphi + (a+\theta))p_H > 0$ so $\varphi(2 + p_H) > -p_H(a+\theta)$. Finally, since $\delta_H = p_H/(2+p_H)$, it clearly implies (A.7), which completes the proof of Lemma 1.

Appendix B: On the characteristic functions.

B.1. Proof of Lemma 2. If Φ_T denotes the characteristic function of U_T under $\mathbf{P}_T$, then it follows from (4.3) that

$$\Phi_T(u) = \exp\left(-\frac{iuc\sqrt{T}}{\sigma_c} + T\left(L_T\left(a_c + \frac{iu}{\sigma_c\sqrt{T}} \right) - L_T(a_c) \right) \right). \tag{B.1}$$

First of all, it is necessary to prove that for T large enough, Φ_T belongs to $L^2(\mathbf{R})$. One can observe that, in contrast to [3], it is impossible here to make use of the Karhunen–Loève expansion of the process (X_t).

LEMMA 6. *For T large enough, Φ_T belongs to $L^2(\mathbf{R})$.*

Proof. The proof is a direct consequence of Proposition 4 in section C.1. We shall now establish an asymptotic expansion for the characteristic function Φ_T, similar to that of Lemma 7.1 of [3].

LEMMA 7. *For any $p > 0$, there exist integers $q(p)$, $r(p)$, and a sequence $(\varphi_{k,l}^H)$ independent of p such that for T large enough*

$$\Phi_T(u) = \exp\left(-\frac{u^2}{2} \right) \left[1 + \frac{1}{\sqrt{T}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \frac{\varphi_{k,l}^H u^l}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max\{1, |u|^{r(p)}\}}{T^{p+1}} \right) \right] \tag{B.2}$$

and the remainder $\mathcal{O}$ is uniform as soon as $|u| \leqq sT^{1/6}$ for some positive constant s.

Proof. It is rather easy to see that for all $k \in \mathbf{N}$, $R_T^{(k)}(a_c) = \mathcal{O}(T^k \exp(-T/c))$. Hence, we infer from (1.9) together with (2.8)–(2.10) that for all $k \in \mathbf{N}$,

$$L_T^{(k)}(a_c) = L^{(k)}(a_c) + \frac{1}{T} H^{(k)}(a_c) + \frac{1}{T} K_T^{(k)}(a_c) + \mathcal{O}\left(T^k \exp\left(-\frac{T}{c} \right) \right). \tag{B.3}$$

Therefore, we find from (B.1) and (B.3) that for any $p > 0$,

$$\log \Phi_T(u) = -\frac{u^2}{2} + T \sum_{k=3}^{2p+3} \left(\frac{iu}{\sigma_c\sqrt{T}} \right)^k \frac{L^{(k)}(a_c)}{k!} + \sum_{k=1}^{2p+1} \left(\frac{iu}{\sigma_c\sqrt{T}} \right)^k \frac{H^{(k)}(a_c) + K_T^{(k)}(a_c)}{k!} + \mathcal{O}\left(\frac{\max\{1, u^{2p+4}\}}{T^{p+1}} \right).$$

We deduce the asymptotic expansion (B.2) by taking the exponential on both sides, remarking that, as soon as $|u| \leqq sT^{1/6}$ for some positive constant s, the quantity $u^l/(\sqrt{T})^k$ in (B.2) remains bounded.

Proof of Lemma 2. It follows from Parseval's formula that B_T, given by (4.7), can be rewritten as

$$B_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{\mathbf{R}} \left(1 + \frac{iu}{a_c \sigma_c \sqrt{T}}\right)^{-1} \Phi_T(u)\, du. \tag{B.4}$$

For some positive constant s, set $s_T = sT^{1/6}$. We can split $B_T = C_T + D_T$, where

$$C_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| \leqq s_T} \left(1 + \frac{iu}{a_c \sigma_c \sqrt{T}}\right)^{-1} \Phi_T(u)\, du, \tag{B.5}$$

$$D_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| > s_T} \left(1 + \frac{iu}{a_c \sigma_c \sqrt{T}}\right)^{-1} \Phi_T(u)\, du. \tag{B.6}$$

From now on, we claim that for some positive constant ν,

$$|D_T| = \mathcal{O}(\exp(-\nu T^{1/3})). \tag{B.7}$$

As a matter of fact, it follows from (B.1) that

$$|\Phi_T(u)| \leqq \left| \exp\left(T\left(L_T\left(a_c + \frac{iu}{\sigma_c \sqrt{T}}\right) - L_T(a_c)\right)\right)\right|.$$

We also deduce from (2.8) that $L(a_c) > 0$ and thus, using Proposition 5 in section C.1, we find that

$$|\Phi_T(u)| \leqq \exp\Big(-TL(a_c)\Big) \exp\left(-\frac{Tu^2}{8\varphi^3(a_c)}\left(1 + \frac{4u^2}{\varphi^4(a_c)}\right)^{-3/4}\right),$$

which leads to (B.7). Finally, we deduce (4.8) from (B.2) and (B.5) together with standard calculus on the $\mathcal{N}(0,1)$ distribution.

B.2. Proof of Lemma 4. If Φ_T stands for the characteristic function of U_T under $\mathbf{P}_T$, we have from (4.17)

$$\Phi_T(u) = \exp\left(-iuc + T\left(L_T\left(a_T + \frac{iu}{T}\right) - L_T(a_T)\right)\right). \tag{B.8}$$

As in the proof of Lemma 2, it follows from Proposition 4 in section C.1 that for T large enough, Φ_T belongs to $L^2(\mathbf{R})$. We shall now propose an asymptotic expansion for Φ_T, slightly different from that of Lemma 7.2 of [3].

LEMMA 8. *For any $p > 0$, there exist integers $q(p)$, $r(p)$, $s(p)$ and a sequence $(\varphi^H_{k,l,m})$ independent of p such that for T large enough*

$$\Phi_T(u) = \Phi(u) \exp\left(-\frac{\sigma_H^2 u^2}{2T}\right) \times \left[1 + \sum_{k=1}^{p} \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\varphi^H_{k,l,m} u^l}{T^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right],$$

where Φ *is given by* (4.31),

$$\gamma_H = c - L'(a_H) = \frac{1+2\theta c\delta_H}{2\theta\delta_H}, \quad \sigma_H^2 = L''(a_H) = -\frac{1}{2\theta^3\delta_H^3},$$
$$\nu_H = \frac{(1-\sin^2(\pi H))\gamma_H}{1-\sin^2(\pi H)-(2H-1)^2\sin(\pi H)(1+2\theta c\delta_H)}.$$

Moreover, the remainder $\mathcal{O}$ *is uniform as soon as* $|u| \leqq sT^{2/3}$ *for some positive constant* s.

Remark 5. One can observe in this asymptotic expansion the limiting χ^2 distribution Φ together with an independent centered Gaussian distribution with small variance σ^2/T.

Proof. First of all, we deduce from (4.9) that

$$L_T(a_T) = L(a_T) + \frac{1}{T}H(a_T) + \frac{1}{T}K(a_T) + \frac{1}{T}\check{R}_T(a_T). \tag{B.9}$$

On the one hand, (2.8) implies that

$$T\left(L\left(a_T + \frac{iu}{T}\right) - L(a_T)\right) = -\frac{T\varphi_T}{2}\left(\left(1 - \frac{iub_T}{T}\right)^{\frac{1}{2}} - 1\right)$$

with $b_T = 2/\varphi_T^2$. Consequently, for all $p \geqq 2$

$$\exp\left(T\left(L\left(a_T + \frac{iu}{T}\right) - L(a_T)\right)\right) = \exp\left(\frac{iu\varphi_T b_T}{4} - \frac{T\varphi_T}{2}\sum_{k=2}^{p} l_k\left(\frac{iub_T}{T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right)\right),$$

where $l_k = -(2k)!/((2k-1)(2^k k!)^2)$, which leads to

$$\exp\left(-iuc + T\left(L\left(a_T + \frac{iu}{T}\right) - L(a_T)\right)\right) = \exp\left(-iu\gamma_H - \frac{\sigma_H^2 u^2}{2T}\right)\left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)}\frac{\varphi_{k,l}^H u^l}{T^k} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{T^{p+1}}\right)\right]. \tag{B.10}$$

On the other hand, we also have from (2.9) that for all $p \geqq 1$

$$\exp\left(H\left(a_T + \frac{iu}{T}\right) - H(a_T)\right) = \left(\frac{\varphi_T - \theta}{\varphi_T - \theta(1 - iub_T/T)^{-\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(1 - \left(\frac{\theta}{\varphi_T - \theta}\right)\sum_{k=1}^{p} h_k\left(\frac{iub_T}{T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^{p+1}}\right)\right)^{-\frac{1}{2}}$$

with $h_k = (2k)!/(2^k k!)^2$. Hence,

$$\exp\left(H\left(a_T + \frac{iu}{T}\right) - H(a_T)\right) = \left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)}\frac{\psi_{k,l}^H u^l}{T^k} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{T^{p+1}}\right)\right]. \tag{B.11}$$

Furthermore, it follows from (4.10) that for all $p \geqq 1$

$$\exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) = \left(\frac{2\varphi_T + (\varphi_T + \theta)p_H}{2\varphi_T + \varphi_T p_H + \theta p_H (1 - iub_T/T)^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$$
$$= \left(1 + \left(\frac{\theta p_H T}{c_T}\right) \sum_{k=1}^{p} h_k \left(\frac{iub_T}{T}\right)^k + \frac{1}{c_T} \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right)\right)^{-\frac{1}{2}},$$

where $c_T = T(2\varphi_T + (\varphi_T + \theta)p_H)$. Therefore, if

$$d_T(u) = 1 + \frac{iu\theta p_H b_T}{2c_T},$$

then we find that for all $p \geqq 2$

$$\exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) = \frac{1}{\sqrt{d_T(u)}} \left(1 + \left(\frac{\theta p_H T}{c_T d_T(u)}\right) \sum_{k=2}^{p} h_k \left(\frac{iub_T}{T}\right)^k + \frac{1}{c_T d_T(u)} \mathcal{O}\left(\frac{|u|^{p+1}}{T^p}\right)\right)^{-\frac{1}{2}}. \tag{B.12}$$

One can easily check that as T goes to infinity, the limits of b_T, c_T, and $d_T(u)$ are given by $2/(\theta\delta_H)^2$, $-(2+p_H)/(1+2\theta c\delta_H)$, and $1 - 2i\gamma_H u$, respectively, where γ_H is given by (4.29). Then, we infer from (B.12) that for all $p \geqq 2$

$$\exp\left(K\left(a_T + \frac{iu}{T}\right) - K(a_T)\right) = \frac{1}{\sqrt{1 - 2i\gamma_H u}} \times \left[1 + \sum_{k=1}^{p} \sum_{l=1}^{q(p)} \sum_{m=1}^{r(p)} \frac{\Psi_{k,l,m}^H u^l}{T^k (1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right]. \tag{B.13}$$

Now, in contrast with [3], the remainder term $\check{R}_T$ plays a prominent role that cannot be neglected. Let $\xi_T = T(\varphi_T + \theta)(r(a_T) - p_H)/c_T$ and

$$\xi_T(u) = \frac{T}{c_T}\left(\varphi_T + \theta\left(1 - \frac{iub_T}{T}\right)^{-\frac{1}{2}}\right)\left(r_T\left(a_T + \frac{iu}{T}\right) - p_H\right).$$

One can observe that ξ_T and $\xi_T(u)$ share the same limit

$$\lim_{T\to\infty} \xi_T(u) = \xi_H = \frac{2(1-\delta_H)(1+2\theta c\delta_H) r_1^H}{\delta_H (2 + p_H) \sin(\pi H)}$$
$$= -\frac{(2H-1)^2 \sin(\pi H)(1 + 2\theta c\delta_H)}{1 - \sin^2(\pi H)}.$$

In addition, it follows from (4.11) that

$$\exp(\check{R}_T(a_T)) = \left(\frac{c_T}{c_T + c_T \xi_T}\right)^{\frac{1}{2}} \left[1 + \mathcal{O}\Big(T \exp(2\theta T \delta_H)\Big)\right]$$
$$= (1 + \xi_T)^{-\frac{1}{2}} \left[1 + \mathcal{O}\Big(T \exp(2\theta T \delta_H)\Big)\right].$$

Moreover, we also have

$$\exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right)\right) = \left(\frac{d_T(u) + e_T(u) + \xi_T(u)}{d_T(u) + e_T(u)}\right)^{-\frac{1}{2}} \times \Big[1 + \mathcal{O}\Big(T\exp(2\theta T\delta_H)\Big)\Big],$$

where

$$e_T(u) = \frac{\theta p_H T}{c_T}\left(\left(1 - \frac{iub_T}{T}\right)^{-\frac{1}{2}} - 1 - \frac{iub_T}{2T}\right).$$

Therefore, along the same lines as in the proof of (B.13), we find that for all $p \geqq 2$

$$\exp\left(\check{R}_T\left(a_T + \frac{iu}{T}\right) - \check{R}_T(a_T)\right) = \frac{\sqrt{1 - 2i\gamma_H u}}{\sqrt{1 - 2i\nu_H u}} \times \left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)}\sum_{m=1}^{r(p)} \frac{\Phi_{k,l,m}^H u^l}{T^k(1 - 2i\nu_H u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right] \tag{B.14}$$

with

$$\nu_H = \frac{\gamma_H}{1 + \xi_H} = \frac{(1 - \sin^2(\pi H))\gamma_H}{1 - \sin^2(\pi H) - (2H-1)^2\sin(\pi H)(1 + 2\theta c\delta_H)}.$$

Finally, Lemma 8 follows from the conjunction of (B.10), (B.11), (B.13), and (B.14).

Proof of Lemma 4. Via Parseval's formula, B_T given by (4.28) can be rewritten as

$$B_T = \frac{1}{2\pi T a_T}\int_{\mathbf{R}} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u)\, du. \tag{B.15}$$

Let $s_T > 0$ be such that $\sqrt{T} = o(s_T)$ as T goes to infinity. We can split $B_T = C_T + D_T$, where

$$C_T = \frac{1}{2\pi T a_T}\int_{|u|\leqq s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u)\, du, \tag{B.16}$$

$$D_T = \frac{1}{2\pi T a_T}\int_{|u|> s_T} \left(1 + \frac{iu}{Ta_T}\right)^{-1} \Phi_T(u)\, du. \tag{B.17}$$

On the one hand, we find from Proposition 4 and the fact that $x \mapsto x(1+x)^{-3/4}$ is increasing that for some positive constant μ

$$|D_T| = \mathcal{O}\left(T(1 + T^{\frac{3}{2}})\exp\left(-\frac{\mu s_T^2}{T}\left(1 + \frac{s_T^2}{T^2}\right)^{-3/4}\right)\right).$$

It clearly leads to

$$|D_T| = \mathcal{O}\left(\exp\left(-\frac{\mu s_T^2}{T}\right)\right).$$

On the other hand, the asymptotic expansion for C_T, which immediately leads to (4.32), follows from Lemma 14, completing the proof of Lemma 4.

B.3. Proof of Lemma 5. The proof follows along the same lines as the proof of Lemma 4. The most important difference is that the scale of Taylor expansion is in $\sqrt{T}$ instead of T. Since Φ_T is the characteristic function of U_T defined by (4.38) under $\mathbf{P}_T$ defined by (4.17), we have

$$\Phi_T(u) = \exp\left(\frac{iu\sqrt{T}}{2\theta\delta_H} + T\left(L_T\left(a_T + \frac{iu}{\sqrt{T}}\right) - L_T(a_T)\right)\right). \tag{B.18}$$

As in the proof of Lemma 2, it follows from Proposition 4 in section C.1 that for T large enough Φ_T belongs to $L^2(\mathbf{R})$. We shall now propose an asymptotic expansion for Φ_T slightly different from that of Lemma 8.

LEMMA 9. *For any $p > 0$, there exist integers $q(p)$, $r(p)$, $s(p)$ and a sequence $(\varphi^H_{k,l,m})$ independent of p such that for T large enough*

$$\Phi_T(u) = \Phi(u)\left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)}\sum_{m=1}^{r(p)} \frac{\varphi^H_{k,l,m}u^l}{(\sqrt{T})^k(1-2i\eta_H u)^m} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{(\sqrt{T})^{p+1}}\right)\right],$$

where Φ is given by (4.41). Moreover, the remainder $\mathcal{O}$ is uniform as soon as $|u| \leqq sT^{1/6}$ for some positive constant s.

Proof. First of all, we deduce from (4.9) that

$$L_T(a_T) = L(a_T) + \frac{1}{T}H(a_T) + \frac{1}{T}K(a_T) + \frac{1}{T}\check{R}_T(a_T). \tag{B.19}$$

On the one hand, (2.8) implies that

$$T\left(L\left(a_T + \frac{iu}{\sqrt{T}}\right) - L(a_T)\right) = -\frac{T\varphi_T}{2}\left(\left(1 - \frac{iub_T}{\sqrt{T}}\right)^{\frac{1}{2}} - 1\right)$$

with $b_T = 2/\varphi_T^2$. Consequently, for all $p \geqq 2$

$$\begin{aligned}&\exp\left(T\left(L\left(a_T + \frac{iu}{\sqrt{T}}\right) - L(a_T)\right)\right)\\ &\quad= \exp\left(\frac{iu\varphi_T b_T\sqrt{T}}{4} - \frac{T\varphi_T}{2}\sum_{k=2}^{p} l_k\left(\frac{iub_T}{\sqrt{T}}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{(\sqrt{T})^{p+1}}\right)\right),\end{aligned}$$

where $l_k = -(2k)!/((2k-1)(2^k k!)^2)$, which leads to

$$\begin{aligned}\exp\left(\frac{iu\sqrt{T}}{2\theta\delta_H} + T\left(L\left(a_T + \frac{iu}{\sqrt{T}}\right) - L(a_T)\right)\right) &= \exp\left(-iu\eta_H - \frac{u^2\sigma_H^2}{2}\right)\\ \times\left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)} \frac{\varphi^H_{k,l}u^l}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{(\sqrt{T})^{p+1}}\right)\right]&.\end{aligned} \tag{B.20}$$

On the other hand, we also have from (2.9) that for all $p \geqq 1$

$$\begin{aligned}\exp\left(H\left(a_T + \frac{iu}{\sqrt{T}}\right) - H(a_T)\right) &= \left(\frac{\varphi_T - \theta}{\varphi_T - \theta(1 - iub_T/\sqrt{T})^{-\frac{1}{2}}}\right)^{\frac{1}{2}}\\ &= \left(1 - \left(\frac{\theta}{\varphi_T - \theta}\right)\sum_{k=1}^{p} h_k\left(\frac{iub_T}{\sqrt{T}}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{(\sqrt{T})^{p+1}}\right)\right)^{-\frac{1}{2}}\end{aligned}$$

with $h_k = (2k)!/(2^k k!)^2$. Hence,

$$\exp\left(H\left(a_T + \frac{iu}{\sqrt{T}}\right) - H(a_T)\right) = 1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)} \frac{\psi_{k,l}^H u^l}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{(\sqrt{T})^{p+1}}\right). \tag{B.21}$$

Furthermore, it follows from (4.10) that for all $p \geqq 1$

$$\begin{aligned}
&\exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right)\\
&\quad= \left(\frac{2\varphi_T + (\varphi_T+\theta)p_H}{2\varphi_T + \varphi_T p_H + \theta p_H (1 - iub_T/\sqrt{T})^{-\frac{1}{2}}}\right)^{\frac{1}{2}}\\
&\quad= \left(1 + \left(\frac{\theta p_H \sqrt{T}}{c_T}\right)\sum_{k=1}^{p} h_k \left(\frac{iub_T}{\sqrt{T}}\right)^k + \frac{1}{c_T}\mathcal{O}\left(\frac{|u|^{p+1}}{(\sqrt{T})^p}\right)\right)^{-\frac{1}{2}},
\end{aligned}$$

where $c_T = \sqrt{T}(2\varphi_T + (\varphi_T + \theta)p_H)$. Therefore, if $d_T(u) = 1 + iu\theta p_H b_T/(2c_T)$, then we find that for all $p \geqq 2$

$$\begin{aligned}
&\exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right) = \frac{1}{\sqrt{d_T(u)}}\\
&\quad\times\left(1 + \frac{\theta p_H T}{c_T d_T(u)}\sum_{k=2}^{p} h_k\left(\frac{iub_T}{\sqrt{T}}\right)^k + \frac{1}{c_T d_T(u)}\mathcal{O}\left(\frac{|u|^{p+1}}{(\sqrt{T})^p}\right)\right)^{-\frac{1}{2}}.
\end{aligned} \tag{B.22}$$

One can easily check that as T goes to infinity, the limits of b_T, c_T, and $d_T(u)$ are given by $2/(\theta\delta_H)^2$, $(2+p_H)\sqrt{-\theta\delta_H}$, and $1 - 2i\eta_H u$, respectively, where η_H is given by (4.40). Then, we infer from (B.22) that for all $p \geqq 2$

$$\begin{aligned}
&\exp\left(K\left(a_T + \frac{iu}{\sqrt{T}}\right) - K(a_T)\right) = \frac{1}{\sqrt{1 - 2i\eta_H u}}\\
&\quad\times\left[1 + \sum_{k=1}^{p}\sum_{l=1}^{q(p)}\sum_{m=1}^{r(p)} \frac{\Psi_{k,l,m}^H u^l}{(\sqrt{T})^k (1 - 2i\eta_H u)^m} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{(\sqrt{T})^{p+1}}\right)\right].
\end{aligned} \tag{B.23}$$

Now, in contrast with [3], the remainder term $\check{R}_T$ plays a prominent role that cannot be neglected. Let $\xi_T = \sqrt{T}(\varphi_T + \theta)(r(a_T) - p_H)/c_T$ and

$$\xi_T(u) = \frac{\sqrt{T}}{c_T}\left(\varphi_T + \theta\left(1 - \frac{iub_T}{\sqrt{T}}\right)^{-\frac{1}{2}}\right)\left(r_T\left(a_T + \frac{iu}{\sqrt{T}}\right) - p_H\right).$$

One can observe that ξ_T and $\xi_T(u)$ share the same limit

$$\lim_{T\to\infty} \xi_T(u) = 0.$$

In addition, it follows from (4.11) that

$$\begin{aligned}
\exp(\check{R}_T(a_T)) &= \left(\frac{c_T}{c_T + c_T\xi_T}\right)^{\frac{1}{2}}\left[1 + \mathcal{O}\left(\sqrt{T}\exp(2\theta T\delta_H)\right)\right]\\
&= (1+\xi_T)^{-\frac{1}{2}}\left[1 + \mathcal{O}\left(\sqrt{T}\exp(2\theta T\delta_H)\right)\right].
\end{aligned}$$

Moreover, we also have

$$\exp\left(\check{R}_T\left(a_T+\frac{iu}{T}\right)\right)=\left(\frac{d_T(u)+e_T(u)+\xi_T(u)}{d_T(u)+e_T(u)}\right)^{-\frac{1}{2}}\times\left[1+\mathcal{O}\Big(\sqrt{T}\exp(2\theta T\delta_H)\Big)\right],$$

where

$$e_T(u)=\frac{\theta p_H T}{c_T}\left(\left(1-\frac{iub_T}{\sqrt{T}}\right)^{-\frac{1}{2}}-1-\frac{iub_T}{2\sqrt{T}}\right).$$

Therefore, along the same lines as in the proof of relation (B.23), we find that for all $p \geqq 2$

$$\exp\left(\check{R}_T\left(a_T+\frac{iu}{T}\right)-\check{R}_T(a_T)\right)$$

$$=1+\sum_{k=1}^{p}\sum_{l=1}^{q(p)}\sum_{m=1}^{r(p)}\frac{\Phi^H_{k,l,m}u^l}{(\sqrt{T})^k(1-2i\eta_H u)^m}+\mathcal{O}\bigg(\frac{\max(1,|u|^{s(p)})}{(\sqrt{T})^{p+1}}\bigg). \tag{B.24}$$

Finally, Lemma 9 follows from the conjunction of (B.20), (B.21), (B.23), and (B.24).

Proof of Lemma 5. Via Parseval's formula, B_T given by (4.28) can be rewritten as

$$B_T=\frac{1}{2\pi T a_T}\int_{\mathbf{R}}\left(1+\frac{iu}{Ta_T}\right)^{-1}\Phi_T(u)\,du. \tag{B.25}$$

For some positive constant s, set $s_T = s\,T^{1/6}$. We can split $B_T = C_T + D_T$, where

$$C_T=\frac{1}{2\pi T a_T}\int_{|u|\leqq s_T}\left(1+\frac{iu}{Ta_T}\right)^{-1}\Phi_T(u)\,du, \tag{B.26}$$

$$D_T=\frac{1}{2\pi T a_T}\int_{|u|> s_T}\left(1+\frac{iu}{Ta_T}\right)^{-1}\Phi_T(u)\,du. \tag{B.27}$$

On the one hand, we find from Proposition 4 that for some constant $\mu > 0$,

$$|D_T|=\mathcal{O}\bigg(\exp\Big(-\frac{\mu s_T^2}{T}\Big)\bigg).$$

On the other hand, the asymptotic expansion for C_T, which immediately leads to (4.32), follows from the same arguments as those in [3, section 7.4].

Appendix C: Technical results.

C.1. Statement of the results. The main interest of the decomposition (4.9) is given by the two following results. They show us that the different functions we deal with are holomorphic and that the behavior of the remainder is negligible in our calculations.

PROPOSITION 3. *Denote*

$$\mathcal{D}_\Delta=\{z\in\mathbf{C}\colon\ \Re z<a_H\},$$
$$\mathcal{D}_1=\left\{z\in\mathbf{C}\colon\ \Re z<a_H-\varepsilon(2+\varepsilon)\,\frac{\theta^2\delta_H^2}{2}\right\},\qquad \varepsilon>0.$$

Then, for T large enough, we have the following assertions.

(a) *The functions φ, L_T, L, H, K, $\check{R}_T$ have analytic extensions to $\mathcal{D}_\Delta$.*

(b) *The function $(z,T) \mapsto \check{R}_T(z)$ is C^∞ on $\mathcal{D}_\Delta \times [T_\Delta, +\infty[$ for T_Δ depending only on H and θ.*

(c) *For $\varepsilon > 0$ and for all $z \in \mathcal{D}_1$*

$$\sqrt{\frac{\delta_H}{(2+p_H)(\delta_H+1)}} \leqq |\exp(H(z)+K(z))| \leqq \frac{4}{\sqrt{2+p_H}}\sqrt{\frac{1+\varepsilon}{\varepsilon}}.$$

(d) *There exists a constant C depending only on θ and H such that for T large enough and for all $z \in \mathcal{D}_1$,*

$$\sqrt{\frac{1}{2} - \frac{C}{T^2\varepsilon(\delta_H\theta)^4}} \leqq |\exp(-\check{R}_T(z))| \leqq C\sqrt{1+\frac{1}{T}+\frac{1}{T\varepsilon}}.$$

Proposition 4. *For T large enough, $\varepsilon > 2C/(T^2(\delta_H\theta)^4)$, $a \in \mathcal{D}_1 \cap \mathbf{R}$, and $u \in \mathbf{R}$, we have*

$$\Big|\exp(T(L_T(a+iu) - L_T(a)))\Big|$$
$$\leqq C(1+T^{\frac{3}{2}})\exp\left(-\frac{Tu^2}{8\varphi^3(a)}\left(1+\frac{4u^2}{\varphi^4(a)}\right)^{-3/4}\right),$$

and the map $u \mapsto \exp(T(L_T(a+iu) - L_T(a)))$ belongs to $L^2(\mathbf{R})$.

Proposition 5. *Let $a \in \mathbf{R}$ be such that $a < a_H$. Then, as T goes to infinity, we have*

$$\exp(-\check{R}_T(a)) = \mathcal{O}\left(\max\left\{1; \frac{-1}{T(\varphi(a)+\delta_H\theta)}\right\}\right).$$

C.2. Proofs of the results. We shall denote the principal determination of the logarithm defined on $\mathbf{C} \setminus]-\infty, 0]$ by

$$\log z = \log|z| + i\,\mathrm{Arg}(z),$$

where

$$\mathrm{Arg}(z) = \begin{cases} \arcsin(|z|^{-1}\Im z) & \text{if } \Re z \geqq 0, \\ \arccos(|z|^{-1}\Re z) & \text{if } \Re z < 0,\ \Im z > 0, \\ -\arccos(|z|^{-1}\Re z) & \text{if } \Re z < 0,\ \Im z < 0. \end{cases}$$

Proof of Proposition 3. Since $T \mapsto S_T$ is a positive increasing process, we have for $a < a_H$

$$\mathbf{E}\Big[\exp(aS_T)\Big] \leqq \lim_{T\to+\infty} \exp(TL_T(a)) < \infty.$$

The Lebesgue dominated convergence theorem yields that $z \mapsto \mathbf{E}\,[\exp(zS_T)]$ has an analytic extension to $\{z \in \mathbf{C},\ \Re z < a_H\}$. In order to prove Proposition 3, we have to obtain the same result for φ, L, H, K, and $\check{R}_T$. The proof is split into steps. First, we study the function φ.

LEMMA 10. *The function* φ *has an analytic extension on* $\{z \in \mathbf{C},\ \Re z < a_H\}$ *still denoted by* φ *such that* $\mathrm{Arg}(\varphi) \in]-\pi/4, \pi/4[$, $\Re\varphi \in]-\delta_H\theta, +\infty[$, $\Im\varphi(z)$ *vanishes if and only if* $\Im z = 0$, *and for* $\varepsilon > 0$

$$\inf_{z \in \mathcal{D}_1} \Re\varphi(z) > -\theta\delta_H(1+\varepsilon).$$

For all $a < a_H$ *and* $u \in \mathbf{R}$,

$$\Re(\varphi(a+iu) - \varphi(a)) \geqq \frac{u^2}{2\varphi^3(a)} \left(1 + \frac{4u^2}{\varphi^4(a)}\right)^{-3/4}. \tag{C.1}$$

Proof of Lemma 10. The properties of φ rely on the properties of the analytic function defined for all $z \in \mathbf{C}$, $\Re z > 0$, by

$$\sqrt{1+z} = \sqrt{|1+z|} \exp\left(\frac{i}{2}\ \mathrm{Arg}(1+z)\right).$$

On the one hand, the imaginary part of $\sqrt{1+z}$ is given by

$$\Im\sqrt{1+z} = \frac{\sqrt{|1+z| - \Re(1+z)}}{\sqrt{2}}\ \mathrm{sign}\,(\Im z).$$

If $\Re z > 0$, then $\mathrm{Arg}(1+z)$ belongs to $]-\pi/2, \pi/2[$, and $\mathrm{Arg}(\sqrt{1+z})$ belongs to $]-\pi/4, \pi/4[$. On the other hand, the real part of $\sqrt{1+z}$ is given by

$$\Re\sqrt{1+z} = \frac{\sqrt{|1+z| + \Re(1+z)}}{\sqrt{2}}. \tag{C.2}$$

If $\Re z > 0$, then $|1+z| \geqq |\Re(1+z)| = \Re(1+z)$, and (C.2) leads us to

$$\Re\sqrt{1+z} \geqq \sqrt{\Re(1+z)}.$$

Thus, for all $z \in \mathbf{C}$ such that $\Re z > \varepsilon(2+\varepsilon)$ we have $\Re(1+z) > (1+\varepsilon)^2$ which clearly implies that

$$\Re(\sqrt{1+z}) \geqq 1 + \varepsilon. \tag{C.3}$$

Now, by the very definition of φ, we have

$$\varphi(z) = -\delta_H\theta \sqrt{1 + \frac{2}{\delta_H^2\theta^2}\,(a_H - z)}.$$

Consequently, the condition $z \in \mathcal{D}_1$ leads to $2\delta_H^{-2}\theta^{-2}\,\Re(a_H - z) > \varepsilon(2+\varepsilon)$, and thus

$$\inf_{z \in \mathcal{D}_1} \Re\varphi(z) > -\theta\delta_H(1+\varepsilon).$$

Furthermore, we have for all $a < a_H$ and $u \in \mathbf{R}$

$$\Re(\varphi(a+iu) - \varphi(a)) = \varphi(a)\,\Re\left(\sqrt{1 - \frac{2iu}{\varphi^2(a)}} - 1\right).$$

Therefore, it follows from (C.2) that

$$\Re(\varphi(a+iu)-\varphi(a)) = \frac{\varphi(a)}{\sqrt{2}}\left(\sqrt{\sqrt{1+\frac{4u^2}{\varphi^4(a)}}+1}-\sqrt{2}\right).$$

Finally, inequality (C.1) follows from the fact that for any $x \geqq 0$

$$\sqrt{\sqrt{1+x}+1}-\sqrt{2} \geqq \frac{x}{4\sqrt{2}(1+x)^{3/4}},$$

which is applied to $x = 4u^2/\varphi^4(a)$.

Hereafter, we study H and K. Observe that if $a \in]-\infty, 0]$, then $H(a) = \widetilde{H}(\varphi(a))$ and $K(a) = \widetilde{K}(\varphi(a))$, where for $z \in \mathbf{C}$ such that $\Re z > -\delta_H\theta$ the functions $\widetilde{H}$ and $\widetilde{K}$ are given by

$$\widetilde{H}(z) = -\frac{1}{2}\log\frac{z-\theta}{2z}, \quad \widetilde{K}(z) = -\frac{1}{2}\log\left(1+\frac{z+\theta}{2z}p_H\right). \tag{C.4}$$

Then, the expected properties of H and K are some consequences of Lemma 10 and the same properties of $\widetilde{H}$ and $\widetilde{K}$ on $\{z \in \mathbf{C}\colon \Re z > -\delta_H\theta\}$ instead of $\{z \in \mathbf{C}\colon \Re z < a_H\}$.

LEMMA 11. *The functions $\widetilde{H}$ and $\widetilde{K}$ admit analytic extensions on $\{z \in \mathbf{C}\colon \Re z > -\delta_H\theta\}$. Moreover, if for all $\varepsilon > 0$, we denote*

$$\mathcal{D}_2 = \left\{z \in \mathbf{C}\colon \Re z > -\delta_H\theta(1+\varepsilon),\ |\operatorname{Arg}(z)| \leqq \frac{\pi}{4}\right\},$$

then for $z \in \mathcal{D}_2$

$$\sqrt{\frac{\delta_H}{(2+p_H)(\delta_H+1)}} \leqq \left|\exp(\widetilde{H}(z)+\widetilde{K}(z))\right| \leqq \frac{4}{\sqrt{2+p_H}}\sqrt{\frac{1+\varepsilon}{\varepsilon}}.$$

Proof of Lemma 11. Using (7.4), it is easy to prove the analytic extensions of $\widetilde{H}$ and $\widetilde{K}$ on $\{z \in \mathbf{C}\colon \Re z > -\delta_H\theta\}$. For $\widetilde{K}$ it is a consequence of the fact that for $z \in \mathbf{C}$, $\Re z > -\delta_H\theta$,

$$\Re\left(1+\frac{z+\theta}{2z}p_H\right) \geqq 1+\frac{p_H}{2}+\frac{\theta p_H}{2\Re z} > 1+\frac{p_H}{2}-\frac{p_H}{2\delta_H} = 0.$$

It remains to prove the inequalities stated in the lemma. We have

$$\begin{aligned}
\widetilde{H}(z)+\widetilde{K}(z) &= -\frac{1}{2}\left(\log(z-\theta)+\log(2z+(z+\theta)p_H)-2\log(2z)\right)\\
&= -\frac{1}{2}\Big(\log(z-\theta)+\log((2+p_H)z+\theta p_H)-2\log(2z)\Big)\\
&= -\frac{1}{2}\left(\log(z-\theta)+\log(2+p_H)+\log\left(z+\frac{\theta p_H}{2+p_H}\right)-2\log(2z)\right)\\
&= -\frac{1}{2}\left(\log(2+p_H)+\log(z-\theta)+\log(z+\delta_H\theta)-\log 4-2\log z\right).
\end{aligned}$$

Thus

$$\exp(\widetilde{H}(z)+\widetilde{K}(z)) = \left(\frac{(2+p_H)(z-\theta)(z+\theta\delta_H)}{4z^2}\right)^{-\frac{1}{2}},$$

where $\sqrt{z} = \sqrt{|z|}\exp(i\,\mathrm{Arg}(z)/2)$. For $z \in \mathbf{C}$ such that $|\,\mathrm{Arg}(z)| \leqq \pi/4$, we have $|\Im z| \leqq \Re z$; if moreover we have $\Re z > -\delta_H\theta(1+\varepsilon)$, then

$$1 \geqq \left|\frac{z+\delta_H\theta}{z}\right| \geqq \frac{\Re z+\delta_H\theta}{\Re z} \geqq \frac{\varepsilon}{(1+\varepsilon)},$$

$$\frac{1+\delta_H}{\delta_H} \geqq \left|\frac{z-\theta}{z}\right| \geqq \frac{\Re z-\theta}{2\Re z} \geqq 1.$$

We have used the fact that the function $x \mapsto (x+\delta_H\theta)/x$ is increasing and the function $x \mapsto (x-\theta)/x$ is decreasing. Assertion (c) of Proposition 3 is then proved.

Now, we focus on $\check{R}_T$. First, observe that for $z \in \mathbf{C}$ such that $\Re z < 0$, we have $\check{R}_T(z) = \tilde{\tilde{R}}_T(\varphi(z))$, where for $z \in \mathbf{C}$ such that $\Re z > -\delta_H\theta$,

$$\begin{aligned}\tilde{\tilde{R}}_T(z) &= -\frac{1}{2}\log\left[1+\frac{(z+\theta)(\tilde{r}_T(z)-p_H)}{(2+p_H)(z+\delta_H\theta)}+\frac{(z+\theta)^2}{(2+p_H)(z-\theta)(z+\delta_H\theta)}\,e^{-2Tz}\right]\\ &= -\frac{1}{2}\log\Bigg[z(2+p_H)(z+\delta_H\theta)+z(z+\theta)(\tilde{r}_T(z)-p_H)\\ &\qquad +\frac{z(z+\theta)^2}{(z-\theta)}\,e^{-2Tz}\Bigg]+\frac{1}{2}\log[z(2+p_H)(z+\delta_H\theta)] \qquad \text{(C.5)}\\ &= -\frac{1}{2}\Big(\log[\tilde{\tilde{R}}_{1,T}(z)]-\log[(2+p_H)z]-\log[z+\delta_H\theta]\Big), \qquad \text{(C.6)}\end{aligned}$$

where

$$\tilde{r}_T(z) = r_H\left(\frac{Tz}{2}\right)e^{-Tz}-1,$$

$$\tilde{\tilde{R}}_{1,T}(z) = z(2+p_H)(z+\delta_H\theta)+z(z+\theta)(\tilde{r}_T(z)-p_H)+\frac{z(z+\theta)^2}{z-\theta}\,e^{-2Tz}.$$

The properties of $\check{R}_T$ are some consequences of Lemma 10 and the following lemma.

LEMMA 12. *Denote*

$$\mathcal{D}_3 = \left\{z \in \mathbf{C}\colon\ \Re z > -\delta_H\theta,\ \mathrm{Arg}(z) \in \left]-\frac{\pi}{4},\frac{\pi}{4}\right[\right\}.$$

For T large enough and for all $z \in \mathcal{D}_3$, $\tilde{\tilde{R}}_{1,T}(z) \in \mathbf{C}\setminus\,]-\infty,0]$.

In fact, Lemma 12 and decomposition (C.6) give us an analytic extension of $\tilde{\tilde{R}}_T$ on $\mathcal{D}_3$. Moreover, there exists T_3 depending only on H and θ such that the function

$$(z,T) \mapsto \frac{\tilde{\tilde{R}}_{1,T}(z)}{(2+p_H)z(z+\delta_H\theta)}$$

is C^∞ and never vanishes on $\{(z,T)\colon\ z \in \mathcal{D}_3,\ T > T_3\}$ and is C^∞ with respect to (z,T). Since $\check{R}_T = \tilde{\tilde{R}}_T(\varphi)$, the function $(z,T) \mapsto \check{R}_T(z)$ is C^∞ on $\{T > T_3\}$ and $\{z \in \mathbf{C}\colon\ \Re z < a_H\}$. Assertion (b) of Proposition 3 is then proved.

Proof of Lemma 12. We recall that for $z \in \mathbf{C}$ such that $\mathrm{Arg}(z) \in\,]-\pi/4,\pi/4\,[$

$$r_H(z)-1 = \frac{\exp(2z)}{\sin(\pi H)}\left(1-\frac{(2H-1)^2}{4z}+\frac{1}{z^2}\,\sin(\pi H)F(z)\right),$$

where F is a continuous bounded function. Then, for $z \in \mathcal{D}_3$,

$$\tilde{r}_T(z) - p_H = \frac{r_1}{Tz} + \frac{1}{T^2 z^2}\, F(Tz).$$

We can exhibit a polynomial term from $\tilde{\breve{R}}_{1,T}(z)$:

$$\begin{aligned} \tilde{\breve{R}}_{1,T}(z) &= z(2+p_H)(z+\delta_H\theta) \\ &\quad + z(z+\theta)\left(\frac{r_1}{Tz} + \frac{1}{T^2z^2}\, F(Tz)\right) + \frac{z(z+\theta)^2}{z-\theta}\, e^{-2Tz} \\ &= z(2+p_H)(z+\delta_H\theta) + (z+\theta)\,\frac{r_1}{T} \\ &\quad + \left(\frac{z+\theta}{T^2 z}\, F(Tz) + \frac{z(z+\theta)^2}{z-\theta}\, e^{-2Tz}\right) \\ &= z(2+p_H)(z+\delta_H\theta) + (z+\theta)\,\frac{r_1}{T} \\ &\quad + \frac{z+\theta}{T^2 z}\left(F(Tz) + \frac{T^2z^2(z+\theta)}{z-\theta}\, e^{-2Tz}\right). \end{aligned} \tag{C.7}$$

Let us denote

$$\begin{aligned} \widetilde{P}_T(z) &= z(2+p_H)(z+\delta_H\theta) + (z+\theta)\,\frac{r_1}{T}, \\ C &= \frac{1-\delta_H}{\delta_H} \sup_{z\in\mathcal{D}_3} \left| F(Tz) + \frac{z+\theta}{z-\theta}\, T^2 z^2 e^{-2zT}\right| < +\infty. \end{aligned} \tag{C.8}$$

Observe that for $z \in \mathcal{D}_3$, on the one hand,

$$\begin{aligned} |\Im \widetilde{P}_T(z)| &= |\Im z| \left[(2+p_H)(2\Re z + \delta_H\theta) + \frac{r_1}{T}\right] \\ &\geqq |\Im z| \left[\frac{2+p_H}{2}\,(-\delta_H\theta) + \frac{r_1}{T}\right]. \end{aligned}$$

Then, for $z \in \mathbf{C}$ such that

$$\Im z > \frac{4}{2+p_H}\,\frac{1}{(-\delta_H\theta)}\left[\frac{C}{T^2} - \frac{r_1}{T}\right],$$

we have $|\Im \tilde{\breve{R}}_{1,T}(z)| > 0$. On the other hand,

$$\begin{aligned} \Re \widetilde{P}_T(z) &= (2+p_H)\Re(z)(\Re z + \delta_H\theta) + \frac{\theta r_1}{T}\,\Re z - (2+p_H)(\Im z)^2, \\ &\geqq -\delta_H\theta^2\,\frac{r_1}{T} - (2+p_H)(\Im z)^2. \end{aligned}$$

Then, for T large enough and for all $z \in \mathcal{D}_3$ such that

$$\Im z > \frac{4}{2+p_H}\,\frac{1}{(-\delta_H\theta)}\left[\frac{C}{T^2} - \frac{r_1}{T}\right],$$

we have $\Re \tilde{\breve{R}}_{1,T}(z) > 0$. We are allowed to conclude that if T is large enough and $z \in \mathcal{D}_3$, then $\tilde{\breve{R}}_{1,T}(z) \in \mathbf{C}\setminus\,]-\infty, 0]$. Lemma 12 is proved.

It remains to prove assertion (d) of Proposition 3. This is given by the following lemma.

LEMMA 13. *Denote*

$$\mathcal{D}_4 = \{z \in \mathbf{C}\colon\ \Re z > -\delta_H\theta(1+\varepsilon)\}.$$

Then, there exist a constant C depending only on θ and the index H such that for T large enough and $z \in \mathcal{D}_4$

$$\sqrt{\frac{1}{4}} \leqq \left|\exp\Big(-\tilde{\check{R}}_T(z)\Big)\right| \leqq \sqrt{C\Big(1+\frac{1}{T}+\frac{1}{T\varepsilon}\Big)}.$$

Proof of Lemma 13. Observe that

$$\left|\exp\Big(-\tilde{\check{R}}_T(z)\Big)\right| = \sqrt{\frac{|\tilde{\check{R}}_{1,T}(z)|}{(2+p_H)|z||z+\delta_H\theta|}}.$$

From (C.7) and (C.8),

$$\begin{aligned}\frac{|\widetilde{P}_T(z)|}{(2+p_H)|z||z+\delta_H\theta|} - \frac{C}{T^2} &\leqq \frac{|\tilde{\check{R}}_{1,T}(z)|}{(2+p_H)|z||z+\delta_H\theta|}\\ &\leqq \frac{|\widetilde{P}_T(z)|}{(2+p_H)|z||z+\delta_H\theta|} + \frac{C}{T^2}.\end{aligned}$$

On the one hand, the very definition of $\widetilde{P}_T$ implies that

$$\frac{\widetilde{P}(z)}{(2+p_H)z(z+\delta_H\theta)} = \frac{1}{z}\left[z+\frac{r_1}{T}+\frac{\theta r_1(1-\delta_H)}{T(z+\delta\theta)}\right],$$

and since $r_1\theta > 0$, we find that for $z \in \mathbf{C}$ such that $\Re z > -\delta_H\theta$, and for $T > 2r_1\theta^{-1}\,\delta_H^{-1}$

$$\left|\frac{\widetilde{P}(z)}{(2+p_H)z(z+\delta_H\theta)}\right| \geqq \frac{1}{|z|}\sqrt{\left[\Re z+\frac{r_1}{T}\right]^2+(\Im z)^2} \geqq \frac{1}{2}.$$

On the other hand,

$$\frac{\widetilde{P}(z)}{(2+p_H)z(z+\delta_H\theta)} = 1+\frac{r_1}{T}\,\frac{1}{(2+p_H)z}+\frac{r_1\theta(1+\delta_H)}{T}\,\frac{1}{(2+p_H)z(z+\delta_H\theta)}$$

and

$$\left|\frac{\widetilde{P}(z)}{(2+p_H)z(z+\delta_H\theta)}\right| \leqq 1+\frac{r_1}{\theta\delta_H T(2+p_H)}+\frac{r_1\theta(1+\delta_H)}{T}\,\frac{1}{(2+p_H)\delta_H^2\theta^2\varepsilon}.$$

Lemma 13 is proved.

Finally, the proof of Proposition 3 follows from the conjunction of Lemmas 10–13.

Proof of Proposition 4. Using the decomposition (4.9), assertion (c) of Proposition 3, the fact that for $a \in \mathcal{D}_1 \cap \mathbf{R}$,

$$0 < \inf_{u\in\mathbf{R}} \exp(\check{R}_T(a+iu)) \leqq \sup_{u\in\mathbf{R}} \exp(\check{R}_T(a+iu)) < \infty,$$

and the equality $\Re(a+iu) = \Re a$, we only have to bound

$$u \mapsto \exp(T(L(a+iu) - L(a))).$$

The bound is clearly given by inequality (C.1).

Proof of Proposition 5. Recall that, by Lemma 12, $\check{R}_T(a) = \tilde{\tilde{R}}_T(\varphi(a))$ with $\tilde{\tilde{R}}_T$ given in (C.5). Moreover, there exist $0 < x_T^- < x_T^+ < -\delta_H\theta$ such that for all $x \in]-\delta_H, +\infty[$

$$\tilde{\tilde{R}}_{1,T}(x) - (2+p_H)(x - x_T^-)(x - x_T^+) = \mathcal{O}(T^{-2}).$$

Since $0 < x_T^- < x_T^+ < -\delta_H\theta$, we have for all $x \in]-\delta_H, +\infty[$

$$\left| \frac{(2+p_H)(x - x_T^-)(x - x_T^+)}{(2+p_H)x(x - \delta_H\theta)} \right| \leqq 1.$$

Consequently,

$$\begin{aligned}\exp(-\check{R}_T(a)) &\leqq \sqrt{1 + \frac{\tilde{\tilde{R}}_{1,T}(\varphi(a)) - (2+p_H)(\varphi(a) - x_T^-)(\varphi(a) - x_T^+)}{(2+p_H)\varphi(a)(\varphi(a) + \delta_H\theta)}} \\ &= \mathcal{O}\left(\max\left\{ 1; \frac{-1}{T(\varphi(a) + \delta_H\theta)} \right\} \right),\end{aligned}$$

which ends the proof of Proposition 5.

C.3. A contour integral for the Gamma function. In order to obtain an asymptotic expansion for B_T, it is necessary to make use of the following lemma which slightly extends Lemma 7.3 of [3]. First of all, let $f_{a,b}$ be the density function of the gamma $\mathcal{G}(a,b)$ distribution, with parameters $a, b > 0$, given by

$$f_{a,b}(x) = \begin{cases} \dfrac{b^a}{\Gamma(a)}\, x^{a-1} \exp(-bx) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{C.9}$$

For all integers $k, \ell \geqq 0$, and for all positive real numbers σ^2, γ, ν, let

$$v_k(a,b,\ell) = \frac{2\pi\sigma^{2k} i^\ell}{2^k k!\, \gamma^{2k+\ell+1}} f_{a,b}^{(2k+\ell)}(1).$$

LEMMA 14. *For any integers $p > 0$ and $\ell \geqq 0$, we have*

$$\int_{\mathbf{R}} \exp\left(-i\gamma u - \frac{\sigma^2 u^2}{2T} \right) \frac{u^\ell}{(1 - 2i\nu u)^a}\, du = \sum_{k=0}^{p} \frac{v_k(a,b,\ell)}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}} \right) \tag{C.10}$$

with $b = \gamma/(2\nu)$.

Proof of Lemma 14. Denote by N_σ the Gaussian kernel with the positive variance σ^2:

$$N_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2} \right).$$

It is well known that the characteristic functions of N_σ and $f_{a,b}$ are given by

$$\widehat{N}_\sigma(x) = \exp\left(- \frac{\sigma^2 x^2}{2}\right) \quad \text{and} \quad \hat{f}_{a,b}(x) = \left(1 - \frac{ix}{b}\right)^{-a},$$

respectively. Then, it follows from [17, p. 177] that for all integers $\ell \geqq 0$ and for all positive real numbers a, b, τ

$$\int_{\mathbf{R}} \exp\left(- ivx - \frac{\tau^2 v^2}{2}\right) v^\ell \hat{f}_{a,b}(v)\, dv = 2\pi i^\ell f_{a,b} * N_\tau^{(\ell)}(x). \tag{C.11}$$

Along the same lines as in [3, section 7.3], it is not hard to see that for any $p > 0$

$$f_{a,b} * N_\tau^{(\ell)}(x) = \sum_{k=0}^{p} \frac{\tau^{2k}}{2^k k!}\, f_{a,b}^{(2k+\ell)}(x) + \mathcal{O}(\tau^{2(p+1)}). \tag{C.12}$$

Hence, we deduce from the conjunction of (C.11) and (C.12) with $\tau^2 = \sigma^2/(T\gamma^2)$ that

$$\begin{aligned} &\int_{\mathbf{R}} \exp\left(- ivx - \frac{\sigma^2 v^2}{2T\gamma^2}\right) v^\ell \hat{f}_{a,b}(v)\, dv \\ &\quad = 2\pi i^\ell \sum_{k=0}^{p} \frac{\sigma^{2k}}{2^k k!\, \gamma^{2k} T^k}\, f_{a,b}^{(2k+\ell)}(x) + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \end{aligned} \tag{C.13}$$

Finally, by taking the values $x = 1$ and $b = \gamma/(2\nu)$ together with the change of variables $u = v/\gamma$ in (C.13), we find that

$$\int_{\mathbf{R}} \exp\left(- i\gamma u - \frac{\sigma^2 u^2}{2T}\right) \frac{u^\ell}{(1 - 2i\nu u)^a}\, du = \sum_{k=0}^{p} \frac{v_k(a,b,\ell)}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right), \tag{C.14}$$

which completes the proof of Lemma 14.

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