On the Usefulness of Persistent Excitation in ARX Adaptive Tracking

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Outline

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Introduction

Consider the d-dimensional $\text{ARX}_d(p, q)$ model given by

$$A(R)X_{n+1} = B(R)U_n + \varepsilon_{n+1}$$

where

1. R the shift-back operator,
2. X_n the system output,
3. U_n the system input,
4. ε_n the driven noise.

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Matrix Polynomials

The polynomials A and B are given for all $z \in \mathbb{C}$ by

$$\begin{aligned} A(z) &= I_d - A_1 z - \cdots - A_p z^p, \\ B(z) &= I_d + B_1 z + \cdots + B_q z^q, \end{aligned}$$

where A_i and B_j are unknown square matrices of order d and I_d is the identity matrix.

Definition

The matrix polynomial B is **causal or minimum phase** if for all $z \in \mathbb{C}$ with $|z| \leq 1$

$$\det(B(z)) \neq 0.$$

The Unknown Parameter

Denote by θ the unknown parameter of the model

$$\theta^t = (A_1, \ldots, A_p, B_1, \ldots, B_q).$$

The ARX model can be rewritten as

$$\boldsymbol{X_{n+1} = \theta^t \Phi_n + U_n + \varepsilon_{n+1}}$$

where the regression vector

$$\Phi_n = \left(X_n^t, \ldots, X_{n-p+1}^t, U_{n-1}^t, \ldots, U_{n-q}^t\right)^t.$$

About the Noise

We assume that (ε_n) is a **martingale difference sequence** adapted to $\mathbb{F} = (\mathcal{F}_n)$ such that for all $n \geq 0$,

$$\mathbb{E}[\varepsilon_{n+1}\varepsilon_{n+1}^t|\mathcal{F}_n] = \Gamma \qquad \text{a.s.}$$

where Γ is a positive definite covariance matrix. Moreover, we assume that (ε_n) has finite conditional moment of order > 2 so

$$\Gamma_n = \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k \varepsilon_k^t \longrightarrow \Gamma \qquad \text{a.s.}$$

Outline

Weighted least squares

The weighted least squares estimator $\widehat{\theta}_n$ of θ satisfies

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + a_n S_n^{-1}(a)\Phi_n \left(X_{n+1} - U_n - \widehat{\theta}_n^t \Phi_n \right)^t$$

$$S_n(a) = \sum_{k=0}^{n} a_k \Phi_k \Phi_k^t + I_\delta$$

The **standard least squares estimator** is given by

$$a_n = 1.$$

Weighted Least Squares

The **weighted least squares estimator** is given for $\gamma > 0$ by

$$a_n = \left(\frac{1}{\log s_n}\right)^{1+\gamma} \quad \text{where} \quad s_n = \sum_{k=0}^{n} \| \Phi_k \|^2 .$$

We always have the decomposition

$$\widehat{\theta}_n - \theta = S_{n-1}^{-1}(a) M_n(a)$$

$$M_n(a) = \sum_{k=0}^{n-1} a_k \Phi_k \varepsilon_{k+1}.$$

Adaptive Control

The control U_n forces X_n to track a given trajectory (x_n). We use the **persistently excited adaptive tracking control**

$$U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n + \xi_{n+1}$$

where (ξ_n) is an exogenous noise with mean 0 and positive definite covariance matrix Δ. The closed-loop system is

$$X_{n+1} - x_{n+1} = \pi_n + \varepsilon_{n+1} + \xi_{n+1}$$

where the prediction error

$$\pi_n = (\theta - \widehat{\theta}_n)^t \Phi_n.$$

We assume that the trajectory (x_n) is bounded and satisfies

$$\sum_{k=1}^{n} \| x_k \|^2 = o(n) \qquad \text{a.s.}$$

and that (ξ_n) satisfies the strong law of large numbers so

$$\Sigma_n = \frac{1}{n}\sum_{k=1}^{n}(\varepsilon_k + \xi_k)(\varepsilon_k + \xi_k)^t \longrightarrow \Gamma + \Delta \qquad \text{a.s.}$$

Definition

The tracking is said to be **residually optimal** if

$$C_n = \frac{1}{n}\sum_{k=1}^{n}(X_k - x_k)(X_k - x_k)^t \longrightarrow \Gamma + \Delta \qquad \text{a.s.}$$

Outline

Preliminars

For all $z \in \mathbb{C}$ such that $|z| \leq r$, where $r > 1$ is strictly less than the smallest modulus of the zeros of $\det(B(z))$, we denote

$$B^{-1}(z) = \sum_{k=0}^{\infty} D_k z^k$$

where $D_0 = I_d$ and for all $k \geq 1$, the matrices D_k are given by

$$\begin{aligned} D_k &= -\sum_{j=0}^{k-1} D_j B_{k-j} \quad \text{if} \quad k \leq q, \\ D_k &= -\sum_{j=1}^{q} D_{k-j} B_j \quad \text{if} \quad k > q. \end{aligned}$$

Let

$$P(z) = B^{-1}(z)(A(z) - I_d) = \sum_{k=1}^{\infty} P_k z^k$$

where all the matrices P_k may be explicitly calculated as functions of the matrices A_i and B_j

$$\begin{aligned} P_k &= -\sum_{j=0}^{k-1} D_j A_{k-j} \quad \text{if} \quad k \leq p, \\ P_k &= -\sum_{j=1}^{p} D_{k-j} A_j \quad \text{if} \quad k > p. \end{aligned}$$

For all $1 \leq i \leq q$, denote by H_i be the square matrix of order d

$$H_i = \sum_{k=i}^{\infty} P_k \Gamma P_{k-i+1}^t + \sum_{k=i-1}^{\infty} Q_k \Delta Q_{k-i+1}^t$$

where $Q_k = D_k + P_k$. In addition, let H be the symmetric square matrix of order dq

$$H = \begin{pmatrix} H_1 & H_2 & \cdots & H_{q-1} & H_q \\ H_2^t & H_1 & H_2 & \cdots & H_{q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{q-1}^t & \cdots & H_2^t & H_1 & H_2 \\ H_q^t & H_{q-1}^t & \cdots & H_2^t & H_1 \end{pmatrix}.$$

For all $1 \leq i \leq p$, let $K_i = P_i\Gamma + Q_i\Delta$ and, if $q \leq p$,

$$K = \begin{pmatrix} 0 & K_1 & K_2 & \cdots & \cdots & K_{p-2} & K_{p-1} \\ 0 & 0 & K_1 & \cdots & \cdots & K_{p-3} & K_{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & K_1 & K_2 & \cdots & K_{p-q+1} \\ 0 & \cdots & \cdots & 0 & K_1 & \cdots & K_{p-q} \end{pmatrix}$$

while, if $p \leq q$,

$$K = \begin{pmatrix} 0 & K_1 & \cdots & K_{p-2} & K_{p-1} \\ 0 & 0 & K_1 & \cdots & K_{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & K_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Denote by L the block diagonal matrix of order dp

$$L = \begin{pmatrix} \Gamma + \Delta & 0 & \cdots & 0 & 0 \\ 0 & \Gamma + \Delta & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \Gamma + \Delta & 0 \\ 0 & 0 & \cdots & 0 & \Gamma + \Delta \end{pmatrix}.$$

Let Λ be the symmetric square matrix of order $\delta = d(p + q)$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{L} & \boldsymbol{K}^t \\ \boldsymbol{K} & \boldsymbol{H} \end{pmatrix}.$$

A Keystone Lemma

Lemma

Let S be the Schur complement of L in Λ

$$\boldsymbol{S = H - KL^{-1}K^t.}$$

If the matrix polynomial B is **minimum phase**, *then the matrices S and Λ are invertible and Λ^{-1} is given by*

$$\Lambda^{-1} = \begin{pmatrix} L^{-1} + L^{-1}K^t S^{-1} K L^{-1} & -L^{-1}K^t S^{-1} \\ -S^{-1}KL^{-1} & S^{-1} \end{pmatrix}.$$

Strong Controllability

Remark

If we use an adaptive control **without excitation**

$$U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n,$$

then S and Λ are not always invertible. It is necessary to add a new concept of **strong controllability**, really not restrictive, which implies that S and Λ are invertible.

▸ Strong Controllability

Remark

In the particular case $d = 1$, the **strong controllability** is equivalent to the **coprimness** of the polynomials $A - I_d$ and B which corresponds to the usual notion of **controllability**.

Outline

Theorem

Assume that B is **minimum phase** *and that* (ε_n) *has finite conditional moment of order* > 2. *Then, for the LS estimator, we have*

$$\lim_{n\to\infty} \frac{S_n}{n} = \Lambda \quad a.s.$$

In addition, the **tracking is residually optimal**

$$\| C_n - \Sigma_n \| = \mathcal{O}\left(\frac{\log n}{n}\right) \qquad a.s.$$

Finally, $\widehat{\theta}_n$ **converges almost surely** *to* θ

$$\| \widehat{\theta}_n - \theta \|^2 = \mathcal{O}\left(\frac{\log n}{n}\right) \qquad a.s.$$

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Theorem

Assume that B is **minimum phase**. *In addition, suppose that either (ε_n) is a white noise or (ε_n) has finite conditional moment of order > 2. Then, for the WLS estimator, we have*

$$\lim_{n\to\infty} (\log n)^{1+\gamma} \frac{S_n(a)}{n} = \Lambda \quad a.s.$$

In addition, the **tracking is residually optimal**

$$\| C_n - \Sigma_n \| = o\left(\frac{(\log n)^{1+\gamma}}{n}\right) \qquad a.s.$$

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Theorem

Assume that B is **minimum phase** *and that* (ε_n) *and* (ξ_n) *have both finite conditional moments of order* $\alpha > 2$. *Then, the* **LS and WLS estimators** *share the same* **CLT**

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Lambda^{-1} \otimes \Gamma)$$

where the inverse matrix Λ^{-1} *may be explicitely calculated and the symbol* $\otimes$ *stands for the matrix Kronecker product.*

Theorem

In addition, the **LS and WLS estimators** *share the same* **LIL** *which means that for any vectors $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^\delta$,*

$$\limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2} v^t(\widehat{\theta}_n-\theta)u$$

$$= -\liminf_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2} v^t(\widehat{\theta}_n-\theta)u$$

$$= \left(v^t\Lambda^{-1}v\right)^{1/2}\left(u^t\Gamma u\right)^{1/2} \quad a.s.$$

In particular,

$$\left(\frac{\lambda_{min}\Gamma}{\lambda_{max}\Lambda}\right) \leq \limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)\|\,\hat{\theta}_n-\theta\,\|^2 \leq \left(\frac{\lambda_{max}\Gamma}{\lambda_{min}\Lambda}\right) \quad a.s.$$

Outline

Simulations Without Excitation

Consider the $\mathrm{ARX}_2(1,1)$ process given by

$$X_{n+1} = AX_n + U_n + BU_{n-1} + \varepsilon_{n+1}$$

where

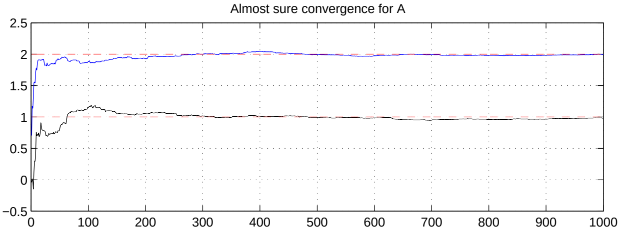
$$U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n,$$

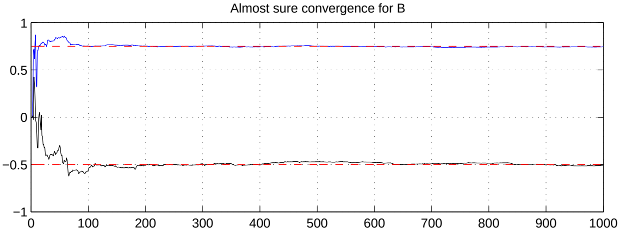
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

It is **strongly controllable** with limiting matrix

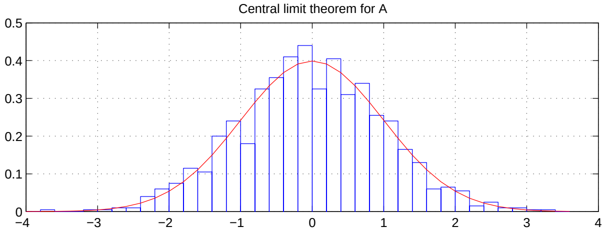
$$\Lambda = \frac{1}{21}\begin{pmatrix} 21 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 \\ 0 & 0 & 192 & 0 \\ 0 & 0 & 0 & 28 \end{pmatrix}.$$

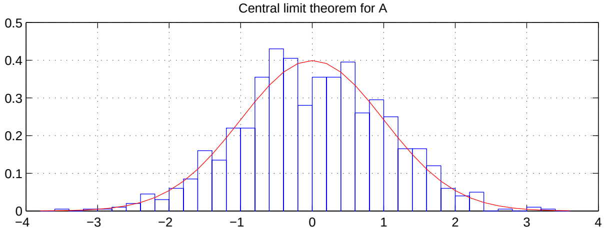
Almost sure convergence





Central limit theorem





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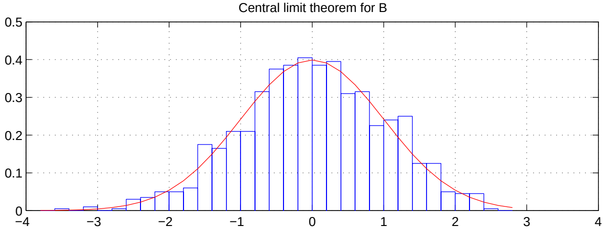
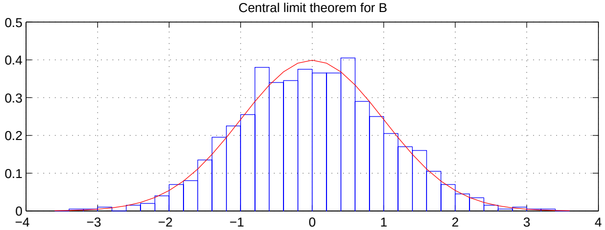




Simulations Without Excitation

Consider the $\text{ARX}_{\mathbf{2}}\mathbf{(1,1)}$ process given by

$$X_{n+1} = AX_n + U_n + BU_{n-1} + \varepsilon_{n+1}$$

where

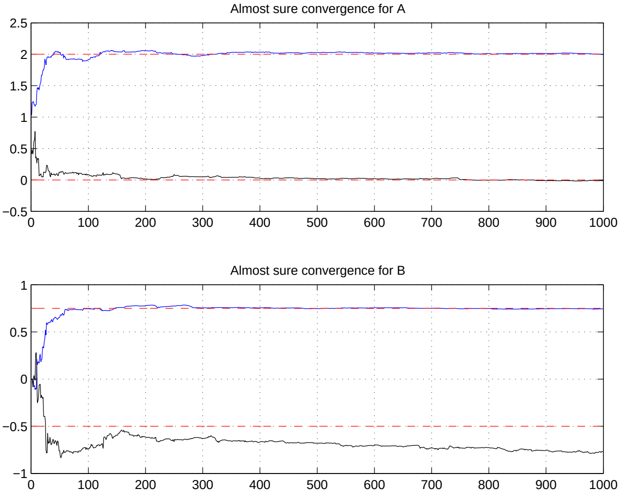
$$\boldsymbol{U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n,}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

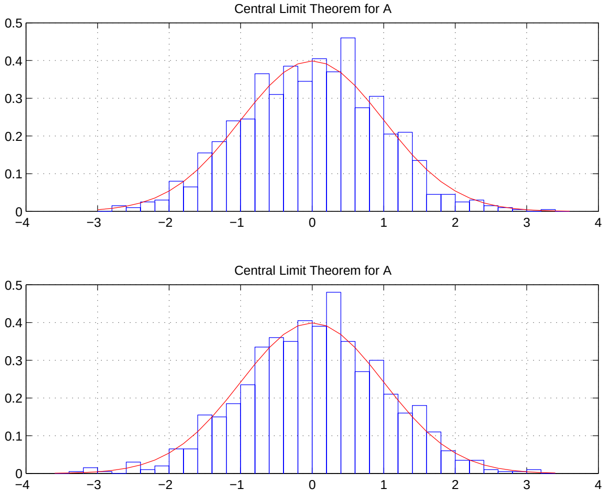
It is **not strongly controllable** because the limiting matrix

$$\Lambda = \frac{1}{21}\begin{pmatrix} 21 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 \\ 0 & 0 & 192 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Almost sure convergence



Central limit theorem



Simulations with Excitation

Consider the $\mathbf{ARX_2(1,1)}$ process given by

$$X_{n+1} = AX_n + U_n + BU_{n-1} + \varepsilon_{n+1}$$

where

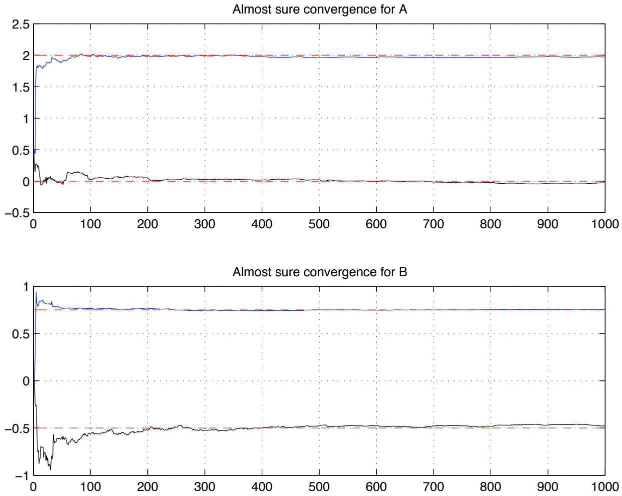
$$\boldsymbol{U_n = x_{n+1} - \widehat{\theta}_n^t \Phi_n + \xi_{n+1}}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thanks to the **exogenous excitation** (ξ_n), Λ is invertible

$$\Lambda = \frac{1}{21}\begin{pmatrix} 42 & 0 & 21 & 0 \\ 0 & 42 & 0 & 21 \\ 21 & 0 & 576 & 0 \\ 0 & 21 & 0 & 28 \end{pmatrix}.$$

Almost sure convergence



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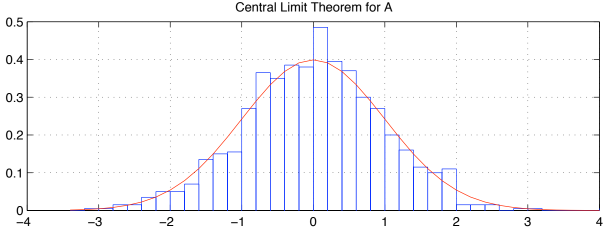

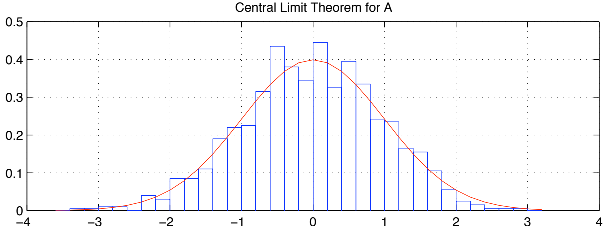

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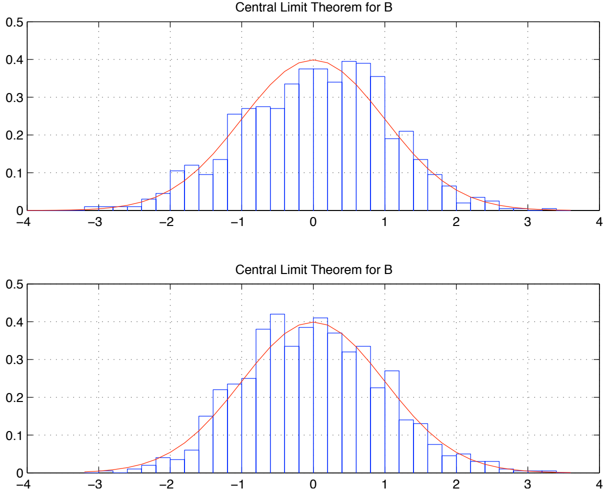

Strong Controllability

Consider the square matrix of order dq given, if $p \geq q$, by

$$\Pi = \begin{pmatrix} P_p & P_{p+1} & \cdots & P_{p+q-2} & P_{p+q-1} \\ P_{p-1} & P_p & P_{p+1} & \cdots & P_{p+q-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{p-q+2} & \cdots & P_{p-1} & P_p & P_{p+1} \\ P_{p-q+1} & P_{p-q+2} & \cdots & P_{p-1} & P_p \end{pmatrix}$$

while, if $p \leq q$, by

$$\Pi = \begin{pmatrix} P_p & P_{p+1} & \cdots & \cdots & P_{p+q-2} & P_{p+q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P_1 & P_2 & \cdots & \cdots & P_{q-1} & P_q \\ 0 & P_1 & P_2 & \cdots & P_{q-2} & P_{q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & P_1 & \cdots & P_p \end{pmatrix}.$$

Strong Controllability

Definition

The $\text{ARX}_d(p, q)$ process is said to be **strongly controllable** if B is **minimum phase** and Π is invertible,

$$\det(\Pi) \neq 0.$$

The concept of strong controllability is not really restrictive.

- If $p = q = 1$, $\longrightarrow \det(A_1) \neq 0$,
- If $p = 2$, $q = 1$, $\longrightarrow \det(A_2 - B_1 A_1) \neq 0$,
- If $p = 1$, $q = 2$, $\longrightarrow \det(A_1) \neq 0$,
- If $p = q = 2$, $\longrightarrow$ Strong Controllability

$$\det \begin{pmatrix} A_1 & A_2 - B_1 A_1 \\ A_2 - B_1 A_1 & -B_1 A_2 + (B_1^2 - B_2) A_1 \end{pmatrix} \neq 0.$$