Numerical method for optimal stopping of piecewise deterministic Markov processes

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Outline

Definition of piecewise deterministic Markov processes

Davis (80's)

General class of non-diffusion dynamic stochastic models: deterministic motion punctuated by random jumps.

Applications

Engineering systems, operations research, management science, economics. . .

Examples

Queuing systems, investment planning, stochastic scheduling, target tracking, insurance analysis, optimal exploitation of resources,. . .

Dynamics

E open subset of $\mathbb{R}^n$, ∂E its boundary and $\overline{E}$ its closure

Local characteristics

- Flow ϕ: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ deterministic motion between jumps, one-parameter group of homeomorphisms
- Intensity λ: $\overline{E} \to \mathbb{R}_+$ intensity of random jumps
- Markov kernel Q on $(\overline{E}, \mathcal{B}(\overline{E}))$ selects the post-jump location

Jumps

- $t^*(x)$ deterministic exit time when the process starts in x:

$$t^*(x) = \inf\{t > 0 : \phi(x,t) \in \partial E\}$$

- law of the first jump time T_1 starting from x

$$\mathbb{P}_x(T_1 > t) = \begin{cases} \mathrm{e}^{-\int_0^t \lambda\left(\phi(x,s)\right) ds} & \text{if} \quad t < t^*(x) \\ 0 & \text{if} \quad t \geq t^*(x) \end{cases}$$

Remark

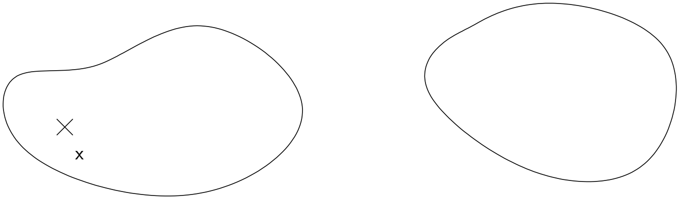
T_1 has a density on $[0, t^*(x)[$ but has an atom at $t^*(x)$:

$$\mathbb{P}_x(T_1 = t^*(x)) > 0$$

Iterative construction

Starting point :

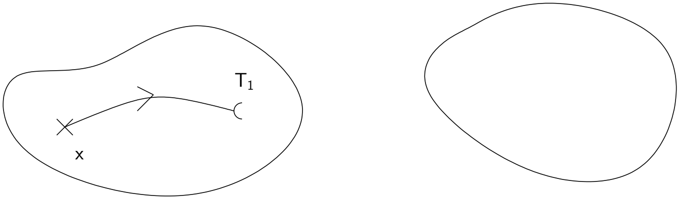
$$X_0 = Z_0 = x$$



Iterative construction

X_t follows the flow until the first jump time $T_1 = S_1$:

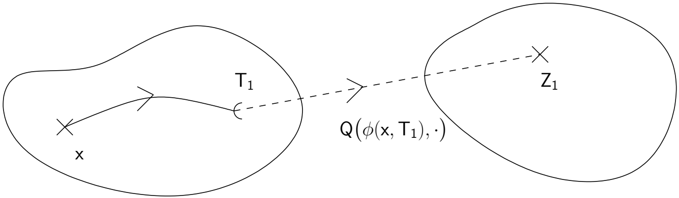
$$X_t = \phi(x, t), \quad t < T_1$$



Iterative construction

Post-jump location Z_1 selected by

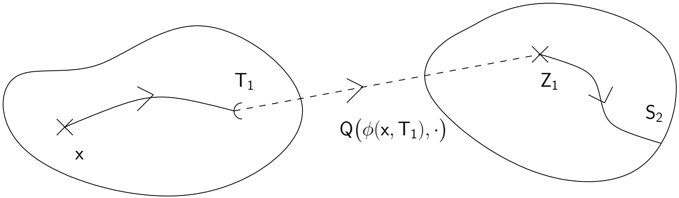
$$Q(\phi(x, T_1), \cdot)$$



Iterative construction

X_t follows the flow until the next jump time $T_2 = T_1 + S_2$:

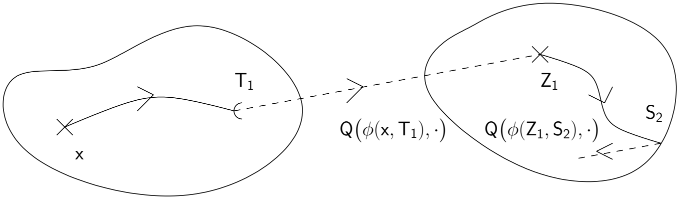
$$X_{T_1+t} = \phi(Z_1, t), \quad t < S_2$$



Iterative construction

Post-jump location Z_2 selected by

$$Q(\phi(Z_1, S_2), \cdot) \ldots$$



Embedded Markov chain

- Z_0 starting point, $S_0 = 0$, $S_1 = T_1$
- Z_n new location after n-th jump, $S_n = T_n - T_{n-1}$, time between two jumps

Proposition

(Z_n, S_n) is a discrete-time Markov chain on $E \times [0, +\infty[$
Only source of randomness of the PDMP

Simple example

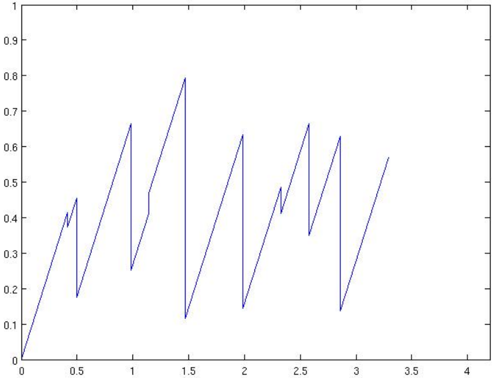
- object moving on $[0;1[$ with constant speed v

Local characteristics

- $\phi(x,t) = x + vt$
- $\lambda(x) = \beta x^{\alpha}$, $\beta > 0$, $\alpha \geq 1$: as the object comes closer to 1 the probablity to jump increases
- $Q(x,\cdot)$ uniform law on $[0;1/2]$

Trajectories

Examples of trajectories for $X_0 = 0$, $v = 1$, $\alpha = 1$, $\beta = 3$ up to the 10-th jump



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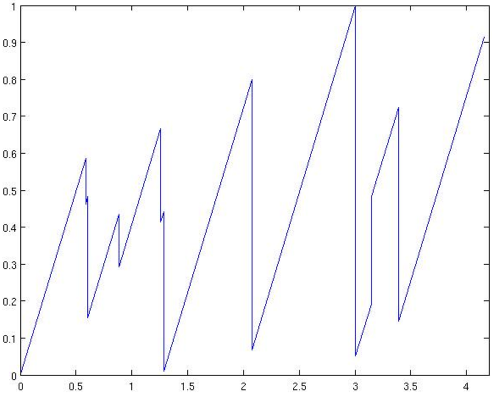

Definition

- Cost function g
- Time horizon N-th jump T_N
- $\mathcal{M}_N$ set of all stopping times $\tau \leq T_N$

Optimal stopping problem

- compute the value function

$$V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)]$$

- find an optimal stopping time τ^* that reaches $V(x)$

Example of application

- X_t state of a machine at time t
- T_n failure of some components

Optimal stopping

Find an optimal balance between

- changing the components too early/often
- avoiding a total breakdown of the machine

Iterative resolution

Gugerli 1986 : $V(x) = v_0(x)$

Backward dynamic programming

- $v_N = g$
- $v_k = L(v_{k+1}, g)$ for $k \leq N-1$

$$L(v,g)(x) = \sup_{u \leq t^*(x)} \left\{ \mathbb{E}\left[v(Z_1)\mathbf{1}_{\{S_1<u\}} + g\big(\phi(x,u)\big)\mathbf{1}_{\{S_1\geq u\}} \;\middle|\; Z_0 = x \right] \right\} \vee \mathbb{E}\left[v(Z_1) \;\middle|\; Z_0 = x \right]$$

Our aim

Objective

Propose a numerical method

- to evaluate the value function
- to compute an optimal stopping rule

with error bounds

Numerical method for diffusion processes

Bally, Pagès, Pham, Printems 98–05

Y_t continuous-time diffusion process

1. time discretization (Euler scheme) : $Y_k = Y_{k\Delta t}$ discrete-time Markov chain
2. quantization : replace Y_k by a random variable $\widehat{Y}_k$ taking values in a finite state space
3. replace the conditional expectations by finite sums

Assumptions + Lipschitz-continuous cost function $\Longrightarrow$ convergence rate of the approximated value function to the original one

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Quantization

Quantization of a random variable X

Find a random variable $\widehat{X}$ such that

- $\widehat{X} \in \Gamma$ with $|\Gamma| = m$
- $\|X - \widehat{X}\|_2$ is minimum

Algorithms

There exist algorithms providing

- Γ
- law of $\widehat{X}$
- transition probabilities for quantization of Markov chains

Specificities of PDMP's

$$L(v,g)(x) = \sup_{u \le t^*(x)} \Big\{ \mathbb{E}\Big[v(Z_1)\mathbf{1}_{\{S_1<u\}} + g\big(\phi(x,u)\big)\mathbf{1}_{\{S_1\ge u\}} \mid Z_0 = x \Big] \Big\} \vee \mathbb{E}\big[v(Z_1) \mid Z_0 = x \big]$$

- jumps at random times
- indicator functions
- supremum

Solution

- use the embedded Markov chain (Z_n, S_n)
- be careful with the time grids

Approximation of the value function

$\widehat{V}(x) = \widehat{v}_0(x)$ with $\widehat{v}_N = g$, and $\widehat{v}_k = \widehat{L}_d(\widehat{v}_{k+1}, g)$

Discretized operator

$$L(v,g)(x) = \sup_{u \leq t^*(x)} \left\{ \mathbb{E}\left[v(Z_1)\mathbf{1}_{\{S_1<u\}} + g\big(\phi(x,u)\big)\mathbf{1}_{\{S_1\geq u\}} \;\middle|\; Z_0 = x \right] \right\} \vee \mathbb{E}\left[v(Z_1) \;\middle|\; Z_0 = x \right]$$

1. discretization of $[0, t^*(z)[$, transformation of sup into max
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Discretized operator

$$
\begin{aligned}
L_d(v,g)(x) \\
= \max_{u \in G(x)} \Big\{ \mathbb{E}\Big[v(Z_1)\mathbf{1}_{\{S_1<u\}} + g\big(\phi(x,u)\big)\mathbf{1}_{\{S_1\geq u\}} \;\Big|\; Z_0 = x \Big] \Big\} \\
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Convergence rate

Theorem

Lipschitz assumptions on ϕ, λ, Q, t^* and g

$$|V(x) - \widehat{V}(x)| \leq C\sqrt{EQ}$$

C explicit constant,
EQ quantization error

$\sqrt{\cdot}$ due to the indicator functions

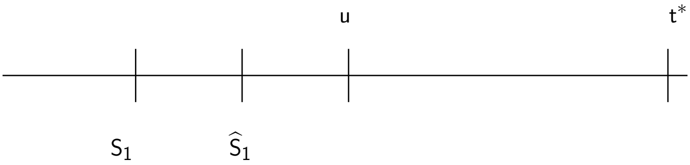
Indicator functions

Dealing with indicator functions

Set η s.t. $\forall s \in G(\widehat{Z}_n)$, $s + \eta < t^*(\widehat{Z}_n)$

$$\left\| \max_{u \in G(\widehat{Z}_n)} \mathbf{E}_x \left[\left| \mathbf{1}_{\{S_{n+1} < u\}} - \mathbf{1}_{\{\widehat{S}_{n+1} < u\}} \right| \middle| \widehat{Z}_n \right] \right\|_2 \leq \frac{1}{\eta} \| S_{n+1} - \widehat{S}_{n+1} \|_2 + C\eta$$

Easy cases:



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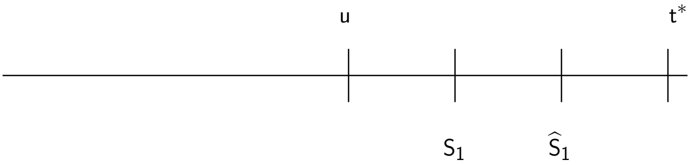
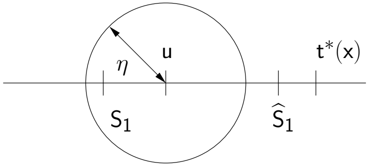


S_1 and $\widehat{S_1}$ are either side of u



$$\left|\mathbf{1}_{\{S_{n+1}<u\}}-\mathbf{1}_{\{\widehat{S}_{n+1}<u\}}\right| \leq \mathbf{1}_{\{|S_{n+1}-u|\leq\eta\}}+\mathbf{1}_{\{|S_{n+1}-\widehat{S}_{n+1}|>\eta\}}$$

$$\mathbf{E}_x\left[\mathbf{1}_{\{u-\eta\leq S_{n+1}\leq u+\eta\}}\middle|\widehat{Z}_n\right] = \int_{u-\eta}^{u+\eta}\lambda(\phi(\widehat{Z}_n,u))du \leq C\eta$$

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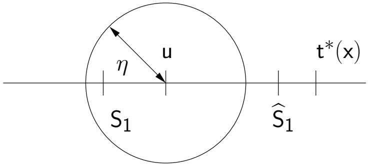

$$\left|\mathbf{1}_{\{S_{n+1}<u\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<u\}}\right| \leq \mathbf{1}_{\{|S_{n+1}-u|\leq\eta\}} + \mathbf{1}_{\{|S_{n+1}-\widehat{S}_{n+1}|>\eta\}}$$

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ϵ-optimal stopping time

Optimal stopping time

$$\mathbb{E}_x[g(X_{\tau^*})] = V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)]$$

Existence?

ϵ-optimal stopping time

$$V(x) - \epsilon \leq \mathbb{E}_x[g(X_\tau)] \leq V(x)$$

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Stopping rule

Proposition of a computable stopping rule $\hat{\tau}$

- explicit iterative construction
- no extra computation
- genuine stopping time in $\mathcal{M}_N$

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Optimality

Theorem

Same assumptions

$$|V(x) - \mathbb{E}_x[g(X_{\hat{\tau}})]| \leq C_1 EV + C_2 \sqrt{EQ}$$

C_1, C_2 explicit constants
EV value function error
EQ quantization error

Provides another approximation of the value function via Monte Carlo simulations

Example

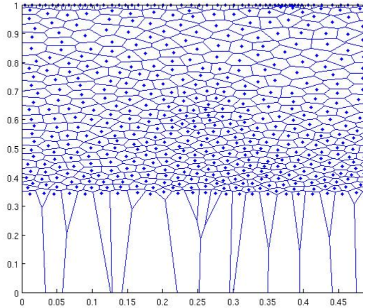
- object moving on $[0;1[$ with constant speed $= 1$

Characteristics

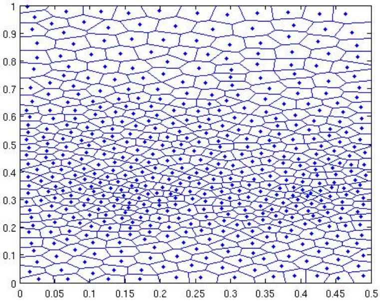
- $\phi(x,t) = x + t$
- $\lambda(x) = 3x$: as the object comes closer to 1 the jump probablity increases
- $Q(x,\cdot)$ uniform distribution on $[0;1/2]$
- horizon : $N = 10$ jumps
- starting point $x = 0$

Quantization grids 500 points

(Z_1, S_1)



(Z_2, S_2)



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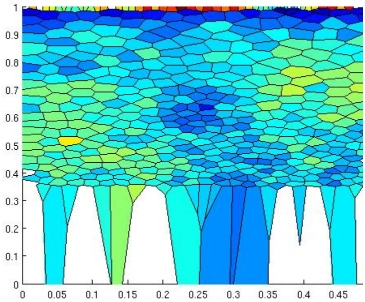

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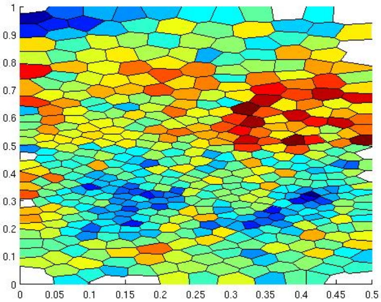
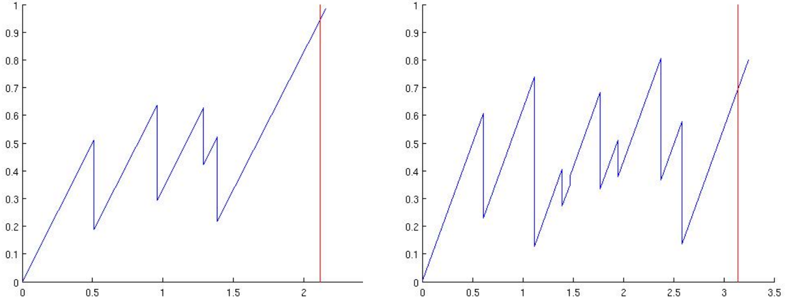


Stopping rule

Examples of stopped trajectories



Evaluations of the value function

V unknown

$$\mathbb{E}_x[g(X_{\hat{\tau}})] \leq V(x) \leq \mathbb{E}[\sup_{0\leq t\leq T_N} g(X_t)] = 0.9878$$

Numerical results

Pt	$\widehat{V}_0$	$\mathbb{E}_x[g(X_{\hat{\tau}})]$	B_1	B_2	B_3
10	0.7760	0.8173	0.1705	74.64	897.0
50	0.8298	0.8785	0.1093	43.36	511.5
100	0.8242	0.8785	0.1028	34.15	400.3
500	0.8432	0.8899	0.0989	21.03	243.1
900	0.8514	0.8968	0.0910	17.98	206.9

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Numerical results

$B_1 = \mathbb{E}[\sup_{0\leq t\leq T_N} g(X_t)] - \mathbb{E}_x[g(X_{\hat{\tau}})]$

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Numerical results

B_2 = upper bound for $|V_0 - \widehat{V}_0|$

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Numerical results

B_3 = upper bound for $|V_0 - \mathbb{E}_x[g(X_{\hat\tau})]|$

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