# SUM-SETS OF SMALL UPPER DENSITY

*by*

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***Abstract.*** — For an infinite set of non-negative integers $A$, we denote by $\overline{d}(A)$ the upper asymptotic density of $A$. If $0 \in A$ and $\gcd(A) = 1$, we can easily prove that $\overline{d}(A+A) \geq \frac{3}{2}\overline{d}(A)$. The goal of this article is to determine the structure of $A$ when both $\overline{d}(A)$ and the quotient $\sigma := \frac{\overline{d}(A+A)}{\overline{d}(A)}$ are small. In particular, we will obtain results when $\frac{3}{2} \leq \sigma < \frac{5}{3}$ and $\overline{d}(A) < \alpha_0$, where $\alpha_0$ is a small absolute constant.

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## 1. Introduction

Let $A \subseteq \mathbb{N}$ be an infinite set on non-negative integers. For $y > x > 0$, we put $A(x) := |A \cap [0;x]|$ and $A(x,y) := |A \cap [x;y]|$. We define the lower asymptotic density $\underline{d}(A)$ and the upper asymptotic density $\overline{d}(A)$ by

$$\underline{d}(A) := \liminf_{x\to\infty} \frac{A(x)}{x}, \quad \overline{d}(A) := \limsup_{x\to\infty} \frac{A(x)}{x}.$$

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Unless the contrary is stated explicitly, we assume that

$$0 \in A, \quad \gcd(A) = 1. \tag{1}$$

We define the sum $X + Y$ of two sets $X, Y \subset \mathbb{R}$ by

$$X + Y = \{x + y \mid x \in X, y \in Y\}.$$

Inverse additive theory describes sets $A$ with "small" sum-set $A + A$. Say, one may ask about sets $A$ of positive lower or upper density with small quotient $\underline{d}(A + A)/\underline{d}(A)$ or $\overline{d}(A + A)/\overline{d}(A)$. For example, let $N \geq 3$ be an integer. Then for the set $A = \{0, 1\} + N\mathbb{N}$ we have

$$\underline{d}(A) = \overline{d}(A) = 2/N, \quad \underline{d}(A + A) = \overline{d}(A + A) = 3/N,$$

so that the above-mentioned quotients are both equal to 3/2. (As we shall see in a while, this is the minimal possible value under the assumption (1).)

Kneser [**7, 4**] gave a complete description of sets $A$ satisfying $\underline{d}(A + A) < 2\underline{d}(A)$. In brief, he showed that $A$ should be "approximately" of the form $K + N\mathbb{N}$, where $N$ is a positive integer and $K$ is a set of residues mod$N$.

Among other things, Kneser's theorem implies that $\underline{d}(A + A) \geq (3/2)\underline{d}(A)$ when $A$ satisfies (1), and equality $\underline{d}(A + A) = (3/2)\underline{d}(A)$ is possible only with $|K| = 2$.

Extending Kneser's results to upper density seems to be a rather difficult problem. The following example, due to Jin [**5**], shows that this time one cannot get away with sets of the type $K + N\mathbb{N}$.

***Example 1.1***. — Let $\alpha$ be a real number satisfying $0 < \alpha < \frac{1}{2}$. Let $(T_n)_{n \geq 1}$ be an increasing sequence of positive integers such that $\lim T_{n+1}/T_n = \infty$. Then the set

$$A = \mathbb{N} \cap \bigcup_{n=1}^{\infty} [\lceil (1 - \alpha)T_n \rceil, T_n].$$

satisfies $\overline{d}(A) = \alpha$ and $\overline{d}(A + A) = (3/2)\alpha$.

In the sequel, we use the notation $\alpha = \overline{d}(A)$ and $\gamma = \overline{d}(A + A)$. We assume that $0 < \alpha \leq 1/2$ and we put $\sigma = \gamma/\alpha$.

It is not difficult to show that $\sigma \geq 3/2$, (see Lemma 2.1 below), but the structure of sets $A$ with $\sigma = 3/2$ was only recently determined by Jin [**6**]. He proved that a set $A$ with $\sigma = 3/2$ is "similar" either to $K + N\mathbb{N}$ with $|K| = 2$, or to the set from Example 1.1. For $\sigma > 3/2$ the problem is open.

In the present article we determine the structure of sets $A$ with $3/2 \leq \sigma < 5/3$ subject to the additional assumption $\alpha < \alpha_0$, where $\alpha_0$

is a small absolute constant. For $\sigma = 3/2$ our result is covered by that of Jin, but for $3/2 < \sigma < 5/3$ our result is new.

Now, we can express the main result of this article.

***Theorem 1.2.*** — *There exists a positive absolute constant $\alpha_0$ such that the following holds. Let $A$ be a set of non-negative integers such that $0 \in A$ and $\gcd(A) = 1$. Put $\alpha = \overline{d}(A)$ and $\gamma = \overline{d}(A+A)$. Assume that $0 < \alpha = \overline{d}(A) \leq \alpha_0$ and that*

$$\gamma = \sigma\alpha,$$

*where $3/2 \leq \sigma < 5/3$. Then we have one of the following cases.*

1. *Non-archimedean case: there exist two positive integers $N$ and $t$ with $\gcd(N,t) = 1$ such that*

$$A \subseteq \{iN; i \in \mathbb{N}\} \cup \{t + iN; i \in \mathbb{N}\},$$

*and*

$$\alpha \geq \frac{6}{(4\sigma - 3)N}.$$

2. *Archimedean case: there exist an increasing sequence of integers $(y_j)_{j\geq 1}$ with*

$$\lim_{j\to\infty} \frac{A(y_j)}{y_j} = \alpha,$$

*and two sequences $(b_j)_{j\geq 1}$ and $(t_j)_{j\geq 1}$ with $0 \leq b_j \leq t_j \leq y_j$ such that, if we define*

$$\lambda_j := \frac{b_j}{y_j - t_j}$$

*and*

$$r_j := \frac{A(t_j, y_j)}{y_j - t_j + 1},$$

*then $A(b_j, t_j) = 0$ for all $j \geq 1$ and*

$$\lim_{j\to\infty} \lambda_j = \lambda, \qquad \lim_{j\to\infty} r_j = r$$

*with*

$$\lambda \leq \frac{2\sigma - 3}{2\sigma - 2}\Big(\frac{1}{2\sigma - 2} - \alpha\Big)^{-1}$$

*and*

$$r \geq \left(\frac{1}{2\sigma - 2} + \lambda\left(\frac{1}{2\sigma - 2} - \alpha\right)\right).$$

***Example 1.3.*** — We cannot extend Theorem 1.2 to the case $\overline{d}(A+A) = \frac{5}{3}\overline{d}(A)$. It suffices to consider the set $A := N\mathbb{N} \cup (1+2N\mathbb{N})$ which verifies this condition but also $\alpha = \frac{3}{2N}$. Putting $\sigma = \frac{5}{3}$ in the Theorem 1.2 would have given $\alpha \geq \frac{18}{11N} > \frac{3}{2N}$.

The following example proves that the lower bound obtained in the non-archimedian case of Theorem 1.2 cannot be refined.

***Example 1.4.*** — Fix $\frac{3}{2} \leq \sigma < \frac{5}{3}$. Let $(T_n)_{n\geq 1}$ be an increasing sequence of positive integers such that

$$\lim_{n\to\infty} T_{n+1}/T_n = \infty$$

and

$$E := \bigcup_{n=1}^{\infty} [\lceil (1-\alpha')T_n \rceil ; T_n],$$

where $\alpha' = \frac{3}{4\sigma-3}$. Let $N$ be a sufficiently large positive integer and

$$A := N.E \cup (1+N.E).$$

We can verify that

$$\alpha = \frac{6}{(4\sigma-3)N} < \alpha_0.$$

Furthermore,

$$\gamma = 3\frac{1+\alpha'}{2N} = \frac{6\sigma}{(4\sigma-3)N}.$$

## 2. General results in additive number theory

Before proving the main theorem, let us show why 3/2 is a lower bound for the quotient $\sigma$.

***Lemma 2.1.*** — *Let $A$ be a set of non-negative integers. Suppose that $0 \in A$ and $\gcd(A) = 1$. Then*

$$\gamma \geq \frac{3}{2}\alpha.$$

We can easily deduce the lemma from the following

***Theorem 2.2.*** — *Let $k \geq 3$ be an integer. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of non-negative integers such that*

$$0 = a_0 < a_1 < \cdots < a_{k-1},$$

$$\gcd(A) = 1.$$

*If $a_{k-1} \geq 2k - 3$, then*

$$|A + A| \geq 3k - 3.$$

*Proof of the theorem.* — See [**9**], p. 23. □

*Proof of the lemma.* — Since $\overline{d}(A) = \alpha$, there exists an increasing sequence of integers $(y_j)_{j \geq 1}$ such that, for all $\varepsilon > 0$, if we define $A_j := A \cap [0; y_j]$ and assume $j$ sufficiently large, we have

$$\alpha - \varepsilon < \frac{|A_j|}{y_j} < \alpha + \varepsilon,$$

$$\gcd(A_j) = 1.$$

In the sequel, we will assume that $y_j \in A_j$.

Under the hypothesis $\alpha < \frac{1}{2}$, we can see that $A_j$ verifies the hypothesis of Theorem 2.2. Then,

$$|A_j + A_j| \geq 3|A_j| - 3,$$

and therefore,

$$\begin{aligned} \frac{(A + A)(2y_j)}{2y_j} &\geq \frac{|A_j + A_j|}{2y_j} \\ &\geq \frac{3}{2}\frac{|A_j|}{y_j} - \frac{3}{2y_j} \\ &\geq \frac{3}{2}\alpha - 2\varepsilon. \end{aligned}$$

It suffices to consider the sequence $(A+A)(2y_j)$ to obtain $\overline{d}(A+A) \geq \frac{3}{2}\alpha$ and conclude the proof. □

In the end of this section, we are going to expose some general results in additive number theory. These are tools which we will use in the following section to prove the main theorem. We will give some references for the proofs of those results.

Let $A$ be a finite set of integers. It is easy to see that $|A + A| \geq 2|A| - 1$, and $|A + A| = 2|A| - 1$ if and only if $A$ is an arithmetical progression.

Freiman [**9**], p.21, generalized this fact.

***Theorem 2.3* (Freiman).** — *Let $A$ be a set of non-negative integers such that $|A| \geq 3$ and $\min(A) = 0$. We denote by $a_k$ the greatest element of $A$. If*

$$a_k \leq 2|A| - 3,$$

*then*

$$|2A| \geq |A| + a_k.$$

This result has been generalized to distinct sets by V.F. Lev and P.Y. Smeliansky in [**8**] and was improved by Y.V. Stanchescu in [**10**]. We will use the following version:

***Theorem 2.4* (Lev,Smelianski).** — *Let $A$ and $B$ be two sets of non-negative integers such that $0 \in A \cap B$. We denote by $l(A) := \max(A) - \min(A)$ the length of $A$ and by $h(A) := l(A) - |A| + 1$ the number of holes in $A$. If*

$$\max(l(A), l(B)) \leq |A| + |B| - 3,$$

*then*

$$|A + B| \geq (|A| + |B| - 1) + \max(h(A), h(B)).$$

Now, let us introduce some notations taken out from [**2**].

***Definition 2.5.*** — Let $A$ and $B$ be two abelian groups and $K \subset A$, $L \subset B$. An application $\varphi : K \to L$ is said to be a *Freiman's homomorphism* or a *$F_2$-homomorphism* if, for all $(x, y, x', y') \in K^4$, we have

$$x + y = x' + y' \Rightarrow \varphi(x) + \varphi(y) = \varphi(x') + \varphi(y').$$

Such a $\varphi$ is said to be a *$F_2$-isomorphism* if it is invertible and if $\varphi^{-1}$ is also a $F_2$-homomorphism.

***Example 2.6.*** — In the sequel, we will use some translations and symmetries around horizontal lines in $\mathbb{Z}^2$ which are both $F_2$-isomorphisms.

We will use the next type of $F_2$-isomorphism too.

***Definition 2.7.*** — Let $l$ a non-negative integer. We call *$F_2$-twisting isomorphism* of order $l$ the following application:

$$\begin{aligned} \Psi_l : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ (u, v) &\mapsto (u + lv, v). \end{aligned}$$

We have clearly the following proposition:

***Proposition 2.8.*** — *A $F_2$-isomorphism $\varphi$ between a set $A$ and $\varphi(A)$ induces a natural one-to-one application between $A+A$ and $\varphi(A)+\varphi(A)$.*

***Remark 2.9.*** — We can define in the same way $F_i$-homomorphisms for any positive integer $i$. Such an application $\varphi : K \longrightarrow L$ has to verify, for all $(x_1, \ldots, x_i, x'_1, \ldots, x'_i) \in K^{2i}$,

$$x_1 + \cdots + x_i = x'_1 + \cdots + x'_i \Rightarrow \varphi(x_1) + \ldots \varphi(x_i) = \varphi(x'_1) + \cdots + \varphi(x'_i).$$

Clearly, such a $F_i$-homomorphism is a $F_2$-homomorphism for all $i \geq 2$.

***Definition 2.10.*** — A set $P$ included in an abelian group is called a *generalized arithmetical progression* of dimension $m$ if it can be written

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$$P = P(x_0; x_1, \ldots, x_m; b_1, \ldots, b_m) = \{x_0+\beta_1 x_1+\cdots+\beta_m x_m; \beta_i = 0, \ldots, b_i-1\}$$

where $x_0, \ldots, x_m$ are elements of the group and $b_1, \ldots, b_m$ are positive integers.

We say that $P$ is a *$F_2$-progression* if the application

$$\theta : \{0, \ldots, b_1-1\} \times \cdots \times \{0, \ldots, b_m-1\} \subset \mathbb{Z}^m \to P$$
$$(\beta_1, \ldots, \beta_m) \mapsto x_0 + \beta_1 x_1 + \cdots + \beta_m x_m,$$

is a $F_2$-isomorphism.

We will heavily use the following fundamental theorem due to G. Freiman whose proof can be found in [**2**] and whose following version is taken out from [**1**]:

***Theorem 2.11* (Freiman).** — *Let $\sigma$ be a positive real number, and $A$ a finite set of non negative integers such that $0 \in A$ and $|A| > k(\sigma)$ where $k$ is a fixed constant depending only on $\sigma$. If*

$$|A+A| \leq \sigma |A|,$$

*then $A$ is a subset of a $F_2$-progression*

$$P = P(0; x_1, \ldots, x_m; b_1, \ldots, b_m)$$

*of dimension $m \leq \lfloor \sigma - 1 \rfloor$ and whose length is bounded from above: $|P| \leq C_1(\sigma)|A|$.*

*Furthermore, if $b_1 \leq b_2 \leq \cdots \leq b_m$, we have*

$$i > \lfloor \log_2 \sigma \rfloor \Rightarrow b_i \leq C_2(\sigma).$$

*where $C_1(\sigma)$ and $C_2(\sigma)$ are constants depending only on $\sigma$.*

Our strategy of proof is simple. First, we are going to transpose the *infinite* problem into a *finite* one. Then, we will use Theorem (2.11) to obtain structure of finite sets. Finally, we will come back to the set $A$ using asymptotic arguments.

In the sequel, Theorem 2.11 will be used with $\sigma \leq 3$ so that it will give rise to $F_2$-progression of dimension at most 2. Then, having a look to Definition 2.10, it will be natural to use results concerning addition of sets in $\mathbb{Z}^2$, particularly the following whose proof can be found in [**3**] p.28:

***Theorem 2.12* (Freiman).** — *Let $A \subset \mathbb{Z}^2$ be a set of at least 12 elements not contained in a single line. We assume that*

$$|A+A| < \frac{10}{3}|A| - 5.$$

*Then $A$ is contained in a set $F_2$-isomorphic to*

$$A^0 = \{(0,0),(0,1),\dots,(0,l_1-1)\} \cup \{(1,0),(1,1),\dots,(1,l_2-1)\},$$

*with $l_1, l_2 \geq 1$ et $l_1 + l_2 = |A+A| - 2|A| + 3$.*

## 3. Proof of the main theorem

With a view to use the theorems of the previous section, let us transpose our problem into a problem of finite sets.

Let $\varepsilon > 0$, we can choose $y_1 \in \mathbb{N}$ sufficiently large and a strictly increasing sequence $(y_j)_{j\geq 1}$ of positive integers such that both the relations below are verified for all $j$:

$$(A+A)(2y_j) \leq (\gamma + \varepsilon) \times 2y_j,$$

$$(\alpha - \varepsilon)y_j \leq A(y_j) \leq (\alpha + \varepsilon)y_j.$$

We will use the notation

$$A_j := \{a \in A,\ a \leq y_j\}.$$

In the sequel, all the notations will depend on the sequence $(y_j)_{j\geq 1}$. Every change of the sequence will naturally changes the sets $A_j$ and all that is related. We will denote by $O(\varepsilon)$ any positive function of $\varepsilon$ bounded above by $C\varepsilon$ where $C$ is an absolut constant.

Now, we are able to precise the structure of the sets $A_j$. We have

$$
\begin{aligned}
\frac{|A_j+A_j|}{|A_j|} &= \frac{|A_j+A_j|}{2y_j} \times 2 \times \frac{y_j}{|A_j|} \\
&\leq \frac{(A+A)(2y_j)}{2y_j} \times 2 \times \frac{y_j}{|A_j|} \\
&\leq 2 \times \frac{\gamma+\varepsilon}{\alpha-\varepsilon} \\
&\leq (2\sigma+\varepsilon') < 4,
\end{aligned} \tag{3}
$$

where $\varepsilon' = O(\varepsilon)$.

Then, for $\varepsilon$ sufficiently small, we can apply the fundamental Theorem 2.11 of Freiman to the sets $A_j$. By a simple calculus, we obtain $m \leq 2$ and $b_2 \leq C_2$. First, we are going to exclude the case where $A_j$ is a subset of an arithmetical progression of dimension $m = 1$ for infinitely many values of $j$.

Suppose it is the case. Then, for $j$ sufficiently large, $A_j \subseteq P_j$ where $P_j$ is an arithmetic progression of difference 1 (for $\gcd(A) = 1$) and first term 0. We can assume it has minimum length. Then, we have, by Theorem 2.11 and since $\{0, y_j\} \subseteq P_j$:

$$|P_j| \geq y_j, \tag{4}$$

$$|P_j| \leq C_1 |A_j|. \tag{5}$$

Now we combine (4) and (5)and we can find a lower bound for $\alpha$:

$$
\begin{aligned}
\alpha &\geq \frac{1}{\sigma}\gamma \\
&\geq \frac{1}{\sigma}\frac{|A_j+A_j|-\varepsilon}{2y_j} \\
&\geq \frac{1}{\sigma}\frac{2|A_j|-1-\varepsilon}{2y_j} \\
&\geq \frac{1}{\sigma}\frac{1}{2y_j}\left(\frac{2y_j}{C_1}-1-\varepsilon\right) \\
&\geq \alpha_0,
\end{aligned} \tag{6}
$$

for an absolut constant $\alpha_0$ (Remember that $\varepsilon$ can be chosen sufficiently small.).

Then, we can exclude this case under hypothesis $\alpha < \alpha_0$ of Theorem 1.2.

***Remark 3.1.*** — The value of $C_1$ (One can find an estimate in [**2**].) implies a very small value for the bound $\alpha_0$. The case where $A$ is included in a single line ($\alpha > \alpha_0$) remains an open question.

Thus, for infinitely many integers $j$, the set $A_j$ is a subset of an arithmetical progression of dimension $m = 2$. By extracting a subsequence, we can assume that it is the case for all the sets $A_j$. Then, for all $j \geq 1$, there is an $F_2$-isomorphism $\theta_j$ between a subset of $\mathbb{Z}^2$ and $A_j$ (See Definition 2.10.). By Proposition 2.8, $\theta_j^{-1}(A_j)$ have the same additive properties than $A_j$. In particular, it satisfies the inequality (3).

At this point, we can apply Theorem 2.12 to $\theta_j^{-1}(A_j)$. We obtain, composing the isomorphisms, that, for all $j \geq 1$, there exists a $F_2$-isomorphism $\varphi_j : \mathbb{Z}^2 \to \mathbb{N}$ such that $A_j \subseteq \varphi_j(A_j^0)$ where $A_j^0 = \{(0,0), (0,1), \ldots, (0, l_{1,j} - 1)\} \cup \{(1,0), (1,1), \ldots, (1, l_{2,j} - 1)\}$. Combining, if necessary, those isomorphisms with translations and twisting isomorphisms (See Definition 2.7.), we can assume that $\varphi_j((0,0)) \in A_j$ and $\varphi_j((1,0)) \in A_j$. Furthermore, we have $l_{1,j} + l_{2,j} = |A_j + A_j| - 2|A_j| + 3$.

We can notice that the number of elements of $\varphi^{-1}(A_j)$ in each line cannot be bounded, otherwise, for all $\varepsilon > 0$, we would have clearly an integer $j_0$ sufficiently large such that $|\varphi^{-1}(A_j) + \varphi^{-1}(A_j)| > (2-\varepsilon)|\varphi^{-1}(A_j)|$ for every $j \geq j_0$ and so, again by Proposition 2.8, $|A_j + A_j| > \frac{10}{3}|A_j|$.

We denote respectively by $d_{1,j}$ and $d_{2,j}$ the differences $\varphi_j((1,0)) - \varphi_j((0,0))$ and $\varphi_j((0,1)) - \varphi_j((0,0))$. Then, we can give explicitly the $F_2$-isomorphism

$$\begin{aligned} \varphi_j : \mathbb{Z} \times \{0,1\} &\longrightarrow \mathbb{N} \\ (x,y) &\mapsto a_j + x d_{1,j} + y d_{2,j}, \end{aligned} \tag{7}$$

where $a_j = \varphi_j((0,0))$.

Since $A \subseteq \mathbb{N}$, the number $d_{1,j}$ has to be positive for infinitely many values of $j$ what we again extract. We can also assume, by switching the lines if necessary, that the differences $d_{2,j}$ are positive integers.

***Lemma 3.2.*** — *The sequence $(d_{1,j})_{j \geq 1}$ is bounded.*

*Proof.* — Assume the contrary. Then, there exists an index $j$ such that $A(d_{1,j}) > 3$ and, consequently, there exists two elements $a$ and $b$ of $A \cap [0, d_{1,j}[$ such that $\varphi_j^{-1}(a)$ and $\varphi_j^{-1}(b)$ lie on the same line. We deduce from (7) that $|b - a| = k d_{1,j}$ where $k$ is a positive integer. This is impossible since $|b - a| < d_{1,j}$. □

Once again, we can again extract a subsequence from $(A_j)_{j\geq 1}$ such that $d_{1,j} = N > 0$ for all $j \geq 1$. $N$ will be chosen maximal.

**3.1. The non-archimedean case.** — In this case, we assume

$$N > 1.$$

We can show that, in this case, the sequence $(d_{2,j})_{j\geq 1}$ can be supposed constant.

***Lemma 3.3.*** — *There exist a positive integer $t$ and a sequence $(y_j)_{j\geq 1}$ such that $d_{2,j} = t$ for all $j \geq 1$.*

*Proof.* — Each set $A_j$ is included in two residue classes $\mod N$. Since those sets verify $A_j \subseteq A_k$ fo $j < k$, the whole set $A$ is included in two residue classes. If we note $t$ the smallest term of the part of $A$ not congruent to $0 \mod N$, we can choose, for each $j \geq 1$, the isomorphism $\varphi_j$ such that $\varphi_j((0,0)) = 0$ and $\varphi_j((1,0)) = t$. □

Hence, we can assume that $d_{2,j} = t$ for all $j \geq 1$ and we can exhibit a $F_2$-isomorphism $\varphi$ between $\mathbb{Z}^2$ and $\mathbb{N}$ such that $\varphi_{|A_j} = \varphi_j$.

$$\begin{aligned} \varphi : \mathbb{Z} \times \{0,1\} &\longrightarrow \mathbb{N} \\ (x,y) &\mapsto xN + yt. \end{aligned}$$

By hypothesis (1), we must have $\gcd(t,N) = 1$ and $A$ is included in two residue classes $\mod N$ which we denote by $B$ and $C$:

$$B = \{a \in A; a \equiv 0 \mod N\}, \qquad C = \{a \in A; a \equiv t \mod N\}.$$

We define $B_j := B(y_j)$ and $C_j := C(y_j)$ and we assume, choosing $y_1$ sufficiently large, that those sets are non empty. Furthermore, we define $b_j := \max(B_j)$ and $c_j := \max(C_j)$. We may assume that $b_j = y_j$, extracting if necessary a subsequence of $(y_j)_{j\geq 1}$.

Let us prove the following result.

***Lemma 3.4.*** — *There exists a sequence $(y_j)_{j\geq 1}$ such that, for all $\varepsilon > 0$, we have for $j$ sufficiently large*

$$|A_j| \geq \frac{1}{\big(2\sigma - 2 + \varepsilon\big)N}(b_j + c_j).$$

*Proof.* — We denote by $t_0$ the smallest element of $A$ non divisible by $N$. We define $S_j := b_j + c_j - t_0 + 2$.

Let $\varepsilon > 0$. We have, using Theorem 2.12,

$$\begin{aligned}\frac{S_j}{N} &\leq |A_j + A_j| - 2|A_j| + 3\\ &\leq \Big(2\sigma - 2 + \varepsilon'\Big)|A_j| + 3\\ &\leq \Big(2\sigma - 2 + \varepsilon''\Big)|A_j|,\end{aligned}$$

where $\varepsilon' = O(\varepsilon)$ and $\varepsilon'' = O(\varepsilon)$. It suffices to choose $j$ sufficiently large to obtain the result. □

We denote again by $(y_j)_{j\geq 1}$ the sequence of integers verifying the last lemma.

Now, we are going to refine the last results. We define

$$X_j := \frac{c_j}{b_j},$$

and

$$\lambda_j := \frac{N|A_j|}{b_j + c_j}.$$

***Lemma 3.5.*** — *There exists a sequence* $(y_j)_{j\geq 1}$ *such that* $\lim_{j\to\infty} X_j = 1$.

*Proof.* — We will only use the definition of the upper asymptotic density of $A$. Given $\varepsilon > 0$, from the sequence $(y_j)_{j\geq 1}$ we can extract a subsequence (also denoted by $y_j$) such that,

$$\frac{A(c_j)}{c_j} \leq \frac{A(b_j)}{b_j} + \varepsilon.$$

Furthermore, we have

$$A(c_j) \geq A(b_j) - \frac{b_j - c_j}{N}.$$

Putting together the last two relations, we obtain:

$$N\varepsilon + \frac{\lambda_j(b_j + c_j)}{b_j} \geq \frac{\lambda_j(b_j + c_j)}{c_j} - \frac{b_j}{c_j} + 1.$$

It yields the following polynomial inequality:

$$\lambda_j X_j^2 - (1 - N\varepsilon)X_j - (\lambda_j - 1) \geq 0. \tag{8}$$

It remains to determine the discriminant and the roots. We obtain

$$\Delta = (2\lambda_j - 1)^2 + \varepsilon(N^2\varepsilon - 2N).$$

Thus, using Lemma 3.4 to bound $\lambda_j$ from below, the roots $X_j' < X_j''$ are

$$X_j' = \frac{1}{2\lambda_j}(1 - N\varepsilon - \sqrt{\Delta}) = \frac{1}{\lambda_j} - 1 + O(\varepsilon),$$

$$X_j'' = \frac{1}{2\lambda_j}(1 - N\varepsilon + \sqrt{\Delta}) = 1 - O(\varepsilon).$$

Clearly, one of the following cases happens.

- The quantity $X_j$ is less than $\frac{1}{\lambda_j} - 1 + O(\varepsilon)$ which is impossible with the lower bound $\lambda_j$ obtained in Lemma 3.4. Indeed, it would implie

$$X_j \leq \frac{1}{\lambda_j} - 1 + O(\varepsilon) \leq 2\sigma - 3 + O(\varepsilon) < \frac{1}{3}$$

  for $\varepsilon$ sufficiently small. We would obtain the relation

$$\frac{(A+A)(b_j)}{b_j} \geq \frac{|B_j| + |B_j| + |C_j + C_j|}{b_j} \geq 2\alpha,$$

  which contradicts the main hypothesis of Theorem 1.2.
- Thus, $X_j \geq 1 - O(\varepsilon)$ which is the conclusion of the lemma.

□

Now we combine the results of the last two lemmas and apply Theorems 2.3 and 2.4 to the sets

$$B_j' := \frac{1}{N} B_j$$

and

$$C_j' := \frac{1}{N}(C_j - t).$$

We have

$$|A_j| \geq \frac{1}{(2\sigma - 2 - \varepsilon)N}(b_j + c_j).$$

We notice that, for $\varepsilon$ sufficiently small, since $\sigma < \frac{5}{3}$,

$$\frac{1}{2\sigma - 2 - \varepsilon} > \frac{3}{4}.$$

Fix $\delta > 0$ such that

$$|A_j| \geq \left(\frac{3}{4} + \delta\right) \frac{(b_j + c_j)}{N}.$$

Using Lemma 3.5, we obtain

$$|A_j| \geq \left(\frac{3}{4}+\delta\right)\frac{(2-\varepsilon')y_j}{N}$$

for $\varepsilon'$ arbitrarily small. Then, there exists a positive constant $\delta'$ such that, for $j$ sufficiently large,

$$|A_j| \geq \left(\frac{3}{2}+\delta'\right)\frac{y_j}{N}.$$

It follows that

$$|B'_j| = |B_j| = |A_j| - |C_j| \geq |A_j| - \frac{y_j}{N} \geq \left(\frac{1}{2}+\delta'\right)\frac{y_j}{N} \geq \left(\frac{1}{2}+\delta'\right)\max(B'_j).$$

Thus we can apply Theorem 2.3 to $B'_j$. We can do the same for $C'_j$. Moreover,we have

$$|B'_j| + |C'_j| = |A_j| \geq (1+2\delta')\frac{y_j}{N},$$

so we can apply Theorem 2.4 too.

For $j$ sufficiently large, we have then

$$\begin{aligned} |A_j + A_j| &= |B_j + B_j| + |B_j + C_j| + |C_j + C_j| \\ &= |B'_j + B'_j| + |B'_j + C'_j| + |C'_j + C'_j| \\ &\geq |B'_j| + \frac{y_j}{N} + (1-\varepsilon')\frac{y_j}{N} + |B'_j| + (1-\varepsilon')\frac{y_j}{N} + |C'_j| \\ &= 2|B_j| + |C_j| + (3-2\varepsilon')\frac{y_j}{N}, \end{aligned} \tag{9}$$

assuming, without loss of generality, that $|B_j| \geq |C_j|$. Here, $\varepsilon'$ is arbitrarily small, by Lemma 3.5.

Now, we have too, for $j$ sufficiently large

$$|A_j + A_j| \leq (2\sigma+\varepsilon')|A_j|,$$

therefore,

$$(2\sigma+\varepsilon')|B_j| + (2\sigma+\varepsilon')|C_j| \geq 2|B_j| + |C_j| + (3-2\varepsilon')\frac{y_j}{N},$$

and finally,

$$(2\sigma - 2 + \varepsilon')|B_j| + (2\sigma - 1 + \varepsilon')|C_j| \geq (3 - 2\varepsilon')\frac{y_j}{N}. \tag{10}$$

We obtain a first conclusion from the inequality (10), using $|B_j| \geq |C_j|$:

$$(4\sigma - 3 + 2\varepsilon')|B_j| \geq (3 - 2\varepsilon')\frac{y_j}{N},$$

and so

$$|B_j| \geq \left(\frac{3}{4\sigma - 3} - \varepsilon''\right)\frac{y_j}{N}, \tag{11}$$

where $\varepsilon'' = O(\varepsilon')$.

We shall again use the relation (10) to bound $|C_j|$ from below and then $|A_j|$:

$$|C_j| \geq \frac{1}{2\sigma - 1 + \varepsilon'}\left((3 - 2\varepsilon)\frac{y_j}{N} - (2\sigma - 2 + \varepsilon)|B_j|\right).$$

We obtain, using (11):

$$\begin{aligned}
|A_j| &\geq |B_j| + |C_j| \\
&\geq \frac{1}{2\sigma - 1 + \varepsilon'}\left((3 - 2\varepsilon')\frac{y_j}{N} + (2 - 2\sigma - \varepsilon')\,|B_j|\right) + |B_j| \\
&\geq \frac{3 - 2\varepsilon'}{2\sigma - 1 + \varepsilon'}\frac{y_j}{N} + \frac{(2 - 2\sigma - \varepsilon') + (2\sigma - 1 + 3\varepsilon')}{2\sigma - 1 + \varepsilon'}|B_j| \\
&\geq \left(\frac{3}{2\sigma - 1} + \frac{3}{(2\sigma - 1)(4\sigma - 3)} - \varepsilon''\right)\frac{y_j}{N} \\
&= \left(\frac{12\sigma - 9 + 3}{(2\sigma - 1)(4\sigma - 3)} - \varepsilon''\right)\frac{y_j}{N} \\
&= \left(\frac{6}{4\sigma - 3} - \varepsilon''\right)\frac{y_j}{N},
\end{aligned} \tag{12}$$

where $\varepsilon'' = O(\varepsilon')$.

It is the conclusion of the non-archimedean case of Theorem 1.2:

$$\overline{d}(A) \geq \frac{6}{(4\sigma - 3)N}.$$

**3.2. The archimedean case.** — Now we assume that

$$N = 1.$$

We can show that, in this case, the sequence $(d_{2,j})_{j\geq 1}$ cannot be bounded. Suppose the contrary, we could extract a subsequence of $(y_j)_{j\geq 1}$ such that $d_{2,j} = t$ for all $j$ and do as in the non-archimedean case, i.e. find a common isomorphism between every $A_j$ and a part of two lines of $\mathbb{Z}^2$. This isomorphism could be written

$$\begin{aligned} \varphi : \mathbb{Z} \times \{0,1\} &\longrightarrow \mathbb{N} \\ (x,y) &\mapsto x + ty. \end{aligned}$$

It is impossible because, for $j$ sufficiently large, we would have an element of $A \cap \varphi(\{y = 0\})$ greater than $t$ (Remember that there is infinitely many elements of $\varphi^{-1}(A)$ on each line.) so that $t$ would have two inverse images by $\varphi$ (One on each line.), which contradicts the definition of a $F_2$-isomorphism.

Then, we can choose $(y_j)_{j\geq 1}$ and consequently $t_j := d_{2,j}$ such that $t_j$ is a strictly increasing sequence. Thus, as in the non-archimedean case, we can have $\varphi_j((0,0)) = 0$, $\varphi_j((1,0)) = 1$, $\varphi_j((1,0)) = t_j$ and

$$\begin{aligned} \varphi_j : \mathbb{Z} \times \{0,1\} &\longrightarrow \mathbb{N} \\ (x,y) &\mapsto x + yt_j. \end{aligned}$$

We shall apply Theorem 2.12 to the sets $A_j$. Then, we can include $A_j$ in a set $A_j^0$ which is the union of two arithmetical progressions $B_j^0$ and $C_j^0$ (of difference $N = 1$ here). We denote as usual by $b_j := l_{2,j} = |B_j^0|$ and $c_j := l_{2,j} = |C_j^0|$ the respective lengths of those two progressions, where $0 \in B_j^0$ and $y_j \in C_j^0$. Indeed, those two elements cannot be in the same progression: in this case, $A$ would be in an arithmetical progression of dimension 1, say $B_j^0$. This case, which is the *single line case*, is already excluded by $\alpha < \alpha_0$. Those lengths being supposed minimal, we have $y_j - t_j = l_{2,j}$ and $\max(B_j^0) = b_j$.

***Lemma 3.6.*** — *There exists a sequence $(y_j)_{j\geq 1}$ such that, for all $\varepsilon > 0$, there exists $j_0 \geq 1$ such that for all $j \geq j_0$, we have*

$$|A_j| \geq \left(\frac{1}{2\sigma - 2} - \varepsilon\right)(l_{1,j} + l_{2,j}).$$

*Proof.* — It suffices to apply Theorem 2.12 for $j$ sufficiently large:

$$\begin{aligned} l_{1,j} + l_{2,j} &\leq |A_j + A_j| - 2|A_j| + 3 \\ &\leq (2\sigma - 2 + \varepsilon')|A_j| + 3 \\ &\leq (2\sigma - 2 + \varepsilon'')|A_j|, \end{aligned}$$

where $\varepsilon'$ is arbitrarily small and $\varepsilon'' = O(\varepsilon')$. □

We denote again by $(y_j)_{j\geq 1}$ the sequence of integers verifying the last lemma.

If $b_j \geq t_j$, then $l_{1,j} + l_{2,j} \geq y_j$ and, by (3.6) and the range of values of $\sigma$, $|A_j| \geq \frac{3}{4} y_j$ what is incompatible with $\alpha < \frac{1}{2}$. Then, we have $b_j < t_j$, and thus

$$A(b_j, t_j) = 0. \tag{13}$$

Now we define $B_j := A \cap [0; b_j]$ and $C_j := A \cap [t_j; y_j]$ with $b_j < t_j$.

The quotient $X_j := \frac{|B_j|}{b_j}$ cannot be too large, otherwise we would obtain, considering the sets $A(b_j)$, a too large value for $\alpha$. Clearly, we have:

$$X := \limsup_{j\to\infty} X_j \leq \alpha. \tag{14}$$

Let us show in which sense $b_j$ is necessarily small compared with $l_{2,j}$.

***Lemma 3.7.*** — *If we define*

$$\lambda_j := \frac{b_j}{l_{2,j}},$$

*we have*

$$\lambda := \limsup_{j\to\infty} \lambda_j \leq \frac{2\sigma - 3}{2\sigma - 2} \left( \frac{1}{2\sigma - 2} - X \right)^{-1}. \tag{15}$$

*Proof.* — We use the lemma (3.6), noting that:

$$|A_j| = |B_j| + |C_j| = X_j \lambda_j l_{2,j} + |C_j|.$$

We obtain, for all $\varepsilon > 0$, for $j$ sufficiently large:

$$X_j \lambda_j l_{2,j} + |C_j| \geq \left( \frac{1}{2\sigma - 2} - \varepsilon \right) (\lambda_j + 1) l_{2,j},$$

and then,

$$|C_j| \geq l_{2,j} \left( \frac{1}{2\sigma - 2} - \varepsilon + \lambda_j \left( \frac{1}{2\sigma - 2} - \varepsilon - X_j \right) \right). \tag{16}$$

Now, we know that $|C_j| \leq l_{2,j}$, therefore we obtain the upper bound

$$\lambda_j \leq \left( \frac{2\sigma - 3}{2\sigma - 2} + \varepsilon \right) \left( \frac{1}{2\sigma - 2} - \varepsilon - X_j \right)^{-1}.$$

It remains to recall us that $X \leq \alpha$ to obtain

$$\lambda \leq \frac{2\sigma - 3}{2\sigma - 2} \left( \frac{1}{2\sigma - 2} - \alpha \right)^{-1}.$$

□

Let us take as a new sequence $(y_j)_{j \geq 1}$ the subsequence such that $\lim_{j \to \infty} \lambda_j = \lambda$. It suffices to look again at the relation (16) to obtain

$$r_j = \frac{|C_j|}{l_{2,j}} \geq \left( \frac{1}{2\sigma - 2} + \lambda \left( \frac{1}{2\sigma - 2} - X \right) \right)$$

for infinitely many values of $j$.

Then, a last extraction of subsequence allows us to suppose the bounded sequence $(r_j)_{j \geq 1}$ having a limit $r$ such that

$$r \geq \left( \frac{1}{2\sigma - 2} + \lambda \left( \frac{1}{2\sigma - 2} - \alpha \right) \right). \tag{17}$$

Hence, putting together the results (13), (14), (3.7) and (17) conclude the proof of the archimedean case of Theorem 1.2.

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