INVERSE ADDITIVE RESULTS FOR SETS WITH SMALL UPPER ASYMPTOTIC DENSITY

by

Guillaume Bordes

Abstract. — For a infinite set of non negative integers A, we denote by $\overline{d}(A)$ the upper asymptotic density of A. If $0 \in A$ and $\gcd(A) = 1$, we can easily prove that $\overline{d}(A + A) \geq \frac{3}{2}\overline{d}(A)$. The goal of this article is to determine the structure of A when both $\overline{d}(A)$ and the growing ratio $\sigma := \frac{\overline{d}(A+A)}{\overline{d}(A)}$ are small. In particular, we will obtain optimal results when $\frac{3}{2} \leq \sigma < \frac{5}{3}$ and $\overline{d}(A) < \alpha_0$ where α_0 is an absolut constant.

Contents

PART I
INTRODUCTION AND PREREQUISITE RESULTS

1. Notations and main results

Let $A \subset \mathbb{N}$ such that $0 \in A$ and $\gcd(A) = 1$.

For all $y > x > 0$, we note $A(x) := |A \cap [0; x]|$ and $A(x, y) := |A \cap [x; y]|$. Then, we can define the *upper asymptotic density* of A by:

$$\overline{d}(A) := \limsup_{x \to \infty} \frac{A(x)}{x}.$$

We note $\alpha := \overline{d}(A)$ and assume that $0 < \alpha < \frac{1}{2}$.

We can define the sum of two sets

$$A + B = \{a + b \mid a \in A, b \in B\},$$

and the dilated of a set

$$\lambda.A := \{\lambda a \mid a \in A\}.$$

Finally, we note $\gamma := \overline{d}(A + A)$ and $\sigma := \frac{\gamma}{\alpha} = \frac{\overline{d}(A+A)}{\overline{d}(A)}$.

It is natural, in additive number theory, to find a lower bound for the *growing ratio* σ. We will prove the

Lemma I.1. — *Let A a set of non negative integers. Suppose that $0 \in A$ and* $\gcd(A) = 1$. *Then*

$$\gamma \geq \frac{3}{2}\alpha.$$

We can easily deduce the lemma from the following

Theorem I.2. — *Let $k \geq 3$ an integer. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ a set of non negative integers such that*

$$0 = a_0 < a_1 < \cdots < a_{k-1},$$

$$\gcd(A) = 1.$$

If $a_{k-1} \geq 2k - 3$, then

$$|A + A| \geq 3k - 3.$$

Proof of the theorem. — See [**7**] p.23. □

Proof of the lemma. — $\overline{d}(A) = \alpha$, so there exists an increasing sequence of integers $(y_j)_{j\geq 1}$ such that, for all $\varepsilon > 0$, if we define $A_j := A \cap [0; y_j]$ and assume j sufficiently large, we have:

$$\alpha - \varepsilon < \frac{|A_j|}{y_j} < \alpha + \varepsilon,$$

$$\gcd(A_j) = 1.$$

In the sequel, we will assume $y_j \in A_j$.

Under the hypothesis $\alpha < \frac{1}{2}$, we can see that A_j verifies the hypothesis of (I.2). Then,

$$|A_j + A_j| \geq 3|A_j| - 3,$$

and therefore,

$$\begin{aligned}\frac{(A+A)(2y_j)}{2y_j} &\geq \frac{|A_j + A_j|}{2y_j} \\ &\geq \frac{3}{2}\frac{|A_j|}{y_j} - \frac{3}{2y_j} \\ &\geq \frac{3}{2}\alpha - 2\varepsilon.\end{aligned}$$

It suffices to consider the sequence $(A+A)(2y_j)$ to obtain $\overline{d}(A+A) \geq \frac{3}{2}\alpha$ and conclude the proof.

□

The lower bound for σ cannot be improved. In the two following examples, taken out from [**5**], it is achieved.

Example I.3. — Let $\alpha < \frac{1}{2}$ and

$$A = \bigcup_{n=1}^{\infty} [[\lceil (1-\alpha)2^{2^n} \rceil ; 2^{2^n}]].$$

Example I.4. — Let m and k some constants such that $k > 4$ and 0, $2m$ et m pairwise not congruent modulo k. Let

$$A = \{m + ik; i \in \mathbb{N}\} \cup \{ik; i \in \mathbb{N}\}.$$

We can choose m and k such that $\gcd(A) = 1$.

Then we have $\alpha = \frac{2}{k} < \frac{1}{2}$ if $k > 4$.

The object of this article is to answer the following question which is a classical inverse problem in additive number theory:

Question: *What is the structure of A when the ratio σ is near of its lower bound?*

For *lower asymptotic density*, this question has been solved by Kneser in 1953 (see [**4**]). Kneser proved that for $\sigma < 2$, the set A is a *large part* of some union of residue classes modulo an integer N. In [**3**], Freiman proved under the additional hypothesis $\alpha < \alpha_0$ that those residue classes were in arithmetical progression $\mod N$. Our goal is to give an analogous theorem of the Freiman's one for upper asymptotic density. We can immediately remark that the structure we will obtain will be different of part of arithmetical progression as we can see in the example (I.3).

We will obtain results by Freiman's methods in the *general case* $\frac{3}{2} \leq \sigma < \frac{11}{6}$ and more precise structure in the *extremal case* $\frac{3}{2} \leq \sigma < \frac{5}{3}$.

***Theorem I.5* (The general case).** — *Let A a set of non negative integers such that $0 \in A$ et $\gcd(A) = 1$. Let $\frac{3}{2} \leq \sigma < \frac{11}{6}$. Let $\alpha := \overline{d}(A)$, $\gamma := \overline{d}(A+A)$ and $2 \leq s \leq 5$ the integer such that*

$$2 - \frac{1}{s} \leq \sigma < 2 - \frac{1}{s+1}.$$

Then, there exists $\alpha_0 \in]0; \frac{1}{2}[$ an absolut constant such that if $0 < \alpha < \alpha_0$ and

$$\gamma = \sigma\alpha, \tag{1}$$

then one of the following cases happens.

1. *Non-archimedian case: there exist $N \in \mathbb{N}$ and $2 \leq r \leq s$ such that*

   A is a subset of the union of r residue classes $\mod N$ whose reduction in $\mathbb{Z}/N\mathbb{Z}$ is an arithmetical progression, and

$$\alpha \geq \frac{r-1}{2N(\sigma-1)}.$$

2. *Archimedian case: there exists $2 \leq r \leq s$, a constant $C_1(\sigma)$ depending only on σ, an increasing sequence of integers $(y_j)_{j\geq 1}$ and sequences*

$$0 \leq h_j^0 \leq g_j^1 \leq h_j^1 \leq g_j^2 \leq \cdots \leq h_j^{r-2} \leq g_j^{r-1} \leq y_j$$

   such that

$$\lim_{j\to\infty} \frac{A(y_j)}{y_j} = \alpha,$$

   and, for all $j \geq 1$,

$$A(h_j^i, g_j^{i+1}) = 0,$$

$$\sum_{i=0}^{i=r-1} \left(h_j^i - g_j^i\right) \leq C_1(\sigma)|A(y_j)|,$$

where $g_j^0 := 0$ *and* $h_j^{r-1} := y_j$.

***Theorem I.6* (The extremal case).** — *Let* A *a set of non negative integers such that* $0 \in A$ *and* $\gcd(A) = 1$. *Let* $\frac{3}{2} \leq \sigma < \frac{5}{3}$. *Let* $\alpha = \overline{d}(A)$ *and* $\gamma = \overline{d}(A+A)$. *Then, there exists* $\alpha_0 \in]0; \frac{1}{2}[$ *an absolut constant such that, if* $0 < \alpha < \alpha_0$ *and*

$$\gamma = \sigma\alpha, \tag{2}$$

one of the following cases happens.

1. *Non-archimedian case: there exist* $N \in \mathbb{N}$ *and* t *coprime such that*

$$A \subseteq \{iN; i \in \mathbb{N}\} \cup \{t + iN; i \in \mathbb{N}\},$$

and

$$\alpha \geq \frac{6}{(4\sigma - 3)N}.$$

2. *Archimedian case: there exist an increasing sequence of integers* $(y_j)_{j\geq 1}$ *with*

$$\lim_{j\to\infty} \frac{A(y_j)}{y_j} = \alpha,$$

and two sequences $(b_j)_{j\geq 1}$ *and* $(t_j)_{j\geq 1}$ *with* $0 \leq b_j \leq t_j \leq y_j$ *such that, if we define*

$$\lambda_j := \frac{b_j}{y_j - t_j},$$

we have $A(b_j, t_j) = 0$ *for all* $j \geq 1$, *and*

$$\lambda := \lim_{j\to\infty} \lambda_j \leq \frac{2\sigma - 3}{2\sigma - 2}\Big(\frac{1}{2\sigma - 2} - \alpha\Big)^{-1},$$

$$\lim_{j\to\infty} \frac{A(t_j, y_j)}{y_j - t_j + 1} \geq \Big(\frac{1}{2\sigma - 2} + \lambda\big(\frac{1}{2\sigma - 2} - \alpha\big)\Big).$$

Remark I.7. —

- We cannot extend (I.6) to the case $\overline{d}(A+A) = \frac{5}{3}\overline{d}(A)$. It suffices to consider the set $A := N\mathbb{N} \cup (1 + 2N\mathbb{N})$ which verifies this hypothesis but also $\alpha = \frac{3}{2N} < \frac{18}{11N}$.
- In the *border case* $\sigma = \frac{3}{2}$, that is to say when the growing ratio achieves its lower bound, we have thus obtained very precisely the structure of A which is essentially either the union of two arithmetical progressions with same difference, or a *lacunar set*, that is to say a set with *big holes* as the one seen in the example (I.4) ($\lambda = 0$ et $\frac{1}{2\sigma - 2} = 1$).
- Comparing the conclusions of the two theorems, we can observ that the last one is really much more precise. For example, in the non archimedian case and for $\sigma = \frac{3}{2}$, it gives $\alpha \geq \frac{2}{N}$ instead of $\alpha \geq \frac{1}{N}$. We will see in the following example in which sense the conclusions of the non-archimedian case of (I.6) cannot be improved.

Example I.8. — Let

$$E := \bigcup_{n=1}^{\infty} [[\lceil (1-\alpha')2^{2^n} \rceil ; 2^{2^n}],$$

where $\alpha' = \frac{3}{4\sigma - 3}$.

Let N a sufficiently large positive integer and

$$A := N.E \cup (1 + N.E).$$

We can verify that

$$\alpha = \frac{6}{(4\sigma - 3)N} < \alpha_0.$$

Furthermore,

$$\gamma = 3\frac{1+\alpha'}{2N} = \frac{6\sigma}{(4\sigma - 3)N}.$$

We can thus conclude that the lower bound obtained in the non-archimedian case of (I.6) cannot be improved.

2. General results in additive number theory

In this section, we are going to expose some general results in additive number theory. These are tools which we will use in the following sections, in the proof of both the main theorems. We will give some references for the proofs of those results.

First, two basic but important results whose proofs can be found in [**7**] p.6 and p.28:

Theorem I.9. — *Let A a finite set of non negative integers. Then, $|A+A| \geq 2|A|-1$. If $|A+A| = 2|A|-1$, then A is an arithmetical progression.*

***Theorem I.10* (Freiman's 3k-4).** — *Let A a finite set of non negative integers. Suppose $|A| > 3$. If*

$$|A + A| \leq 3|A| - 4,$$

then A is a subset of an arithmetical progression of length $|A+A| - |A| + 1$.

Now, let us introduce some notations taken out from [**2**].

Definition I.11. — Let A and B two abelian groups and $K \subset A$, $L \subset B$. An application $\varphi : A \to B$ is said to be a *Freiman's homomorphism* or a *F_2-homomorphism* if, for all $(x, y, x', y') \in K^4$, we have

$$x + y = x' + y' \Rightarrow \varphi(x) + \varphi(y) = \varphi(x') + \varphi(y').$$

Such a φ is said to be a *F_2-isomorphism* if it is invertible and if φ^{-1} is also a F_2-homomorphism.

Example I.12. — In the sequel, we will use some translations and axial symmetries in $\mathbb{Z}^2$ which are both F_2-isomorphisms.

We will use the next type of F_2-isomorphism too.

Definition I.13. — Let l a non negative integer. We call F_2-twisting isomorphism of order l the following application:

$$\Psi_l : (u, v) \longrightarrow (u + lv, v).$$

Definition I.14. — A set P included in an abelian group is called a *generalized arithmetical progression* of dimension m if it can be written

$$P = P(x_0; x_1, \dots, x_m; b_1, \dots, b_m) = \{x_0 + \beta_1 x_1 + \cdots + \beta_m x_m; \beta_i = 0, \dots, b_i - 1\}$$

where $x_0, \dots, x_m$ are elements of the group and $b_1, \dots, b_m$ are positive integers.

P is said to be a *F_2-progression* if the application

$$\{0, \dots, b_1\} \times \cdots \times \{0, \dots, b_m - 1\} \to P$$
$$(\beta_1, \dots, \beta_m) \mapsto x_0 + \beta_1 x_1 + \cdots + \beta_m x_m,$$

is a F_2-isomorphism.

Remark I.15. — So, a F_2-progression can be seen as a subset of $\mathbb{Z}^m$, that can explain the use of multidimensionnal additive theorems.

We will mainly use the following fundamental theorem due to G. Freiman whose proof can be found in [**2**] and whose following version is taken out from [**1**]:

Theorem I.16. — *Let σ a positive real number, and A a finite set of non negative integers such that $0 \in A$ and $|A| > k(\sigma)$ where k is a fixed constant depending only on σ. If*

$$|A + A| \leq \sigma |A|,$$

then A is a subset of a F_2-progression

$$P = P(0; x_1, \dots, x_m; b_1, \dots, b_m)$$

of dimension $m \leq \lfloor \sigma - 1 \rfloor$ and whose length is bounded above: $|P| \leq C_1(\sigma)|A|$. Furthermore, if the b_i's constitue an increasing sequence, we have

$$i > \lfloor \log_2 \sigma \rfloor \Rightarrow b_i \leq C_2(\sigma).$$

where $C_1(\sigma)$ and $C_2(\sigma)$ are constants depending only on σ.

Remark I.17. — When the set A is a subset of an arithmetical progression of dimension 2, it can be represented in the affine plane by way of a F_2-isomorphism. Then, we will say that A is included in r lines when $b_2 = r$. This terminology will be also used for infinite sets.

Our strategy of proof is simple. First, we are going to transpose the *infinite* problem into a *finite* one. Then, we will use the last theorem to obtain structure of finite sets. Finally, we will come back to the set A by using asymptotic arguments. Assuming the hypothesis (1) or (2), we will see that -in the finite problem- σ will be less than 3 so the arithmetical progressions will have dimension 2 or less. Then, it will be natural to use results concerning addition of sets in $\mathbb{Z}^2$, particularly the following:

Theorem I.18. — *Let A a finite subset of $\mathbb{Z}^2$. Let $s \geq 2$ an integer and $C(s)$ a constant depending only on s. If $|A| > C(s)$ is sufficiently large and if*

$$|A+A| < \big(4 - \frac{2}{s+1}\big)|A| - (2s+1),$$

then there exist s parallel lines which cover the set A.

Proof. — Voir [**9**]. □

PART II
PROOF OF THE GENERAL CASE: $\sigma < \frac{11}{6}$

With a view to use the theorems of the previous section, let us transpose our problem into a problem of finite sets.

Let $\varepsilon > 0$, we can choose $y_1 \in \mathbb{N}$ sufficiently large and a strictly increasing sequence $(y_j)_{j\geq 1}$ of positive integers such that both the relations below are verified for all j:

$$(A+A)(2y_j) \leq (\gamma + \varepsilon) \times 2y_j,$$

$$(\alpha - \varepsilon)y_j \leq A(y_j) \leq (\alpha + \varepsilon)y_j.$$

We will use the notation

$$A_j := \{a \in A,\ a \leq y_j\}.$$

In the sequel, all the notations will be all related to the sequence $(y_j)_{j\geq 1}$. Every change of the sequence will naturally changes the sets A_j and all that is related.
Now, we are able to precise the structure of the sets A_j.

We have:

$$
\begin{aligned}
\frac{|A_j + A_j|}{|A_j|} &= \frac{|A_j + A_j|}{2y_j} \times 2 \times \frac{y_j}{|A_j|} \\
&\le \frac{(A+A)(2y_j)}{2y_j} \times 2 \times \frac{y_j}{|A_j|} \\
&\le 2 \times \frac{\gamma + \varepsilon}{\alpha - \varepsilon} \\
&\le \left(2\sigma + \varepsilon'\right) < 4,
\end{aligned} \tag{3}
$$

where $\varepsilon' = O(\varepsilon)$.

Then, for ε sufficiently small, we can apply the fundamental theorem of Freiman (I.16) to the sets A_j. By a simple calculus, we obtain $m \le 2$ and $b_2 \le C_2$. First, we are going to exclude the case where A_j is a subset of an arithmetical progression of dimension $m = 1$ for infinitely many values of j.

Suppose it is the case. Then, for j sufficiently large, $A_j \subseteq P_j$ where P_j is an arithmetic progression of difference 1 (for $\gcd(A) = 1$) and first term 0. We can assume it has minimum length. Then, we have, by (I.16) and since $\{0, y_j\} \subseteq P_j$:

$$|P_j| \ge y_j, \tag{4}$$

$$|P_j| \le C_1 |A_j|. \tag{5}$$

Combining (4) et (5) and using the usual notation $\sigma = 2 - \delta$, we can find a lower bound for α:

$$
\begin{aligned}
\alpha &\ge \frac{1}{\sigma}\gamma \\
&\ge \frac{1}{\sigma}\frac{|A_j + A_j| - \varepsilon}{2y_j} \\
&\ge \frac{1}{\sigma}\frac{2|A_j| - 1 - \varepsilon}{2y_j} \\
&\ge \frac{1}{\sigma}\frac{1}{2y_j}\left(\frac{2y_j}{C_1} - 1 - \varepsilon\right) \\
&\ge \alpha_0,
\end{aligned} \tag{6}
$$

for an absolut constant α_0 (remember that ε can be chosen sufficiently small).

Then, we can exclude this case under hypothesis $\alpha < \alpha_0$ of theorem (I.5).

Remark II.1. — The value of C_1 - one can find an estimate in [**2**] - implies a very small value of the bound α_0. Anyway, one can see that our method cannot be employed with $\alpha \geq \frac{1}{2} > \alpha_0$. The case where A is included in a single line ($\alpha > \alpha_0$) remains an open question.

So, for infinitely many integers j, A_j is a subset of an arithmetical progression of dimension $m = 2$. By extracting a subsequence, we can assume that it is the case for all the sets A_j. At this point, we have obtained that, for all $j \geq 1$,

$$A_j \subseteq P_j = P(0; d_j^1, d_j^2; b_j^1, b_j^2),$$

where

$|P_j| \leq C_1|A_j|$ and the sequence $(b_j^2)_{j\geq 1}$ is bounded.

Since $A \subseteq \mathbb{N}$, $d_j^1 N$ has to be positive for infinitely many values of j (what we extract). We can also assume, by shifting the lines if necessary (the first term of the progression is not 0 yet), that the differences d_j^2 are positive integers. In fact, there is a F_2-isomorphism which sends A_j into a rectangle in $\mathbb{Z}^2$.

Lemma II.2. — *The sequence $(d_j^1)_{j\geq 1}$ is bounded.*

Proof. — Assume the contrary. Then, there exists an index j such that $A(d_j^1) > \max_j(b_j^2)$ and, consequently, such that two elements of $A \cap [0, d_j^1[$ denoted by a and b lie on the same line. This is impossible under the hypothesis $b - a < d_j^1$. □

Once again, we can extract a subsequence from $(A_j)_{j\geq 1}$ (which we denote by the same notation) such that $d_j^1 = N > 0$ for all $j \geq 1$, we will choose N maximum.

1. The non-archimedian case

In this section, we assume:

$$N > 1.$$

We can show that, in this case, the sequence $(d_j^2)_{j\geq 1}$ is also bounded.

Lemma II.3. — *The sequence $(d_j^2)_{j\geq 1}$ is bounded.*

Proof. — We have seen that A is included in a certain number of residue classes mod N (at least two since $\gcd(A) = 1$). We can extract from $(y_j)_{j\geq 1}$ a subsequence such that the horizontal lines containing each of these classes are in the same order. We denote by (a_i, b_i) the coordinates of the image of the smallest term c_i of each class, for $1 \leq i \leq r$ ($b_i < b_{i+1}$). If there exists i such that, for infinitely many values of j, $a_i \leq a_{i+1}$, then $d_j^2 \leq c_{i+1} - c_i$ thus the lemma. Otherwise, if $a_i > a_{i+1}$ for all i, we can apply a valuable *twisting isomorphism* (see (I.13)) to A_j to be in the conditions of the latter case without changing the conclusions of the theorem (I.16). □

Hence, we can assume, by extracting again a subsequence, that $d_j^2 = d$ for all $j \geq 1$. d will be chosen maximum.

For all $j \geq 1$, there is a F_2-isomorphism ϕ_j which sends A_j onto a subset of a rectangle $R_j \subseteq \mathbb{Z}^2$ such that, for all point $(u,v) \in R_j$, one has

$$\phi^{-1}(u,v) - \phi^{-1}(u-1,v) = N,$$

$$\phi^{-1}(u,v) - \phi^{-1}(u,v-1) = d.$$

Furthermore, we have ever seen that the heigt of the rectangle R_j is bounded. For all j, we can assume $\phi_j(0) = (0,0)$. Hence, if $j < k$, one has $A_j \subseteq A_k$ and the isomorphisms ϕ_j and ϕ_k coincide on A_j. Then we can define a F_2-isomorphism ϕ such that $\phi_{|A_j} \equiv \phi_j$. This isomorphism sends A onto a subset of a half-strip R of $\mathbb{Z}^2$ which can be denote by

$$R = \{(u,v) \in \mathbb{Z}^2 \mid x \geq x_0,\ v = v^0, v^0+1, \ldots, v^0+r-1\}.$$

We can assume that the *border lines* $v = v^0$ and $v = v^0 + r - 1$ contain at least one element of A (otherwise, this line is not in R).
At this point, we can apply the two-dimensionnal theorem of Stanchescu (I.18) to all the $\phi(A_j)$ which verify the same additive properties than A_j, that is to say, in view of the definition of s and for j sufficiently large:

$$|\phi(A_j) + \phi(A_j)| < \big(4 - \frac{2}{s}\big)|\phi(A_j)| - (2s-1).$$

Then, there exist s lines covering $\phi(A_j)$. Since the sets $\phi(A_j)$ form an increasing sequence of sets, we can claim that there exist s parallel lines covering A. They have to be horizontal, otherwise, their intersection with the half-strip R would yield a finite set A.

We have just proved that the number t of horizontal lines intersecting A in the half-strip R satisfies $t \leq r \leq s \leq 5$. We denote by $v_1, \ldots, v_t$ the ordinates of those lines. For example, we have $v_1 = v^0$ and $v_t = v^0 + r - 1$.

Lemma II.4. — *For all $v^0 \leq j \leq v^0 + r - 1$, the line $\{(u,v) \mid v = j\}$ intersects A in at least one point.*

Proof. — Let $D_{i,j}$ the following set:

$$D_{i,j} := \{(u,v) \in \phi(A_j) \mid v = i\}.$$

We know that $\phi(A)$ is covered by 3, 4 or 5 horizontal lines. In the sequel, we are going to distinguish the three cases and, in each one, prove that, if the lemma is false, we have

$$|\bigcup_{i=v_1}^{v_t} D_{i,j} + \bigcup_{i=v_1}^{v_t} D_{i,j}| \geq \frac{11}{3}\Big(\sum_{i=v_1}^{v_t} |D_{i,j}|\Big) - C, \tag{7}$$

where C is an absolut constant. By an asymptotic argument and under the hypothesis (1), it is impossible.

1. We assume that A is covered by three lines with ordinates v_1, v_2 and v_3.

   We can assume $v_1 = 0$ and then, by minimality of d, we have $\gcd(v_1, v_2, v_3) = 1$ and thus, assuming the lemma false

   $$v_1 + v_3 \neq 2v_2.$$

   Then, we have, for all $j \geq 1$, that the sets $D_{a,j} + D_{b,j}$ corresponding to the following values of a and b are pairwise separated:

   $$(a, b) \in \{(v_1, v_1), (v_1, v_2), (v_2, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_3)\}.$$

   We obtain the contradiction (7) with $C = 6$ using the theorem (I.9).
2. We assume that A is covered by four lines with ordinates v_1, v_2, v_3 and v_4. Assuming the lemma false implies that the v_i's are not in atithmetical progression.
   We are going to use the same arguments and suppose there is a *hole* in the sequence $(v_i)_{1 \leq i \leq 4}$. So, we are able to exhibit separated sumsets as in the three lines case. We suppose -without loss of generality- that v_1, v_2 and v_3 are not an arithmetical progression. The separated sumsets are the sets $D_{a,j} + D_{b,j}$ corresponding to the following values of a and b:

$$(8) \qquad (a, b) \in \{(v_1, v_1), (v_1, v_2), (v_2, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_3), (v_3, v_4), (v_4, v_4)\}.$$

   If we can add to this list one of the couple (v_1, v_4) or (v_2, v_4), then we obtain the contradiction (7) by using (I.9). On the contrary, if we cannot add those couples, then, necessarily, $v_2 + v_4 = 2v_3$. Hence, (v_2, v_3, v_4) is an arithmetical progression. Since (v_1, v_2, v_3, v_4) is not one, we have $v_1 + v_4 = 2v_2$ and the ordinates of the lines are $(v_1, v_1 + 2a, v_1 + 3a, v_1 + 4a)$. We will say that A is of the *integer type* $(0, 2, 3, 4)$.

   It remains to work on this this set of ordinates or on the set $0, 2, 3, 4$ to exhibit appropriate separated sumsets. Indeed, to each list like (8), we can associate a relation using (I.9). For example, using *integer notations*, to the list

   $$E_1 := \{(0, 0), (0, 2), (0, 3), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\},$$

   we can associate the relation

   $$|\bigcup_{i=v_1}^{v_t} D_{i,j} + \bigcup_{i=v_1}^{v_t} D_{i,j}| \geq R_1 := 4|D_{v_1,j}| + 4|D_{v_2,j}| + 5|D_{v_3,j}| + 3|D_{v_4,j}| - 8.$$

   Considering the sets:

$$E_2 := \{(0,0),(0,2),(0,3),(0,4),(2,3),(3,3),(3,4),(4,4)\},$$

$$E_3 := \{(0,0),(0,2),(0,3),(2,2),(2,3),(2,4),(3,4),(4,4)\},$$

we obtain the following relations

$$|\bigcup_{i=v_1}^{v_t} D_{i,j} + \bigcup_{i=v_1}^{v_t} D_{i,j}| \geq R_2 := 5|D_{v_1,j}| + 3|D_{v_2,j}| + 3|D_{v_3,j}| + 5|D_{v_4,j}| - 8,$$

$$|\bigcup_{i=v_1}^{v_t} D_{i,j} + \bigcup_{i=v_1}^{v_t} D_{i,j}| \geq R_3 := 4|D_{v_1,j}| + 5|D_{v_2,j}| + 3|D_{v_3,j}| + 4|D_{v_4,j}| - 8.$$

It suffices to note that

$$\frac{1}{9}(4R_1 + 3R_2 + 2R_3) \geq \frac{35}{9}\Big(\sum_{i=v_1}^{v_t} |D_{i,j}|\Big) - 8,$$

to obtain (7).

3. We assume that A is covered by five lines with ordinates v_1, v_2, v_3, v_4 and v_5 with $v_1 = 0$ and $\gcd(v_i) = 1$.

As we have done in the four lines case, we have to exhibit first the different integer types of the sets A, then the separated subsets and the relations related.

Let $Y := (v_1, v_2, v_3)$ and $Z := (v_2, v_3, v_4, v_5)$. We have to distinguish three cases depending on Y and Z to be or not arithmetical progressions (indeed, they cannot be both arithmetical progression by hypothesis).

If neither Y nor Z are arithmetical progressions, then, using the four lines case for Z, we obtain the relation (7) with multiplicative coefficient $\frac{35}{9}$. The other cases yields a lot integer types of sets A but the reader will easily convince him that only one is interesting: $(0, 2, 3, 4, 5)$. The others permit to obtain the contradiction (7) with multiplicative coefficient 4. As done in the four lines case, it suffices to study this particular set. We only give the coefficient of $|D_{v_k,j}|$ in expressions R_i for $1 \leq i \leq 4$ and $1 \leq k \leq 5$. It will be an exercise for the reader to find the different separated sumsets related to each set of coefficients.

We have obtained $R_1 = (5, 3, 4, 4, 4)$, $R_2 = (4, 4, 5, 4, 3)$, $R_3 = (4, 5, 3, 5, 3)$ and $R_4 = (5, 4, 3, 3, 5)$. To obtain the contradiction, it suffices to find a good linear combination of the R_i's, say $2R_1 + 2R_2 + 1R_3 + 2R_4$ which gives (7) with multiplicative coefficient $\frac{27}{7}$.

□

From the last lemma, we have deduced $r = t$. It remains to measure the *density* of the set A in those r lines. We are going to use the following theorem:

Theorem II.5. — *Let r a positive integer. Let A a subset of $\mathbb{Z}^2$ included in*

$$K := \bigcup_{i=0}^{r-1} \{(u,v) \mid v = i,\ u \geq 0\},$$

such that each of those half-lines contains at least one point of A and verifying

$$|A + A| \leq (4 - \varepsilon)|A| - C$$

where ε and C are fixed constants. We assume A sufficiently large, $|A| > k_{\varepsilon,r,C}$.

We denote by q_i the greatest common divisor of the mutual differences $a - b$ with $(a;i),(b;i) \in \{(u,v) \mid v = i\}$ and $q := \gcd(q_0, \ldots, q_{r-1})$.
We denote by p the maximum abscissa of a point of A. We can assume -by shifting the lines if necessary- that at least one point with abscissa p has an ordinate $v_p < r - 1$. We denote by u_1 the minimum abscissa of a point of $\{(u,v) \mid v = v_p + 1\}$, u_2 the minimum abscissa of $\{(u,v) \mid v = r-1\}$ and u_0 the minimum abscissa of $\{(u,v) \mid v = 0\}$.
If $q = 1$, then

$$u_2 - u_0 + (r-1)(p - u_1) < |A + A| - 2|A| + C'_{C,\varepsilon,r},$$

where C' is a constant depending only on C, ε and r.

For the proof of this theorem, it is necessary to study a transformation introduced by Freiman in [**3**] p.27. In this article, we will use it only in a particular case, I mean in the plane.

Definition II.6. — Let A a subset of $\mathbb{Z}^2$. Let i an integer. Let D the horizontal line with equation $v = i$. Let s_u the number of elements of A with abscissa u. Then, we call *projection of K onto D* and we denote by A^0 the set

$$A^0 := \{(u,v) \mid u \in \mathbb{Z}, i \leq v \leq i - 1 + s_u\}.$$

If $s_u \leq 1$ for all u, it is the usual orthogonal projection.

With the notations of the last definition, Freiman has proved in [**3**] the relation

$$|A^0 + A^0| \leq |A + A|. \tag{9}$$

Proof of the theorem (II.5). — We are not going to apply directly this last inequality to the set A but to a set which is F_2-isomorph. First, we assume that $v_p = 0$, translating vertically A if necessary. Then, we are going to apply it a *twisting isomorphism* (see (I.13)).
We can choose l maximum such that $\Psi_l(A)$ has 2 points on the same vertical line and not more than 2 points on any another vertical line. Thus we have, for all $j > l$ and all $i \in \mathbb{Z}$, $|\Psi_j(A) \cap \{u = i\}| \leq 1$. We can remark that

$$l \geq p - u_1.$$

As usually, we denote by A^0 the projected of $\Psi_l(A)$ onto $\{v = 0\}$. A^0 is constitued with two parts:

- A_1^0 which contains between 1 and r points on the line $\{v = 1\}$, say $|A_1^0| = c$,
- A_0^0 included in $\{v = 0\}$, which contains at least the points $(u_0 - lb; 0)$ and $(u_2 + l(r-1-b); 0)$, where $-b$ is the smallest ordinate of a point of A. Furthermore, by definition of q, A_0^0 cannot be an arithmetical progression of difference greater than 1.

Hence, one has:

$$\begin{aligned}|A^0 + A^0| &= |A_0^0 + A_0^0| + |A_0^0 + A_1^0| + |A_1^0 + A_1^0| \\ &\geq |A_0^0 + A_0^0| + |A_0^0| + 2c - 1 \\ &\geq |A_0^0 + A_0^0| + |A| + c - 1.\end{aligned}$$

Now, using the relation (9), one has

$$|A^0 + A^0| \leq |A + A|,$$

therefore

$$|A_0^0 + A_0^0| \leq |A + A| - |A| - c + 1.$$

Under hypothesis on A, we obtain finally

$$|A_0^0 + A_0^0| \leq (3 - \varepsilon)|A_0^0| + (2 - \varepsilon)c + 1.$$

Since $|A|$ is sufficiently large, we can apply the classic *3k-4 theorem* (I.10) to A_0^0. So we can conclude:

$$u_2 - u_0 + l(r-1) \leq |A_0^0 + A_0^0| - 2|A_0^0| + 1 \leq |A + A| - 2|A| + C',$$

what we have to prove, remarking that $l \geq (p - u_1)$.

□

In order to use the theorem (II.5), we have to verify the hypothesis concerning q.

We denote by $Q_{i,j}$ the greatest common divisor of the differences $a - b$ where $a, b \in D_{i,j}$ and we define $Q_j := \gcd(Q_{v_0}, \ldots, Q_{v_0+r-1})$. The integer sequence $(Q_j)_{j \geq 1}$ decreases, so we can assume it is constant, equal to Q, choosing j sufficiently large. We have to prove that $Q = N$. Q is clearly a multiple of N.

By the lemma (II.4), it cannot exist an index i_0 such that

$$a_{i_0} + a_{i_0+2} \neq 2a_{i_0+1} \mod Q,$$

with any $a_i \in D_{i,j}$.

Hence, modulo Q, the elements of $D_{i,j}$ (their representants so) are in arithmetical progression. Now, N has been chosen maximum with this property, therefore $Q = N$.

At this point, we can use (II.5), remarking that the values u_0, u_1 and u_2 are fixed, so are $o(p_j)$. It suffices to assume j sufficiently large to obtain

$$(10) \qquad o(p_j) + (r-1)(p_j - o(p_j)) < |A_j + A_j| - 2|A_j| + C',$$

where p_j is the maximum abscissa of a point of $\phi(A_j)$ and C' is a constant not depending on j.

Since $y_j \in A_j$ and A_j is a subset of a half-strip of bounded height, we have

$$(11) \qquad lim_{j\to\infty} \frac{Np_j}{y_j} = 1.$$

Combining both the relations (10) and (11), we can deduce that, for all $\varepsilon > 0$, there exists j_0 such that, for all $j \geq j_0$:

$$-\varepsilon + (r-1)(1-\varepsilon)\frac{y_j}{N} \leq 2(\gamma+\varepsilon)y_j - 2(\alpha-\varepsilon)y_j \leq 2y_j(\sigma-1)\alpha,$$

and so, what we have to prove

$$\alpha \geq \frac{r-1}{2N(1-\delta)}.$$

2. The archimedian case

In this case, we will assume

$$N = 1.$$

We can show that, in this case, the sequence $(d_j^2)_{j\geq 1}$ cannot be bounded. Indeed, in this case, we could do as in the non-archimedian case and find a common isomorphism between every A_j and a part of a half-strip of $\mathbb{Z}^2$. It is impossible because, with $N = 1$, the smaller term of a line would have another image on another line, which contradicts the definition of a F_2-isomorphism.

The conclusions of (I.5) in the archimedian case come clearly from the two-dimensionnal structure of A. The holes come from the definition of F_2-isomorphism too.

Remark II.7. — The structure given here is less precise than in the other cases. We cannot use the projection tool because there is no control on d_j^2. Thus, there is no control on the parameters u_0, u_1 and u_2 of the theorem (II.5).
In the extremal case, we will see that more informations on the two-dimensionnal structure of each A_j (even in the archimedian case) will allow us to precise the structure of A.

PART III
PROOF OF THE EXTREMAL CASE: $\sigma < \frac{5}{3}$

This case is a particular situation of the general case and will use the same notations as in the last part. Then, we can use the beginnnig of the proof of (I.5). We know that there is a F_2-isomorphism between A and a subset of two horizontal lines of $\mathbb{Z}^2$ since $s = 2$.

Under the additional hypothesis $\sigma < \frac{5}{3}$, we have some new informations on the structure of the sets A_j. Indeed, we are going to use the following two-dimensional theorem whose proof can be found in [**3**] p.28:

Theorem III.1. — *Let $A \subset \mathbb{Z}^2$ of cardinal more than 11 not contained in a single line. We assume that*

$$|A+A| < \frac{10}{3}|A| - 5.$$

Then A is contained in a set F_2-isomorphic to

$$A_0 = \{(0,0),(0,1),\ldots,(0,l_1-1)\} \cup \{(1,0),(1,1),\ldots,(1,l_2-1)\},$$

with $l_1, l_2 \geq 1$ et $l_1 + l_2 = |A+A| - 2|A| + 3$.

For more precise result, we will also use the following additive result of Freiman, whose proof can be found in [**7**] p.21:

Theorem III.2. — *Let A a set of non negative integers of cardinal more than 3 such that $\min(A) = 0$. Let $a_k := \max(A)$. If*

$$a_k \leq 2|A| - 3,$$

then

$$|2A| \geq |A| + a_k.$$

This result has been generalized to distinct sets by V.F. Lev et P.Y. Smeliansky in [**6**] then improved by Y.V. Stanchescu in [**8**].We will use the following version:

Theorem III.3. — *Let A and B two sets of non negative integers containing 0. We denote by $l(A) := \max(A) - \min(A)$ the length of A and by $h(A) := l(A) - |A| + 1$ the number of holes in A. If*

$$\max(l(A), l(B)) \leq |A| + |B| - 3,$$

then

$$|A+B| \geq (|A| + |B| - 1) + \max(h(A), h(B)).$$

1. The non-archimedian case

In this case, we assume

$$N > 1.$$

As we have seen in the proof of the general case, there exist a positive integer t such that $\gcd(t, N) = 1$ and A is included in two residue classes $\mod N$ which we denote by B and C:

$$B = \{a \in A; a \equiv 0 \mod N\},$$

$$C = \{a \in A; a \equiv t \mod N\}.$$

We define $B_j := B(y_j)$ and $C_j := C(y_j)$ and we assume, chosing y_1 sufficiently large, that those sets are non empty. Furthermore, we define $b_j := \max(B_j)$ and $c_j := \max(C_j)$. We assume without loss of generality, extracting again a subsequence, that $b_j = y_j$.

Lemma III.4. — *There exists a sequence $(y_j)_{j\geq 1}$ such that, for all $\varepsilon > 0$, we have for j sufficiently large*

$$|A_j| \geq \frac{1}{\big(2\sigma - 2 + \varepsilon\big)N}(b_j + c_j).$$

Proof. — We denote by t_0 the smallest element of A non multiple of N. We define $S_j := b_j + c_j - t_0 + 2$.

Let $\varepsilon > 0$. We have, using (III.1),

$$\begin{aligned}\frac{S_j}{N} &\leq |A_j + A_j| - 2|A_j| + 3\\ &\leq \Big(2\sigma - 2 + \varepsilon'\Big)|A_j| + 3\\ &\leq \Big(2\sigma - 2 + \varepsilon''\Big)|A_j|,\end{aligned}$$

where $\varepsilon' = O(\varepsilon)$ and $\varepsilon'' = O(\varepsilon)$. It suffices to choose j sufficiently large to obtain the result.

□

We denote again by $(y_j)_{j\geq 1}$ the sequence of integers verifying the last lemma.

Now, we are going to precise the last results. We define

$$X_j := \frac{c_j}{b_j},$$

and

$$\lambda_j := \frac{N|A_j|}{b_j + c_j}.$$

Lemma III.5. — *There exists a sequence* $(y_j)_{j\geq 1}$ *such that* $\lim_{j\to\infty} X_j = 1$.

Proof. — We will only use the definition of the upper asymptotic density of A. Let $\varepsilon > 0$, from the sequence $(y_j)_{j\geq 1}$ we can extract a subsequence (which we also denote by y_j) such that,

$$\frac{A(c_j)}{c_j} \leq \frac{A(b_j)}{b_j} + \varepsilon.$$

Furthermore, we have

$$A(c_j) \geq A(b_j) - \frac{b_j - c_j}{N}.$$

Putting together the last two relations, we obtain:

$$N\varepsilon + \frac{\lambda_j(b_j + c_j)}{b_j} \geq \frac{\lambda_j(b_j + c_j)}{c_j} - \frac{b_j}{c_j} + 1.$$

It yields the following polynomial inequation:

$$\lambda_j X_j^2 - (1-\varepsilon)X_j - (\lambda_j - 1) \geq 0. \tag{12}$$

It remains to determine discriminant et roots to see that one of two following case happens:

- X_j is less than $\frac{1}{\lambda_j} - 1 + O(\varepsilon)$ which is impossible with the lower bound λ_j obtained (III.4). Indeed, it would implies

  $$X_j \leq \frac{1}{\lambda_j} - 1 + O(\varepsilon) \leq 2\sigma - 3 + O(\varepsilon) < \frac{1}{3}.$$

  This relation would implie

  $$\frac{(A+A)(b_j)}{b_j} \geq \frac{|B_j| + |B_j| + |C_j + C_j|}{b_j} \geq 2\alpha,$$

  which contradicts the main hypothesis of (I.6).
- $X_j \geq 1 - O(\varepsilon)$ which is the conclusion of the lemma.

□

Combining the results of the last two lemmas, we are able to prove the following one, which gives a lower bound for α and will allow us to apply (III.2) and (III.3) to $B'_j := \frac{1}{N}B_j$ and $C'_j := \frac{1}{N}(C_j - t)$.

Lemma III.6. — *There exists a sequence* $(y_j)_{j\geq 1}$ *such that, for all* $\varepsilon > 0$ *and for* j *sufficiently large, we have:*

$$|A_j| \geq \Big(\frac{1}{(\sigma-1)N} - \varepsilon\Big)y_j.$$

Proof. — The proof only consists in putting together the two previous lemmas and taking j sufficiently large. It follows:

$$b_j + c_j \geq (2-\varepsilon)y_j,$$

and

$$|A_j| \geq \frac{1}{(2\sigma - 2 - \varepsilon)N}(b_j + c_j).$$

□

We denote again by $(y_j)_{j\geq 1}$ the sequence of integers verifying the last lemma.

We notice that, for ε sufficiently small, since $\delta > \frac{1}{3}$,

$$\frac{1}{2\sigma - 2 - \varepsilon} > \frac{3}{4}.$$

Let us fix such a $\varepsilon > 0$. The importance of the last remark is that it necessarily implies, for j sufficiently large,

$$|B_j| = |B'_j| > \frac{1}{2}\max(B'_j) > (\frac{1}{2} + \varepsilon)\frac{y_j}{N}$$

et

$$|C_j| = |C'_j| > \frac{1}{2}\max(C'_j) > (\frac{1}{2} + \varepsilon)\frac{y_j}{N}.$$

Hence, we can use (III.2) and (III.3) to obtain, again for j sufficiently large,

$$\begin{aligned} |A_j + A_j| &= |B_j + B_j| + |B_j + C_j| + |C_j + C_j| \\ &= |B'_j + B'_j| + |B'_j + C'_j| + |C'_j + C'_j| \\ &\geq |B'_j| + \frac{y_j}{N} + (1-\varepsilon)\frac{y_j}{N} + |B'_j| + (1-\varepsilon)\frac{y_j}{N} + |C'_j| \\ &= 2|B_j| + |C_j| + (3 - 2\varepsilon)\frac{y_j}{N}, \end{aligned} \tag{13}$$

assuming, without loss of generality, that $|B_j| \geq |C_j|$.

Now, we have too

$$|A_j + A_j| \leq (2\sigma + \varepsilon)|A_j|,$$

therefore,

$$(2\sigma + \varepsilon)|B_j| + (2\sigma + \varepsilon)|C_j| \geq 2|B_j| + |C_j| + (3 - 2\varepsilon)\frac{y_j}{N},$$

and finally,

$$(2\sigma - 2 + \varepsilon)|B_j| + (2\sigma - 1 + \varepsilon)|C_j| \geq (3 - 2\varepsilon)\frac{y_j}{N}. \tag{14}$$

We obtain a first conclusion from (14), using $|B_j| \geq |C_j|$:

$$(4\sigma - 3 + 2\varepsilon)|B_j| \geq (3 - 2\varepsilon)\frac{y_j}{N},$$

and so

$$|B_j| \geq \Big(\frac{3}{4\sigma - 3} - \varepsilon'\Big)\frac{y_j}{N}, \tag{15}$$

where $\varepsilon' = O(\varepsilon)$.

We are going to use again (14) to bound below $|C_j|$ and then $|A_j|$:

$$|C_j| \geq \frac{1}{2\sigma - 1 + 3\varepsilon}\Big((3 - 2\varepsilon)\frac{y_j}{N} - (2\sigma - 2 + \varepsilon)|B_j|\Big).$$

On obtient, en utilisant (15):

$$\begin{aligned}
|A_j| &\geq |B_j| + |C_j| \\
&\geq \frac{1}{2\sigma - 1 + 3\varepsilon}\Big((3 - 2\varepsilon)\frac{y_j}{N} + \big(2 - 2\sigma - \varepsilon\big)|B_j|\Big) + |B_j| \\
&\geq \frac{3 - 2\varepsilon}{2\sigma - 1 + 3\varepsilon}\frac{y_j}{N} + \frac{(2 - 2\sigma - \varepsilon) + (2\sigma - 1 + 3\varepsilon)}{2\sigma - 1 + 3\varepsilon}|B_j| \\
&\geq \Big(\frac{3}{2\sigma - 1} + \frac{3}{(2\sigma - 1)(4\sigma - 3)} - \varepsilon'\Big)\frac{y_j}{N} \\
&= \Big(\frac{12\sigma - 9 + 3}{(2\sigma - 1)(4\sigma - 3)} - \varepsilon'\Big)\frac{y_j}{N} \\
&= \Big(\frac{6}{4\sigma - 3} - \varepsilon'\Big)\frac{y_j}{N},
\end{aligned} \tag{16}$$

where $\varepsilon' = O(\varepsilon)$.

It is the conclusion of the non-archimedian case of (I.6):

$$\overline{d}(A) \geq \frac{6}{(4\sigma - 3)N}.$$

2. The archimedian case

In this case, we assume as in the general archimedian case that

$$N = 1.$$

We have ever seen that the sequence $(d_j^2)_{j\geq 1}$ could not be bounded.

We are going to apply (III.1) to the sets A_j. Then, we can include A_j in a set $A_{0,j}$ which is the union of 2 arithmetical progressions Q_j^1 and Q_j^2 (of difference 1 here). We denote by $b_j := l_{1,j} := |Q_j^1|$ and $l_{2,j} := |Q_j^2|$ the respective lengths of those two progressions, where $0 \in Q_j^1$ and $y_j \in Q_j^2$. Indeed, those two elements cannot be in the same progression, A would be in an arithmetical progression of dimension 1. Those lengths being supposed minimum, we have $y_j - t_j = l_{2,j}$ and the

Lemma III.7. — *There exists a sequence $(y_j)_{j\geq 1}$ such that, for all $\varepsilon > 0$, there exists $j_0 \geq 1$ such that for all $j \geq j_0$, we have*

$$|A_j| \geq \Big(\frac{1}{2\sigma - 2} - \varepsilon\Big)(l_{1,j} + l_{2,j}).$$

Proof. — It suffices to apply (III.1) for j sufficiently large:

$$\begin{aligned} l_{1,j} + l_{2,j} &\leq |A_j + A_j| - 2|A_j| + 3 \\ &\leq (2\sigma - 2 + \varepsilon')|A_j| + 3 \\ &\leq (2\sigma - 2 + \varepsilon'')|A_j|, \end{aligned}$$

where ε', defined in (3), is sufficiently small and $\varepsilon'' = O(\varepsilon')$.

□

We denote again by $(y_j)_{j\geq 1}$ the sequence of integers verifying the last lemma.

If $b_j \geq t_j$, then $l_{1,j}+l_{2,j} \geq y_j$ and, by (III.7) and the range of values of σ, $|A_j| \geq \frac{3}{4}y_j$ what is incompatible with $\alpha < \frac{1}{2}$. Then, we have $b_j < t_j$, and thus

$$A(b_j, t_j) = 0. \tag{17}$$

Let us define $B_j := A \cap [0; b_j]$ and $C_j := A \cap [t_j; y_j]$.

The ratio $X_j := \frac{|B_j|}{b_j}$ cannot be too large, otherwise, we would obtain, considering the sets $A(b_j)$, a too large value for α. Clearly, we have:

$$X := \limsup_{j\to\infty} X_j \leq \alpha. \tag{18}$$

Let us show in which sense b_j is necessarily small compared with $l_{2,j}$.

Lemma III.8. — *If we define*

$$\lambda_j := \frac{b_j}{l_{2,j}},$$

we have

$$\lambda := \limsup_{j\to\infty} \lambda_j \leq \frac{2\sigma - 3}{2\sigma - 2}\Big(\frac{1}{2\sigma - 2} - X\Big)^{-1}. \tag{19}$$

Proof. — We use the lemma (III.7), noting that:

$$|A_j| = |B_j| + |C_j| = X_j\lambda_j l_{2,j} + |C_j|.$$

We obtain, for all $\varepsilon > 0$, for j sufficiently large:

$$X_j\lambda_j l_{2,j} + |C_j| \geq \Big(\frac{1}{2\sigma-2} - \varepsilon\Big)(\lambda_j+1)l_{2,j},$$

and then,

$$|C_j| \geq l_{2,j}\Big(\frac{1}{2\sigma-2} - \varepsilon + \lambda_j\big(\frac{1}{2\sigma-2} - \varepsilon - X_j\big)\Big). \tag{20}$$

Now, we know that $|C_j| \leq l_{2,j}$, therefore we obtain the upper bound

$$\lambda_j \leq \Big(\frac{2\sigma-3}{2\sigma-2} + \varepsilon\Big)\Big(\frac{1}{2\sigma-2} - \varepsilon - X_j\Big)^{-1}.$$

□

It remains to recall us that $X \leq \alpha$ to obtain

$$\lambda \leq \frac{2\sigma-3}{2\sigma-2}\Big(\frac{1}{2\sigma-2} - \alpha\Big)^{-1}.$$

Let us take as a new sequence $(y_j)_{j\geq 1}$ the subsequence such that $\lim_{j\to\infty}\lambda_j = \lambda$. It suffices to look again at the relation (20) to obtain

$$\liminf_{j\to\infty}\frac{|C_j|}{l_{2,j}} \geq \Big(\frac{1}{2\sigma-2} + \lambda\big(\frac{1}{2\sigma-2} - X\big)\Big). \tag{21}$$

A last extraction of subsequence allows us to convert the lower limit symbol to a simple limit symbol.

Hence, the conclusions (17), (18), (III.8) et (21) conclude the proof of the archimedian case of the theorem (I.6).

PART IV
CONCLUSION

1. The important step in all those arguments is the fundamental theorem of Freiman which permits to give the multi-dimensional structure of the sets A_j. In the extremal case, we have obtained more precise results by using structural two-dimensional theorems. To obtain this type of results with greater values of σ, it is necessary to develop theorems concerning more than two lines. This type of work can be found in [**9**] concerning three lines.
2. In [**3**], p.83, Freiman achieves the same type of theorem (I.5) for lower asymptotic density but under hypothesis $\sigma = 2 - \delta$ with $\delta > 0$. What is the difference -the obstacle- in our upper density case? Let us consider only the non-archimedian case (it is the single one in Freiman's theorem). In fact, the difficulty is to prove that the lines covering A are in arithmetical progression (see

lemma (II.4)). To do that, Freiman uses the fact that, if A and B are distinct sets of non negative integers,

$$\underline{d}(A \cup B) \geq \underline{d}(A) + \underline{d}(B).$$

In the upper density case, this inequality does not hold. that is why we have been restricted to a low number of lines and we have studied the situations case by case. Maybe an algorithmic computation would allow us to consider some greater values of σ and a few more lines but it would not solve the general case $\sigma < 2$.

Here, we have seen in the four lines case that for all finite set A contained in four lines not in arithmetical progression, we have $|A+A| \geq \frac{35}{9}|A|$. It is easy to find finite sets which realize the equality but it is more difficult to find a infinite set A contained in four lines not in arithmetical progression which verifies

–

$$\lim_{j\to\infty} \frac{A(y_j)}{y_j} = \overline{d}(A),$$

–

$$|A_j + A_j| < 4|A_j|.$$

It is the same problem for a greater number of lines. We could extend our general case to $\sigma < 2$ if the following conjecture were true:

Conjecture IV.1. — *Let A a set of non negative integers such that $0 \in A$ and $\gcd(A) = 1$. Assume that A is F_2-isomorphic to a subset of $\mathbb{Z}^2$ contained in s lines not in arithmetical progression. We denote by $(y_j)_{j\geq 1}$ an increasing sequence of integers such that*

$$\lim_{j\to\infty} \frac{A(y_j)}{y_j} = \overline{d}(A).$$

Then, $|A_j + A_j| > 4|A_j|$.

3. What about $\sigma = 2$? As in the lower density case, it is impossible to extend our results to the case of growing ration σ greater or equal than 2. Indeed, we find in [**1**] an elegant example of set which is not covered by the cases exposed in this article and such that $\overline{d}(A+A) = 2\overline{d}(A)$.

Example IV.2. — Let η an irrational number and $0 < \varepsilon < 1$. We define

$$A := \{n \in \mathbb{N} \mid n\eta \in \big(-\frac{\varepsilon}{2}; \frac{\varepsilon}{2}\big) \mod 1\}.$$

Then, $\overline{d}(A+A) = \underline{d}(A+A) = 2\overline{d}(A) = 2\underline{d}(A)$. The proof, based on the uniform repartition of the sequence $n\eta \mod 1$ is left to the reader.

References

[1] Y. BILU *Addition of sets of integers of positive density.*
Journal of Number Theory **64** p. 233-275 (1997)

[2] Y. BILU *Structure of sets with small sumset.*
Astérisque **258** p. 77-108 (1999)

[3] G.A. FREIMAN *Foundations of a structural theory of set addition.*
Translations of Mathematical Monographs **37** (1973)

[4] H. HALBERSTAM AND K.F. ROTH *Sequences.*
2nd edition, Springer-Verlag, New-York/Berlin (1983)

[5] R. JIN *Inverse problem for upper asymptotic density.*
Transactions of the American Mathematical Society **355 vol. 1** p. 57-78 (2002)

[6] V.F. LEV AND P.Y. SMELIANSKY *On addition of two distinct sets of integers.*
Acta Arithmetica **70** p. 85-91(1995)

[7] M.B. NATHANSON *Additive number theory. Inverse problems and the geometry of sumsets.*
Graduate Texts in Mathematics **165** (1996)

[8] Y. STANCHESCU *On addition of two distinct sets of integers.*
Acta Arithmetica **75** p. 191-194 (1996)

[9] Y. STANCHESCU *Multidimensional Inverse Additive Problems.*
Thesis submitted to the senate of Tel-Aviv University (1998)

2004/03/16

GUILLAUME BORDES • *E-mail :* bordes@math.u-bordeaux.fr