A uniform open image theorem for ℓ-adic representations of étale fundamental groups

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Let k be a field of characteristic 0, X a smooth, separated, geometrically connected scheme over k with generic point η. An ℓ-adic representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is said to be geometrically strictly rationnally perfect (GSRP for short) if $\mathrm{Lie}(\rho(\pi_1(X_{\overline{k}})))^{ab} = 0$. Typical examples of such representations are those arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X or, more generally, from the action of $\pi_1(X)$ on the ℓ-adic etale cohomology groups $H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell)$, $i \geq 0$ of the geometric generic fiber of a smooth proper scheme Y over X. Let G denote the image of ρ. Any closed point x on X induces a splitting $x : \Gamma_{\kappa(x)} := \pi_1(\mathrm{Spec}(\kappa(x))) \to \pi_1(X_{\kappa(x)})$ of the canonical restriction epimorphism $\pi_1(X_{\kappa(x)}) \to \Gamma_{\kappa(x)}$ (here, $\kappa(x)$ denotes the residue field at x) so one can define the closed subgroup $G_x := \rho \circ x(\Gamma_{\kappa(x)}) \subset G$ (up to inner automorphisms).

The main result we are going to discuss in this series of lectures is the following uniform open image theorem.

Theorem 1 *Assume that k is a finitely generated field of characteristic 0 and that X is a curve. Then,*

1. *for any representation $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ and any integer $d \geq 1$, the set $X_{\rho,d,\geq 3}$ of all closed points $x \in X$ such that G_x has codimension ≥ 3 in G and $[\kappa(x) : k] \leq d$ is finite.*
2. *Furthermore, if $\rho : \pi_1(X) \to \mathrm{GL}_m(\mathbb{Z}_\ell)$ is GSRP, then the set $X_{\rho,d,\geq 1}$ of all closed points $x \in X$ such that G_x has codimension ≥ 1 in G and $[\kappa(x) : k] \leq d$ is finite, and there exists an integer $B_{\rho,d} \geq 1$ such that $[G : G_x] \leq B_{\rho,d}$ for any closed point $x \in X \setminus X_{\rho,d,\geq 1}$ such that $[\kappa(x) : k] \leq d$.*

The lectures will be divided into four sections:

1. General strategy
   - (a) Short review of etale fundamental groups.
   - (b) Short review of compact ℓ-adic Lie groups.
   - (c) Notation and statements.
   - (d) The GSRP property.
   - (e) Reduction to a diophantine problem: non-existence of rational points of certain "moduli spaces".
   - (f) Main ingredients (projective system argument, Faltings' theorems).
2. Case $d = 1$
   - (a) Explicit Riemann-Hurwitz formula.
   - (b) Reduction of the problem to counting points on reduction modulo ℓ of ℓ-adic analytic subspaces of $\mathbb{Z}_\ell^N$ (Serre-Oesterlé's asymptotic bounds).

   (c) From geometry to arithmetic *via* Faltings' theorem (Mordell conjecture).

3. Case $d \geq 1$

   (a) Growth of gonality along projective systems of Galois covers.
   (b) Induced representation argument.
   (c) From geometry to arithmetic *via* Faltings' theorem (Lang conjecture for abelian varieties).

4. Further developments in the case of torsion on abelian schemes

Among ℓ-adic representations of $\pi_1(X)$, those arising from the action of $\pi_1(X)$ on the generic ℓ-adic Tate module $T_\ell(A_\eta)$ of an abelian scheme A over X are of particular interest. A corollary of theorem 1 is the following uniform boundedness statement for the ℓ-primary torsion of abelian varieties paramatrized by curves.

Corollary 2 *Assume that k is a finitely generated field of characteristic 0. For any integer $d \geq 1$, there exists an integer $N := N(d, A)$ such that $A_x[\ell^\infty](\kappa(x)) \subset A_x[\ell^N]$ for any closed point $x \in X$ such that $[\kappa(x) : k] \leq d$.*

This corollary is a consequence of the following geometric statement.

Lemma 3 *Assume that k is an algebraically closed field of characteristic 0 and that A_η contains no nontrivial isotrivial abelian subvariety. For each integer $N \geq 1$, write $g(N)$ for the minimal genus of the $\kappa(v)$, $v \in A_\eta[N]$ of order exactly N. Then $\lim_{n\to\infty} g(\ell^n) = +\infty$.*

In the concluding section, we would like to explain how this lemma can be partly extended:

(a) To the case when X is a surface, with Kodaira dimension replacing the genus.
(b) To the case of mod ℓ representations (ℓ varying). More precisely, in that case, we will sketch the proof of the fact that, *if X has genus ≥ 1 or if X has genus 0 and A_η has semistable reduction everywhere except possibly over one point, then $\lim_{\ell\to\infty} g(\ell) = +\infty$.*