A uniform open image theorem for ℓ-adic representations of étale fundamental groups (Talk 1: preliminaries)

Anna Cadoret - I.M.B. (Univ. Bordeaux 1)
Akio Tamagawa - R.I.M.S. (Kyoto Univ.)

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1. (Slides) Recall classical facts about:
   (a) Etale fundamental group;
   (b) ℓ-adic Lie groups
2. State our main result and some of its corollaries
3. Sketch the proof

Etale fundamental groups of schemes: Definition

X: connected scheme
$\mathcal{C}_X^{(1)}$: category of finite étale covers of X

$\overline{x} : \Omega \to X$: geometric point $\curvearrowright$ $F_{\overline{x}}$: $\begin{array}{ccc} \mathcal{C}_X^{(1)} & \to & FSETS \\ Y \xrightarrow{p} X & \mapsto & Y_{\overline{x}} \end{array}$

Thm.: $\mathcal{C}_X^{(1)}$ *is a Galois category with fiber functor* $F_{\overline{x}} : \mathcal{C}_X^{(1)} \to FSETS$.

$\pi_1(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$ is endowed with a profinite group structure:

$$\pi_1(X, \overline{x}) = \varprojlim \Pi_{\overline{x},p},$$

where $\Pi_{\overline{x},p} = \mathrm{Im}(\pi_1(X, \overline{x}) \to \mathcal{S}(Y_{\overline{x}}))$ and the projective limit is over all connected (Galois) $Y \xrightarrow{p} X$.

The profinite group $\pi_1(X, \overline{x})$ is the *étale fundamental group of* X *with base point* $\overline{x}$.

$$\begin{array}{ccc} \mathcal{C}_X^{(1)} & \xrightarrow{F_{\overline{x}}} & FSETS \\ {\scriptstyle eq.}\updownarrow & \nearrow_{For} & \\ \mathcal{C}(\pi_1(X, \overline{x})) & & \end{array}$$

Facts:

1. Two fiber functors are always isomorphic hence $\pi_1(X, \overline{x})$ does not depend on $\overline{x}$ up to a unique inner isomorphism.

2. Functoriality: the functor from Galois categories pointed with fiber functors and fundamental functors to profinite groups is an equivalence of categories.

   $f : X' \to X$
   $\curvearrowright f^* : \mathcal{C}_X^{(1)} \to \mathcal{C}_{X'}^{(1)}$
   $\curvearrowright \pi_1(f) : \pi_1(X', \overline{x}') \to \pi_1(X, f(\overline{x}'))$

Etale fundamental groups of schemes: Additional assumptions

1. $\underline{X\text{: locally noetherian, normal, connected scheme}}$ with generic point η

   $\mathcal{C}_X^{(2)}$: category of finite separable $k(\eta)$-algebras étale over X
   $k(\eta) \hookrightarrow \overline{k(\eta)}$: fixed algebraic closure (geometric generic point)

   $$\mathcal{C}_X^{(1)} \quad \underset{Normalization}{\overset{Rational\ function\ ring}{\rightleftarrows}} \quad \mathcal{C}_X^{(2)} \quad \underset{Galois\ theory}{\overset{'Usual'}{\longleftrightarrow}} \quad \mathcal{C}(\mathrm{Gal}(M_X|k(\eta)))\,,$$

   where $k(\eta) \hookrightarrow M_X$ is the maximal algebraic subextension of $k(\eta) \hookrightarrow \overline{k(\eta)}$ unramified over X.

   $\pi_1(X, \overline{\eta}) = \mathrm{Gal}(M_X|k(\eta))$.

2. k: field
   $\underline{X \to k \text{ of finite type and geometrically connected}}$

$$(*) \quad 1 \longrightarrow \pi_1(X_{\overline{k}}) \longrightarrow \pi_1(X) \longrightarrow \Gamma_k \longrightarrow 1$$

Etale fundamental groups of schemes: Facts (elementary)

1. $U <_{op} \pi_1(X) \longleftrightarrow X_U \to X_{k_U}$ connected;
   $U \lhd_{op} \pi_1(X) \longleftrightarrow X_U \to X_{k_U}$ Galois,

   where $k \hookrightarrow k_U = M_X^U \cap \overline{k}$ finite field extension.

2. $U <_{op} \pi_1(X)$

   (a) $\curvearrowright U^{geo} := U \cap \pi_1(X_{\overline{k}}) <_{op} \pi_1(X_{\overline{k}}) \longleftrightarrow X_U \times_{k_U} \overline{k} \to X_{\overline{k}}$.

   (b) For any $x \in X(k)$,

   $$(*) \quad 1 \longrightarrow \pi_1(X_{\overline{k}}) \longrightarrow \pi_1(X) \underset{s_x}{\rightleftarrows} \Gamma_k \longrightarrow 1$$

   and $s_x(\Gamma_k) < U$ if and only if

   $$\begin{array}{ccc} X_U & \overset{x_U}{\longleftarrow\cdots} & \\ \downarrow & & \\ X & \underset{x}{\longleftarrow} & k \end{array}$$

Notation: If $\dim(X) = 1$,

g_U: genus of (a smooth compactification of) X_U.
γ_U : $\overline{k}$-gonality

ℓ-adic Lie groups: Definition and facts (non elementary)

X: topological space.

Chart(X): set of triples (U, ϕ, n), where $U \subset_{op} X$;
$\phi : U \tilde{\to} U_0 \subset_{op} \mathbb{Z}_\ell^n$ homeomorphism.

Notation: $C = (U_C, \phi_C, \dim(C))$.

$C \sim C' \Leftrightarrow$ $\phi_{C'} \circ \phi_C^{-1} : \phi_C(U_C \cap U_{C'}) \tilde{\to} \phi_{C'}(U_C \cap U_{C'})$ are ℓ-adic analytic maps.
$\phi_C \circ \phi_{C'}^{-1} : \phi_{C'}(U_C \cap U_{C'}) \tilde{\to} \phi_C(U_C \cap U_{C'})$

Atlas(X): set of subsets $A \subset$Chart(X) such that $C,\ C' \in A \Rightarrow C \sim C'$;
$X = \bigcup_{C \in A} U_C$.

$A \sim A' \Leftrightarrow C \in A,\ C' \in A' \Rightarrow C \sim C'$.

$\mathcal{M}_\ell$: category of *ℓ-adic manifolds*:

Objects: $(X, \mathcal{A})$ with X a topological space and $A \in$Atlas$(X)/\sim$;
Morphisms: Continuous maps which are ℓ-adic analytic maps in the charts.

$\curvearrowright \mathcal{G}_\ell < \mathcal{M}_\ell$: subcategory of *$\ell$-adic Lie groups*.

Facts:

1. The forgetful functor $\mathcal{G}_\ell \to TOPGRP$ is fully faithful;

2. Let $G \in \mathcal{G}_\ell$ then:

   (a) For any $H <_{cl} G$, $H \in \mathcal{G}_\ell$ and $H \hookrightarrow G \in$Hom$_{\mathcal{G}_\ell}$;

   (b) For any $H \lhd_{cl} G$, $G/N \in \mathcal{G}_\ell$ and $G \twoheadrightarrow G/N \in$Hom$_{\mathcal{G}_\ell}$.

$\ell > 2$ (for simplicity);

G: pro-ℓ group;
$P_1(G) := G$, $P_{n+1}(G) := \overline{P_n(G)^\ell[P_n(G), G]}$, $n \geq 1$.

G is a *uniform pro-ℓ group* if:

1. G is topologically finitely generated;
2. $G/\overline{G^\ell}$ is commutative;
3. $[P_n(G) : P_{n+1}(G)] = [G : P_2(G)]$, $n \geq 1$.

Assume that G is a uniform pro-ℓ group and let $d := d(G)$ be the minimal number of generators of G:

$$G = \overline{< g_1, \dots, g_d >}$$

Facts: $\begin{array}{ccc} G & \tilde{\rightarrow} & P_{n+1}(G) \\ g & \mapsto & g^{\ell^n} \end{array}$ is an homeomorphism.

$\curvearrowright$ Define: $\begin{array}{l} g +_G g' = \lim_{n\to+\infty} (g^{\ell^n} {g'}^{\ell^n})^{\ell^{-n}} \\ (g, g')_G = \lim_{n\to+\infty} [g^{\ell^n} {g'}^{\ell^n}]^{\ell^{-2n}} \end{array}$

1.
$$\alpha\colon \begin{array}{ccc} (\mathbb{Z}_\ell^d, +) & \tilde{\rightarrow} & (G, +_G) \\ (\lambda_1, \dots, \lambda_d) & \mapsto & \lambda_1 g_1 +_G \dots +_G \lambda_d g_d \end{array}$$
is a continuous group isomorphism (in particular, this endows G with the structure of a ℓ-adic Lie group).

2. $(G, +_G, (,)_G)$ is a $\mathbb{Z}_\ell$-Lie algebra.

Thm: *Let G be a topological group. Then $G \in \mathcal{G}_\ell$ if and only if G contains an open subgroup which is a a uniform pro-ℓ group $U <_{op} G$.*

In that case, the chart $\alpha^{-1} : U \rightarrow \mathbb{Z}_\ell^d$ is compatible with the structure of ℓ-adic manifold on U induced from the one on G and $d(U)$ does not depend on U: we call it the dimension of G.

ℓ-adic Lie groups: Lie correspondance

$G,\ G' \in \mathcal{G}_\ell$;

$\mathcal{H}(G,G')$:set of pairs (U,f), where $U <_{op} G$ and $f : U \to G' \in \mathrm{Hom}_{\mathcal{G}_\ell}$.

$(U,f) \sim (V,g) \Leftrightarrow$ there exists $W <_{op} U \cap V$ such that $f|_W = g|_W$.

$\curvearrowright \mathcal{G}_\ell \to \tilde{\mathcal{G}}_\ell$, where $\tilde{\mathcal{G}}_\ell$ is the category with:

Objects: $Ob(\mathcal{G}_\ell)$;
Morphisms: $\mathrm{Hom}_{\tilde{\mathcal{G}}_\ell}(G,G') = \mathcal{H}(G,G')/\sim$.

$\curvearrowright \mathcal{L}_\ell$: category of finite dimensional $\mathbb{Q}_\ell$ Lie algebras.

Define a functor:

$$Lie : \mathcal{G}_\ell \to \mathcal{L}_\ell$$

as follows:

- $G \in \mathcal{G}_\ell$, $U <_{op} G$ uniform pro-ℓ $\curvearrowright$ $Lie^0(U) := (U, +_U, (,)_U)$;
  $\curvearrowright$ $Lie(G) := Lie^0(U) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$;
- $f : G \to G' \in\mathrm{Hom}_{\mathcal{G}_\ell}$, $U' <_{op} G'$ uniform pro-ℓ, $U <_{op} f^{-1}(U')$ uniform pro-ℓ
  $\curvearrowright$ $Lie^0(f) := f|_U : (U, +_U, (,)_U) \to (U', +_{U'}, (,)_{U'})$;
  $\curvearrowright$ $Lie(f) := Lie^0(f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell : Lie(G) \to Lie(G')$.

Thm:

$$\begin{array}{ccc} \mathcal{G}_\ell & \xrightarrow{Lie} & \mathcal{L}_\ell \\ \downarrow & \nearrow_{eq.} & \\ \tilde{\mathcal{G}}_\ell & & \end{array}$$

Facts:

1. $1 \to K \to G \to Q \to 1$ short exact sequence in $\mathcal{G}_\ell$
   $\Rightarrow 1 \to Lie(K) \to Lie(G) \to Lie(Q) \to 1$ short exact sequence in $\mathcal{L}_\ell$;
2. There exists $U <_{op} G$ such that $Lie(U^{ab}) = Lie(U)^{ab} (= Lie(G)^{ab})$.

$U := Id + \ell M_d(\mathbb{Z}_\ell) <_{op} G := GL_d(\mathbb{Z}_\ell)$ is a uniform pro-ℓ subgroup and the logarithm:

$$\begin{array}{rlcl} \ell^{-1}\text{ln}: & (U, +_U, (,)_U) & \tilde{\rightarrow} & (M_d(\mathbb{Z}_\ell), +, [,]) \\ & Id + \ell u & \mapsto & \sum_{n\geq 1} \frac{(-1)^{n+1}\ell^{n-1}}{n} u^n \end{array}$$

induces an isomorphism of $\mathbb{Z}_\ell$-Lie algebras.

$\Rightarrow$ Lie$(G) = (M_d(\mathbb{Q}_\ell), +, [,])$.

Facts:

1. For any $g_0 \in G$, with:

$$\begin{array}{rlcl} \phi_{g_0}: & G & \tilde{\rightarrow} & G \\ & g & \mapsto & g_0 g g_0^{-1} \end{array}$$

   one has:

$$\begin{array}{rlcl} Ad(g_0) := Lie(\phi_{g_0}): & Lie(G) & \tilde{\rightarrow} & Lie(G) \\ & u & \mapsto & [g_0, u] = g_0 u - u g_0. \end{array}$$

2. Any compact subgroup of $GL_d(\mathbb{Q}_\ell)$ is conjugate to a subgroup of $GL_d(\mathbb{Z}_\ell)$.