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BOUNDARY LAYER FOR A PENALIZATION METHOD FOR VISCOUS INCOMPRESSIBLE FLOW

Gilles Carbou and Pierre Fabrie
Mathématiques Appliquées de Bordeaux, UMR 5466 et Université Bordeaux 1
351 cours de la Libération, 33405 Talence cedex, France

(Submitted by: Reza Aftabizadeh)

Abstract. To compute the flow around an obstacle, it is now quite classical to add in the equations a penalization term on this obstacle. From a computational point of vue, this method gives very accurate results and avoid to use unstructured mesh to discretize the equations in complex geometry. The aim of this paper is to give a mathematical explanation of such good results. This study is performed for the incompressible Navier Stokes equations.

1. Introduction and setting of the problem

About twenty years ago, there were several attempts to penalize the no-slip boundary condition on the boundary of an obstacle surrounded by a viscous fluid. The aim was to avoid body-fitted unstructured mesh in order to use accurate and fast spectral [17] or finite volumes approximation on cartesian meshes [16]. A way to do that is to add a penalized velocity term in the incompressible Navier-Stokes equations. Following the former works of Peskin [19], [20], several authors (for instance Goldstein and all [14]) add both a time integral of the velocity and a velocity penalization term only on the boundary. Some others studies show that the penalization has to be extended to the volume of the obstacle to give correct solution for large Reynolds numbers [21]. In independent works, Arquis and Caltagirone [4] add a penalization term on the velocity defined on the volume of the obstacle. In a further work [11], it is suggested that this model is able to give the drag and lift coefficients by integration of the penalization term over the obstacle. These ideas were also used with success by Angot [1] [2] to deal with fluid-porous-solid systems. Various works use the same methodology to compute incompressible flows around a cylinder or behind a step [2], [5], [6], [7], [16].

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In a previous work, P. Angot, C-H bruneau and P. Fabrie [3] establish error estimates for an L^2 penalization. These estimates are not in agreement with experimental results [3]. In fact when one penalizes the Navier-Stokes equation in the following way :

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial t} - \frac{1}{Re}\Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{1}{\varepsilon^2}\chi_\omega u^\varepsilon + \nabla p^\varepsilon = f, \\ &\text{div } u^\varepsilon = 0, \end{aligned}$$

where ω in the obstacle, the authors found a bound of the error in the fluid region in $\varepsilon^{\frac{1}{2}}$, whereas in the numerical experiments the bound seems to be of order ε. The aim of this paper is to bring to the fore the right estimate which is prooved in Theorem 1.3. We show that it appears a boundary layer in the obstacle, and not in the fluid region. It is the key point of this paper, and we think that it is the right explanation of so good results in numerical experiments.

To obtain these results we perform an asymptotic expansion of the solution in the spirit of BKW method (see for example [15]). This idea was successfuly used by the same authors for a model of ferromagnetism to show long time existence of regular solution [10]. The weakness of this method, and in general the BKW method is that one has to obtain very regular solutions on the limit problem.

1.1. **Main results.** Let Ω be a regular bounded domain of $\mathbb{R}^d$, $d = 2$ ou 3 and ω be a regular open subset of Ω such that $\overline{\omega} \subset \Omega$. We will use the following notations:

- $\mathcal{U} = \Omega \setminus \overline{\omega}$
- $\mathbf{V} = \Big\{ v \in H^1_0(\mathcal{U};\mathbb{R}^3), \quad \text{div } v = 0 \Big\}$
- $\mathbf{H} = \Big\{ v \in L^2(\mathcal{U};\mathbb{R}^3), \quad \text{div } v = 0 \text{ and } v \cdot n = 0 \text{ on } \partial\mathcal{U} \Big\}$
- P is the orthogonal projection for the L^2 scalar product onto $\mathbf{H}$.
- A is the operator with domain $\mathbf{H} \cap H^2(\mathcal{U})$ defined by $A = -P \circ \Delta$, that is if $f \in \mathbf{H}$,

$$AV = f \iff \exists \pi \in H^1(\mathcal{U})/\mathbb{R}, \ -\Delta V + \nabla \pi = f$$

Our first result concerns the existence of regular solutions for the Navier Stokes equations around the obstacle ω

Proposition 1.1. *Let $v_0 \in H^5(\mathcal{U}) \cap \mathbf{V}$. Let $f \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathcal{U})$ with space support inclused in $\mathcal{U}$. We assume that the initial data v_0 satisfies the following* *compatibility condition:*

$$\left\{\begin{array}{l} v_0' := -Av_0 - \Pi\big((v_0\cdot\nabla)v_0 - f_{|t=0}\big) \in \mathbf{V} \\ \\ v_0'' := -Av_0' - \Pi\left((v_0'\cdot\nabla)v_0 + (v_0\cdot\nabla)v_0' - \dfrac{\partial f}{\partial t}_{|t=0}\right) \in \mathbf{V} \end{array}\right. \tag{1.1.1}$$

There exists a time $T^*>0$ and there exists V^0 defined on $[0,T^*[\times\mathcal{U}$ such that

$$\begin{cases} \dfrac{\partial V^0}{\partial t} - \Delta V^0 + (V^0\cdot\nabla)V^0 + \nabla p^0 = f & \textit{in } [0,T^*[\times\mathcal{U} \\ \operatorname{div} V^0 = 0 & \textit{in } [0,T^*[\times\mathcal{U} \\ V^0 = 0 & \textit{on } [0,T^*[\times\partial\mathcal{U} \\ V^0(t=0) = v_0 & \textit{in } \mathcal{U}. \end{cases}$$

For all $T<T^*$, this solution V^0 is in $L^\infty(0,T;H^5(\mathcal{U}))\cap L^2(0,T;H^6(\mathcal{U}))$, with $T^\star=+\infty$ if $d=2$.

In order to approximate V^0 we consider the solution u^ε of a penalized problem in wich we penalize the obstacle ω:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla\pi^\varepsilon + \dfrac{1}{\varepsilon^2}\chi_\omega u^\varepsilon = f & \text{in } [0,T^*[\times\Omega, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } [0,T^*[\times\Omega, \\ u^\varepsilon = 0 & \text{on } [0,T^*[\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x) & \text{in } \Omega. \end{cases} \tag{1.1.2}$$

We will precise later the initial data u_0^ε.

We will perform an asymptotic expansion of u^ε which describes the boundary layer in the penalized set ω.

We introduce $\varphi:\omega\longrightarrow\mathbb{R}^+$ defined by $\varphi(x)=\operatorname{dist}(x,\partial\omega)$. We remark that since ω is regular, φ is smooth in a neigbourhood $\omega_1\subset\omega$ of $\partial\omega$.

The boundary layer and the asymptotic expansion of u^ε is described in the following theorem :

Theorem 1.1. *Let v_0, V^0 and T^* as in Proposition* 1.1. *There exists two functions V^1 and V^2 defined on $[0,T^*[\times\mathcal{U}$, there exists two profiles W^1 and W^2 defined on $[0,T^*)\times\omega\times\mathbb{R}^+$ such that if u_0^ε is an initial data of the form*

$$u_0^\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon V^1(0,x) + \varepsilon^{\frac32} r^\varepsilon(x) & \textit{if } x\in\mathcal{U} \\ \varepsilon W^1(0,x,\dfrac{\varphi(x)}{\varepsilon}) + \varepsilon^{\frac32} r^\varepsilon(x) & \textit{if } x\in\omega, \end{cases}$$

where $\|r^\varepsilon\|_{L^2(\Omega)} \leq K$ *and such that* $\operatorname{div} u_0^\varepsilon = 0$ *on* Ω, *then there exists* u^ε *a solution of the penalized problem* (1.1.2) *which satisfies*

$$u^\varepsilon(t,x) = \begin{cases} V^0(t,x) + \varepsilon V^1(t,x) + \varepsilon^2 V^2(t,x) + \varepsilon^{\frac{3}{2}} v_\varepsilon^r(t,x) & \textit{for } x \in \mathcal{U} \\ \varepsilon W^1(t,x,\dfrac{\varphi(x)}{\varepsilon}) + \varepsilon^2 W^2(t,x,\dfrac{\varphi(x)}{\varepsilon}) + \varepsilon^{\frac{3}{2}} w_\varepsilon^r(t,x) & \textit{for } x \in \omega, \end{cases}$$

where v_ε^r *and* w_ε^r *are bounded in* $L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \forall T < T^\star$.

Remark 1.1. As we will see in the last part, it is necessary to perform the asymptotic expansion at order 2 to obtain a remainder term at order $\frac{3}{2}$.

We can obtain a similar theorem when the initial data is less well prepared:

Theorem 1.2. *Let* v_0, V^0 *and* T^* *as in Proposition* 1.1. *There exists two functions* V^1 *and* V^2 *defined on* $[0,T^*[\times\mathcal{U}$, *there exists two profiles* W^1 *and* W^2 *defined on* $[0,T^*) \times \omega \times \mathbb{R}^+$ *such that if* u_0^ε *is an initial data of the form*

$$u_0^\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon r^\varepsilon(x) & \textit{if } x \in \mathcal{U} \\ \varepsilon r^\varepsilon(x) & \textit{if } x \in \omega, \end{cases}$$

where $\|r^\varepsilon\|_{L^2(\Omega)} \leq K$ *and such that* $\operatorname{div} u_0^\varepsilon = 0$ *on* Ω, *then there exists* u^ε *a solution of the penalized problem* (1.1.2) *which satisfies*

$$u^\varepsilon(t,x) = \begin{cases} V^0(t,x) + \varepsilon V^1(t,x) + \varepsilon^2 V^2(t,x) + \varepsilon \nu_\varepsilon^r(t,x) & \textit{for } x \in \mathcal{U} \\ \varepsilon W^1(t,x,\dfrac{\varphi(x)}{\varepsilon}) + \varepsilon^2 W^2(t,x,\dfrac{\varphi(x)}{\varepsilon}) + \varepsilon \varpi_\varepsilon^r(t,x) & \textit{for } x \in \omega, \end{cases}$$

where ν_ε^r *and* ϖ_ε^r *are bounded in* $L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \forall T < T^\star$.

Remark 1.2. In Theorem 1.1 the initial data is well prepared and we obtain a remainder term of order $\varepsilon^{\frac{3}{2}}$ in the asymptotic expansion whereas this remainder term is of order ε in Theorem 1.2 with less well prepared data. The second Theorem is sufficient to obtain the following result.

Theorem 1.3. *Let* v_0, V^0 *and* T^* *as in Proposition* 1.1. *We consider a perturbation of the initial data on the form*

$$u_0^\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon r^\varepsilon(x) & \textit{if } x \in \mathcal{U} \\ \varepsilon r^\varepsilon(x) & \textit{if } x \in \omega, \end{cases}$$

where $\operatorname{div} r^\varepsilon = 0$ *and* $\|r^\varepsilon\|_{L^2(\Omega)} \leq K$. *Then there exists* u^ε *a solution of the penalized problem* (1.1.2) *satisfying for all* $T < T^*$, *there exists a constant* C *independant of* ε *such that*

$$\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

The solution u^ε is a good approximation of V^0 since for all $T < T^*$ there exists C such that

$$\|u^\varepsilon - V^0\|_{L^\infty(0,T;L^2(\mathcal{U}))} + \|u^\varepsilon - V^0\|_{L^2(0,T;H^1(\mathcal{U}))} \leq C\varepsilon.$$

Remark 1.3. In the three Theorems, for all $\varepsilon > 0$ the maximal time of existence for u^ε is T^* given by Proposition 1.1.

In the following part we will build the asymptotic expansion of u^ε using BKW method. All the calculations of this part are formal. The third section is devoted to prove the existence and the regularity of the different terms of the asymptotic expansion. In the last part we estimate the remainder term and we prove Theorems 1.1 and 1.2. Theorem 1.3 is a direct consequence of Theorem 1.2

2. Ansatz

We seek u^ε satisfying

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla \pi^\varepsilon + \dfrac{1}{\varepsilon^2}\chi_\omega u^\varepsilon = f & \text{in } \mathbb{R}_t^+ \times \Omega, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times \Omega, \\ u^\varepsilon = 0 & \text{on } \mathbb{R}_t^+ \times \partial\Omega, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x) & \text{in } \Omega. \end{cases} \tag{2.0.3}$$

We denote by v^ε (resp. p^ε) the restriction of u^ε (resp. π^ε) in $\mathcal{U} = \Omega \setminus \overline{\omega}$ and w^ε (resp. q^ε) the restriction of u^ε (resp. π^ε) on ω.

Equation (2.0.3) is equivalent to the following system on v^ε and w^ε :

$$\begin{cases} (1) \quad \dfrac{\partial w^\varepsilon}{\partial t} - \Delta w^\varepsilon + (w^\varepsilon \cdot \nabla) w^\varepsilon + \nabla q^\varepsilon + \dfrac{1}{\varepsilon^2} w^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times \omega \\ (2) \quad \operatorname{div} w^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times \omega \\ (3) \quad \dfrac{\partial v^\varepsilon}{\partial t} - \Delta v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla p^\varepsilon = f & \text{in } \mathbb{R}_t^+ \times \mathcal{U} \\ (4) \quad \operatorname{div} v^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times \mathcal{U} \\ (5) \quad v^\varepsilon = w^\varepsilon & \text{on } \mathbb{R}_t^+ \times \partial\omega \\ (6) \quad -\dfrac{\partial v^\varepsilon}{\partial n} + p^\varepsilon n = -\dfrac{\partial w^\varepsilon}{\partial n} + q^\varepsilon n & \text{on } \mathbb{R}_t^+ \times \partial\omega \\ (7) \quad v^\varepsilon = 0 & \text{on } \mathbb{R}_t^+ \times \partial\Omega \end{cases} \tag{2.0.4}$$

where n is the outward unitary normal at $\partial\mathcal{U}$.

Remark 2.1. The boundary condition (6) in the previous equation comes from the variational formulation of Equation (2.0.3) since :

$$\int_{\mathcal{U}}(-\Delta u^\varepsilon + \nabla\pi^\varepsilon)\cdot\psi = \int_{\mathcal{U}}\big((\nabla u^\varepsilon\cdot\nabla\psi) - \pi^\varepsilon \mathrm{div}\,\psi\big) + \int_{\partial\omega}\big(-\frac{\partial u^\varepsilon}{\partial n} + \pi^\varepsilon n\big)\cdot\psi$$

and as the same equation occurs in ω.

We perform an asymptotic expansion of v^ε, w^ε, p^ε and q^ε of the form :

$$\begin{aligned}
&v^\varepsilon(t,x) = V^0(t,x) + \varepsilon V^1(t,x) + \dots,\\
&p^\varepsilon(t,x) = \frac{1}{\varepsilon^2}p^{-2}(t,x) + \frac{1}{\varepsilon}p^{-1}(t,x) + p^0(t,x) + \dots,\\
&w^\varepsilon(t,x) = W^0(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon W^1(t,x,\frac{\varphi(x)}{\varepsilon}) + \dots,\\
&q^\varepsilon(t,x) = \frac{1}{\varepsilon^2}q^{-2}(t,x,\frac{\varphi(x)}{\varepsilon}) + \frac{1}{\varepsilon}q^{-1}(t,x,\frac{\varphi(x)}{\varepsilon}) + q^0(t,x,\frac{\varphi(x)}{\varepsilon}) + \dots
\end{aligned}$$

We assume that the terms $W^i(t,x,z)$ and $q^i(t,x,z)$ satisfy

$$W^i(t,x,z) = \overline{W}^i(t,x) + \widetilde{W}^i(t,x,z)$$

where $\widetilde{W}^i(t,x,z)$ and all its derivative tend to zero when z tends to $+\infty$.

In (2.0.4) we will formaly replace u^ε, v^ε, p^ε and w^ε by their asymptotic expansion and we will identify the different powers of ε.

2.1. **Formal asymptotic expansion.** We will use the following notations:

- $x = (x_1, x_2, x_3)$ are the coordinates in $\mathbb{R}^3$,
- $W_z = \dfrac{\partial W}{\partial z}$ and $W_{zz} = \dfrac{\partial^2 W}{\partial z^2}$,
- $\nabla W = \begin{pmatrix} \dfrac{\partial W}{\partial x_1} \\ \dfrac{\partial W}{\partial x_2} \\ \dfrac{\partial W}{\partial x_3} \end{pmatrix}$ and $\nabla W_z = \begin{pmatrix} \dfrac{\partial^2 W}{\partial x_1\partial z} \\ \dfrac{\partial^2 W}{\partial x_2\partial z} \\ \dfrac{\partial^2 W}{\partial x_3\partial z} \end{pmatrix}$,
- $\Delta W = \dfrac{\partial^2 W}{\partial x_1^2} + \dfrac{\partial^2 W}{\partial x_2^2} + \dfrac{\partial^2 W}{\partial x_3^2}$.

We remark that if $w(t,x) = W(t,x,\frac{\varphi(x)}{\varepsilon})$,

$$\nabla w(t,x) = \nabla W(t,x,\frac{\varphi(x)}{\varepsilon}) + \frac{1}{\varepsilon}\nabla\varphi(x) W_z(t,x,\frac{\varphi(x)}{\varepsilon}),$$

and

$$\begin{aligned}\Delta w(t,x) = \ & \frac{1}{\varepsilon^2}|\nabla\varphi|^2 W_{zz}(t,x,\frac{\varphi(x)}{\varepsilon})\\ & +\frac{1}{\varepsilon}\left(2(\nabla\varphi(x)\cdot\nabla W_z(t,x,\frac{\varphi(x)}{\varepsilon}))+\Delta\varphi(x)W_z(t,x,\frac{\varphi(x)}{\varepsilon})\right)\\ & +\Delta W(t,x,\frac{\varphi(x)}{\varepsilon}).\end{aligned}$$

We recall that $|\nabla\varphi| = 1$ in ω_1, that $\nabla\varphi = n$ and that $\dfrac{\partial\varphi}{\partial n} = 1$ on $\partial\omega$.

2.1.1. *Asymptotic expansion of Equation* (1) *in* (2.0.4).

Order ε^{-3}**:** $q_z^{-2}\nabla\varphi = 0$, and since $\widetilde{q^{-2}}$ tends to zero when z tends to $+\infty$ we deduce that

$$\widetilde{q^{-2}} = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega\times\mathbb{R}_z^+. \tag{2.1.1}$$

Order ε^{-2} **:** $-W_{zz}^0 + W^0 + q_z^{-1}\nabla\varphi + \nabla q^{-2} = 0$, hence taking the limit of this equation when z tends to $+\infty$,

$$\overline{W}^0 + \nabla\overline{q^{-2}} = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega \tag{2.1.2}$$

and by difference, we deduce

$$-\widetilde{W}_{zz}^0 + \widetilde{W}^0 + q_z^{-1}\nabla\varphi = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega\times\mathbb{R}_z^+. \tag{2.1.3}$$

Order ε^{-1} **:**

$$-W_{zz}^1 + W^1 - 2(\nabla\varphi\cdot\nabla W_z^0) - \Delta\varphi W_z^0 + (W^0\cdot\nabla\varphi)W_z^0 + \nabla q^{-1} + q_z^0\nabla\varphi = 0,$$

hence we get

$$\overline{W^1} + \nabla\overline{q^{-1}} = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega \tag{2.1.4}$$

and

$$\begin{aligned} & -\widetilde{W}_{zz}^1 + \widetilde{W}^1 - 2(\nabla\varphi\cdot\nabla\widetilde{W}_z^0) - \Delta\varphi\widetilde{W}_z^0 \\ & \qquad +(W^0\cdot\nabla\varphi)\widetilde{W}_z^0 + \nabla\widetilde{q^{-1}} + \widetilde{q}_z^0\nabla\varphi = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega\times\mathbb{R}_z^+.\end{aligned} \tag{2.1.5}$$

Order ε^0 **:**

$$\frac{\partial W^0}{\partial t} - W_{zz}^2 - 2(\nabla\varphi\cdot\nabla W_z^1) - \Delta\varphi W_z^1 - \Delta W^0$$

$$+(W^0\cdot\nabla)W^0 + (W^1\cdot\nabla\varphi)W_z^0 + (W^0\cdot\nabla\varphi)W_z^1 + \nabla q^0 + q_z^1\nabla\varphi + W^2 = 0.$$

Hence we obtain that

$$\frac{\partial\overline{W}^0}{\partial t} - \Delta\overline{W}^0 + (\overline{W}^0\cdot\nabla)\overline{W}^0 + \nabla\overline{q^0} + \overline{W}^2 = 0 \qquad \text{in } \mathbb{R}_t^+\times\omega \tag{2.1.6}$$

and

$$\begin{aligned}&\frac{\partial \widetilde{W}^0}{\partial t} - \widetilde{W}^2_{zz} - 2(\nabla\varphi\cdot\nabla)\widetilde{W}^1_z - \Delta\varphi\widetilde{W}^1_z - \Delta\widetilde{W}^0\\ &+(\widetilde{W}^0\cdot\nabla)\overline{W}^0 + (W^0\cdot\nabla)\widetilde{W}^0 + (W^1\cdot\nabla\varphi)\widetilde{W}^0_z + (W^0\cdot\nabla\varphi)\widetilde{W}^1_z\\ &+\nabla\widetilde{q}^0 + \widetilde{q}^1_z\nabla\varphi + \widetilde{W}^2 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega\times\mathbb{R}^+_z.\end{aligned} \tag{2.1.7}$$

2.1.2. *Asymptotic expansion of equation* (2) *in* (2.0.4).

Order ε^{-1} **:** $\nabla\varphi\cdot\widetilde{W}^0_z = 0$ hence

$$\widetilde{W}^0\cdot n = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega\times\mathbb{R}^+_z. \tag{2.1.8}$$

Order ε^0 **:** $n\cdot W^1_z + \operatorname{div} W^0 = 0$ so

$$\operatorname{div}\overline{W}^0 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega, \tag{2.1.9}$$

$$n\cdot\widetilde{W}^1_z + \operatorname{div}\widetilde{W}^0 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega\times\mathbb{R}^+_z. \tag{2.1.10}$$

Order ε **:** $n\cdot W^2_z + \operatorname{div} W^1 = 0$, therefore

$$\operatorname{div}\overline{W}^1 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega, \tag{2.1.11}$$

$$n\cdot\widetilde{W}^2_z + \operatorname{div}\widetilde{W}^1 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega\times\mathbb{R}^+_z. \tag{2.1.12}$$

Order ε^2 **:** $n\cdot W^3_z + \operatorname{div} W^2 = 0$ and

$$\operatorname{div}\overline{W}^2 = 0 \qquad \text{in } \mathbb{R}^+_t\times\omega. \tag{2.1.13}$$

2.1.3. *Asymptotic expansion of equation* (3) *in* (2.0.4).

Order ε^{-2} **:** $\nabla p^{-2} = 0$, hence

$$p^{-2} = 0 \qquad \text{in } \mathbb{R}^+\times\mathcal{U}. \tag{2.1.14}$$

Order ε^{-1} **:** $\nabla p^{-1} = 0$ so

$$p^{-1} = 0 \qquad \text{in } \mathbb{R}^+\times\mathcal{U}. \tag{2.1.15}$$

Order ε^0 **:**

$$\frac{\partial V^0}{\partial t} - \Delta V^0 + V^0\cdot\nabla V^0 + \nabla p^0 = f \qquad \text{in } \mathbb{R}^+\times\mathcal{U}. \tag{2.1.16}$$

Order ε**:**

$$\frac{\partial V^1}{\partial t} - \Delta V^1 + V^0\cdot\nabla V^1 + V^1\cdot\nabla V^0 + \nabla p^1 = 0 \qquad \text{in } \mathbb{R}^+\times\mathcal{U}. \tag{2.1.17}$$

Order ε^2 **:**

$$\frac{\partial V^2}{\partial t} - \Delta V^2 + V^0\cdot\nabla V^2 + V^1\cdot\nabla V^1 + V^2\cdot\nabla V^0 + \nabla p^2 = 0 \quad \text{in } \mathbb{R}^+\times\mathcal{U}. \tag{2.1.18}$$

2.1.4. *Asymptotic expansion of equation* (4) *in* (2.0.4). At all orders we obtain that

$$\operatorname{div} V^0 = \operatorname{div} V^1 = \operatorname{div} V^2 = 0 \qquad \text{in } \mathbb{R}^+ \times \mathcal{U}. \tag{2.1.19}$$

2.1.5. *Asymptotic expansion of equation* (5) *in* (2.0.4). At the different orders we obtain

$$V^0(t,x) = W^0(t,x,0), \quad V^1(t,x) = W^1(t,x,0), \quad V^2(t,x) = W^2(t,x,0) \tag{2.1.20}$$

on $\mathbb{R}^+ \times \partial\omega$.

2.1.6. *Asymptotic expansion of equation* (6) *in* (2.0.4).

Order ε^{-2} **:**

$$q^{-2} n = p^{-2} n \text{ on } \mathbb{R}^+ \times \partial\omega. \tag{2.1.21}$$

Order ε^{-1} **:**

$$-W_z^0 + q^{-1} n = p^{-1} n \text{ on } \mathbb{R}^+ \times \partial\omega. \tag{2.1.22}$$

Order ε^{0} **:**

$$-\frac{\partial W^0}{\partial n} - W_z^1 + q^0 n = p^0 n - \frac{\partial V^0}{\partial n} \text{ on } \mathbb{R}^+ \times \partial\omega. \tag{2.1.23}$$

Order ε **:**

$$-\frac{\partial W^1}{\partial n} - W_z^2 + q^1 n = p^1 n - \frac{\partial V^1}{\partial n} \text{ on } \mathbb{R}^+ \times \partial\omega. \tag{2.1.24}$$

2.2. **Equation satisfied by the different terms.** We are going now to determine completely the different terms of the asymptotic expansion.

2.2.1. *Determination of* q^{-2}. With (2.1.1) we already know that $\widetilde{q^{-2}} = 0$. With (2.1.21) we obtain using (2.1.14) that on $\partial\omega$, $\overline{q^{-2}} = q^{-2} = p^{-2} = 0$.

Now taking the divergence of equation (2.1.2), as $\operatorname{div} \overline{W}^0 = 0$ (see (2.1.9)), we obtain that $\Delta\overline{q^{-2}} = 0$. Therefore, since $\overline{q^{-2}} = 0$ is zero on the boundary, $\overline{q^{-2}} = 0$ and so $q^{-2} = 0$ on $\mathbb{R}^+ \times \omega \times \mathbb{R}^+$.

2.2.2. *Determination of* q^{-1}. With (2.1.15) we know that $p^{-1} = 0$ on $\partial\omega$. Furthermore, taking the scalar product of (2.1.22) with n, since $\widetilde{W}^0 \cdot n = 0$ (see (2.1.8)), we obtain that $q^{-1} = 0$ on $\partial\omega$.

Taking the divergence of (2.1.4) and using (2.1.11) we have

$$\Delta\overline{q^{-1}} = 0 \text{ in } \mathbb{R}^+ \times \omega. \tag{2.2.1}$$

We have proved that $\overline{q^{-2}} = 0$ hence with (2.1.2), $\overline{W}^0 = 0$ in $\mathbb{R}^+ \times \omega$. Taking the scalar product of (2.1.3) with n, since with (2.1.8) $\widetilde{W}^0 \cdot n = 0$, we have

$\widetilde{q_z^{-1}} = 0$, hence $\widetilde{q^{-1}} = 0$ on $\mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+$. We obtain then that $\overline{q^{-1}} = 0$ on $\partial\omega$ and with (2.2.1) we have $\overline{q^{-1}} = 0$ on $\mathbb{R}^+ \times \omega$. With (2.1.4) this implies that

$$\overline{W}^1 = 0 \quad \text{in} \quad \mathbb{R}^+ \times \omega. \tag{2.2.2}$$

2.2.3. *Determination of* W^0. We already know that $\overline{W}^0 = 0$. With (2.1.3), since $\widetilde{q^{-1}} = 0$, we have

$$-\widetilde{W}^0_{zz} + \widetilde{W}^0 = 0 \quad \text{in} \quad \mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+.$$

Now from (2.1.22), since q^{-1} and p^{-1} are zero we obtain $\widetilde{W}^0_z = 0$ in $\mathbb{R}_t^+ \times \partial\omega \times \{z = 0\}$.

We extend the boundary condition to ω and we suppose that $\widetilde{W}^0_z = 0$ in $\mathbb{R}^+ \times \omega \times z = 0$. This implies that $\widetilde{W}^0 = 0$, hence since $\overline{W}^0 = 0$,

$$W^0 = 0 \quad \text{in} \quad \mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+. \tag{2.2.3}$$

2.2.4. *Determination of* V^0 *and* p^0. Using (2.2.3) and (2.1.20) we deduce that $V^0 = 0$ on $\mathbb{R}^+ \times \partial\omega$. Hence with equations (2.1.16) and (2.1.19), V^0 and p^0 are uniquely determined by

$$\begin{cases} \dfrac{\partial V^0}{\partial t} - \Delta V^0 + V^0 \cdot \nabla V^0 + \nabla p^0 = f & \text{in } \mathbb{R}^+ \times \mathcal{U} \\ \text{div } V^0 = 0 & \text{in } \mathbb{R}^+ \times \mathcal{U} \\ V^0 = 0 & \text{on } \mathbb{R}^+ \times \partial\mathcal{U} \\ V^0(t = 0) = v_0 & \text{in } \mathcal{U}. \end{cases}$$

2.2.5. *Determination of* $\widetilde{W}^1$. With (2.1.5), since $W^0 = 0$ and $q^{-1} = 0$, we have

$$-\widetilde{W}^1_{zz} + \widetilde{W}^1 + \widetilde{q}^0_z n = 0. \tag{2.2.4}$$

On the other hand, with (2.1.10) we have $\widetilde{W}^1_z \cdot n = 0$ and then

$$\widetilde{W}^1 \cdot n = 0. \tag{2.2.5}$$

Therefore, taking the scalar product of (2.2.4) with n we obtain that $\widetilde{q}^0_z = 0$ that is

$$\widetilde{q}^0 = 0 \quad \text{in} \quad \mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+. \tag{2.2.6}$$

Equation (2.2.4) reduce to

$$-\widetilde{W}^1_{zz} + \widetilde{W}^1 = 0 \quad \text{in} \quad \mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+,$$

hence since $\widetilde{W}^1$ tends to zero when z tends to $+\infty$, we can write

$$\widetilde{W}^1(t,x,z) = w^1(t,x)e^{-z} \quad \text{in} \quad \mathbb{R}_t^+ \times \omega \times \mathbb{R}_z^+,$$

where w^1 is the value of $\widetilde{W}^1$ at $z=0$.

Le us determine w^1. With (2.1.23) as $W^0 = 0$ and as $\widetilde{q^0} = 0$, we have at the boundary :

$$-\widetilde{W}_z^1 + \overline{q^0}n = -\frac{\partial V^0}{\partial n} + p^0 n.$$

Taking the scalar product of this equation with n, since $\widetilde{W}^1$ is tangent to the boundary of ω, we obtain

$$\overline{q^0} = -(\frac{\partial V^0}{\partial n} \cdot n) + p^0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\omega. \tag{2.2.7}$$

Thus, at the boundary (when $z=0$ and $x \in \partial\omega$),

$$\widetilde{W}_z^1 = -w^1 = -\frac{\partial V^0}{\partial n} + (\frac{\partial V^0}{\partial n} \cdot n)n.$$

We extend w_1 inside ω setting, for x in a neigbourhood of $\partial\omega$, $w^1(t,x) = w^1(t,P(x))$, where $P(x)$ is the orthogonal projection of x on $\partial\omega$. Therefore,

$$\begin{cases} \widetilde{W}^1(t,x,z) = w^1(t,x)e^{-z} & \text{in } \mathbb{R}^+ \times \omega \times \mathbb{R}^+ \\ w^1(t,x) = \dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n} \cdot n)n & \text{on } \mathbb{R}^+ \times \partial\omega. \end{cases} \tag{2.2.8}$$

Remark 2.2. Equation (2.2.8) means that the boundary layer is tangent to $\partial\omega$.

Remark 2.3. Since w^1 is tangent to the boundary $\partial\omega$ and since $\dfrac{\partial w^1}{\partial n} = 0$ in a neibourhood of $\partial\omega$, we have

$$\int_{\partial\omega} \operatorname{div} w^1 = 0 \tag{2.2.9}$$

using Stokes formula on $\partial\omega$ which is a compact manifold without boundary.

2.2.6. *Determination of* q^0. We know that $\widetilde{q^0} = 0$. With (2.1.6), since $W^0 = 0$ we obtain

$$\nabla\overline{q^0} + \overline{W}^2 = 0. \tag{2.2.10}$$

Taking the divergence of this equation and using (2.1.19) and (2.2.7) we determine completly $\overline{q^0}$ by

$$\begin{cases} \Delta\overline{q^0} = 0 \text{ in } \mathbb{R}^+ \times \omega \\ \overline{q^0} = -(\dfrac{\partial V^0}{\partial n} \cdot n) + p^0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\omega. \end{cases} \tag{2.2.11}$$

Furthermore, knowing $\overline{q^0}$ we determine $\overline{W}^2$ setting $\overline{W}^2 = -\nabla\overline{q^0}$.

2.2.7. *Determination of* V^1. With (2.1.20) and (2.2.8) since $\overline{W}^1 = 0$, V^1 and p^1 satisfy

$$\begin{cases} \dfrac{\partial V^1}{\partial t} - \Delta V^1 + (V^0 \cdot \nabla)V^1 + (V^1 \cdot \nabla)V^0 + \nabla p^1 = 0 & \text{in } \mathbb{R}^+ \times \mathcal{U} \\ \text{div } V^1 = 0 & \text{in } \mathbb{R}^+ \times \mathcal{U} \\ V^1 = \dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n} \cdot n)n & \text{in } \mathbb{R}^+ \times \partial\omega \\ V^1 = 0 & \text{on } \ \mathbb{R}^+ \times \partial\Omega. \end{cases}$$

We remark that $V^1 \cdot n = 0$ on $\mathbb{R}^+ \times \partial\omega$.

2.2.8. *Determination of* $\widetilde{W}^2$ *and* $\widetilde{q}^1$. We recall that we have

$$-\widetilde{W}^2_{zz} - 2(n \cdot \nabla)\widetilde{W}^1_z + \widetilde{q}^1_z n + \widetilde{W}^2 = 0. \tag{2.2.12}$$

With (2.1.12), we know that $(\widetilde{W}^2_z \cdot n) = -\text{div } \widetilde{W}^1$ and by integration we determine $(\widetilde{W}^2 \cdot n)$ by

$$(\widetilde{W}^2(t,x,z) \cdot n) = \int_z^{+\infty} \text{div } \widetilde{W}^1(t,x,\zeta)d\zeta. \tag{2.2.13}$$

Taking the scalar product of (2.2.12) with n, we determine $\widetilde{q}^1_z$ by

$$\widetilde{q}^1_z = (\widetilde{W}^2_{zz} \cdot n) - (\widetilde{W}^2 \cdot n) + 2\left((n \cdot \nabla)\widetilde{W}^1_z\right) \cdot n,$$

hence $\widetilde{q}^1$ is given by

$$\widetilde{q}^1 = -\int_z^{+\infty} \left((\widetilde{W}^2_{zz} \cdot n) - (\widetilde{W}^2 \cdot n) + 2\left((n \cdot \nabla)\widetilde{W}^1_z\right) \cdot n\right). \tag{2.2.14}$$

It remains to determine the tangent part of $\widetilde{W}^2$. Taking (2.1.24)$\wedge n$, and (2.2.12)$\wedge n$ we obtain that $\widetilde{W}^2 \wedge n$ satisfies the following equation:

$$\begin{cases} \widetilde{W}^2_{zz} \wedge n + 2(n \cdot \nabla)\widetilde{W}^1_z \wedge n - \widetilde{W}^2 \wedge n = 0 \text{ in } \mathbb{R}^+ \times \omega \times \mathbb{R}^+ \\ \widetilde{W}^2_z \wedge n = -\dfrac{\partial W^1}{\partial n} \wedge n + \dfrac{\partial V^1}{\partial n} \wedge n \quad \text{on} \quad \mathbb{R}^+ \times \partial\omega \times \{z = 0\}. \end{cases} \tag{2.2.15}$$

2.2.9. *Determination of V^2 and p^2.* With equations (2.1.18), (2.1.19) and (2.1.20), since W^2 is known, V^2 and p^2 are completly determined by the system

$$\begin{cases} \dfrac{\partial V^2}{\partial t} - \Delta V^2 + (V^0 \cdot \nabla) V^2 + (V^1 \cdot \nabla) V^1 + (V^2 \cdot \nabla) V^0 \\ \qquad\qquad\qquad\qquad + \nabla p^2 = 0 \qquad \text{in } \mathbb{R}^+ \times \mathcal{U} \\ \text{div } V^2 = 0 \qquad \text{in } \mathbb{R}^+ \times \mathcal{U} \\ V^2 = W^2 \text{ on } \mathbb{R}^+ \times \partial\omega \\ V^2 = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega. \end{cases}$$

Remark 2.4. The boundary condition $V^2 = W^2$ on $\partial\omega$ is compatible with the divergence free condition div $V^2 = 0$ in $\mathcal{U}$ since

$$\int_{\partial\omega} W^2 \cdot n = 0.$$

As a matter of fact,

$$\int_{\partial\omega} W^2 \cdot n = \int_{\partial\omega} \widetilde{W}^2 \cdot n + \int_{\partial\omega} \overline{W}^2 \cdot n.$$

Now, from (2.2.13) and (2.2.9),

$$\int_{\partial\omega} W^2 \cdot n = \int_{\partial\omega} \int_{z=0}^{+\infty} \text{div } (w^1(t,x) e^{-z}) = \int_{\partial\omega} \text{div } (w^1(t,x)) = 0$$

Furthermore,

$$\int_{\partial\omega} \overline{W}^2 \cdot n = \int_{\partial\omega} \frac{\partial \nabla \overline{q^0}}{\partial n} = -\int_{\omega} \Delta \overline{q^0} = 0.$$

3. Existence and regularity of the terms of the expansion

3.1. **Stokes operator.** First we recall some basic properties of Stokes operator.

We consider

$$\mathbf{H} = \left\{ V \in L^2(\mathcal{U}; \mathbb{R}^3) \text{ such that div } V = 0 \text{ in } \mathcal{U} \text{ and } \; V \cdot n = 0 \;\text{ on }\; \partial\mathcal{U} \right\}$$

and

$$\mathbf{V} = \left\{ V \in H_0^1(\mathcal{U}; \mathbb{R}^3) \text{ such that } \text{div } V = 0 \text{ in } \mathcal{U} \right\}.$$

Let P be the orthogonal projection for the L^2 scalar product onto $\mathbf{H}$.

We denote by A the operator with domain $\mathbf{H} \cap H^2(\mathcal{U})$ defined by $A = -P \circ \Delta$, that is if $f \in \mathbf{H}$,

$$AV = f \iff \exists \pi \in H^1(\mathcal{U})/\mathbb{R}, \;\; -\Delta V + \nabla \pi = f.$$

We recall the results due to Cattabriga (see [12])

Proposition 3.1. *There exists C such that for all $V \in D(A)$,*

$$\|V\|_{H^2(\mathcal{U})} + \|\pi\|_{H^1(\mathcal{U})/\mathbb{R}} \leq C\|AV\|_{L^2(\mathcal{U})},$$

$$\|V\|_{H^1(\mathcal{U})} + \|\pi\|_{L^2(\mathcal{U})/\mathbb{R}} \leq C\|AV\|_{H^{-1}(\mathcal{U})}.$$

3.2. **Existence and regularity of V^0.**

Proposition 3.2. *Let $v_0 \in H^5(\mathcal{U}) \cap D(A)$ and let $f \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathcal{U}))$. We suppose that v_0 and f satisfy the following compatibility conditions:*

$$\begin{cases} v_0' := -Av_0 - \Pi\left((v_0 \cdot \nabla)v_0 - f_{|t=0}\right) \in \mathbf{V} \\ \\ v_0'' := -Av_0' - \Pi\left((v_0' \cdot \nabla)v_0 + (v_0 \cdot \nabla)v_0' - \dfrac{\partial f}{\partial t}_{|t=0}\right) \in \mathbf{V} \end{cases} \tag{3.2.1}$$

There exists a time $T^* > 0$, there exists a unique V^0 and a pressure p^0 solution of

$$\begin{cases} \dfrac{\partial V^0}{\partial t} - \Delta V^0 + (V^0 \cdot \nabla)V^0 + \nabla p^0 = f \;\; \textit{in } [0, T^*[\times \mathcal{U} \\ \\ \operatorname{div} V^0 = 0 \;\; \textit{in } [0, T^*[\times \mathcal{U} \\ \\ V^0 = 0 \;\; \textit{on } [0, T^*[\times \partial\mathcal{U} \\ \\ V^0(t = 0) = v_0 \;\; \textit{in } \mathcal{U}. \end{cases} \tag{3.2.2}$$

and satisfying for all $T < T^*$

$$\begin{cases} V^0 \in \mathcal{C}^0(0,T;H^5(\mathcal{U})) \cap L^2(0,T;H^6(\mathcal{U})), \\ \\ \dfrac{\partial V^0}{\partial t} \in \mathcal{C}^0(0,T;H^3(\mathcal{U})) \cap L^2(0,T;H^4(\mathcal{U})), \\ \\ p^0 \in \mathcal{C}^0(0,T;H^4(\mathcal{U})) \cap L^2(0,T;H^5(\mathcal{U})), \\ \\ \dfrac{\partial p^0}{\partial t} \in \mathcal{C}^0(0,T;H^3(\mathcal{U})) \cap L^2(0,T;H^4(\mathcal{U})), \\ \\ \dfrac{\partial^2 V^0}{\partial t^2} \in \mathcal{C}^0(0,T;H^1(\mathcal{U})) \cap L^2(0,T;H^2(\mathcal{U})). \end{cases}$$

Remark 3.1. *It is well known that $T^* = +\infty$ in the 2-dimensional case, and that $T^* < +\infty$ in 3-d.*

Proof : we consider a Galerkin approximation of equation (3.2.2). This approximation is based on the eigenspaces of the Stokes Operator A.

Let us introduce the eigenfunctions $(e_1, \ldots, e_p, \ldots)$ of the Stokes operator:

$$\begin{cases} e_i \in \mathbf{V} \cap H^2 \\ \\ A(e_i) = \lambda_i e_i \end{cases}$$

such that the family (e_i) is an hilbertian basis of H and an orthogonal basis of V and $D(A)$.

We set $W_i = \operatorname{span}\{e_1, \ldots, e_i\}$ and we denote by Π_i the orthogonal projection from H onto W_i. Using Cauchy-Lipschitz theorem, we know that there exists a unique $V_i^0 \in \mathcal{C}^\infty(0,T_i;W_i)$ such that

$$\frac{\partial V_i^0}{\partial t} + AV_i^0 + \Pi_i(V_i^0 \cdot \nabla V_i^0) = \Pi_i(f) \tag{3.2.3}$$

We will perform estimates on the Galerkin approximation V_i^0 and on its derivative in time. These estimates will be independant on i and will allow us to take the limit in appropriate spaces. The existence of strong solution in $L^\infty(0,T^*;\mathbf{V}) \cap L^2(0,T^*;D(A))$ is classical. We focus our attention on the regularity properties at higher order.

In order to simplify the notations, we denote by $v = V_i^0$ in the following estimates. In addition, we denote π the pressure associated to the Stokes operator (that is $-\Delta v + \nabla\pi = Av$).

L^2 **estimate :** we multiply (3.2.3) by v and we integrate on $\mathcal{U}$. Since $v \in H_0^1(\mathcal{U})$ and div $v = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq \|f\|_{L^2}\|v\|_{L^2}$$

Using Gronwall Lemma, we obtain that for all T there exists C independant on i such that

$$\|v\|_{L^\infty(0,T;L^2(\mathcal{U}))} + \|v\|_{L^2(0,T;H^1(\mathcal{U}))} \leq C.$$

H^1 **estimates :** we multiply (3.2.3) by Av and we integrate on $\mathcal{U}$. We obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \|Av\|_{L^2}^2 \leq & \int_{\mathcal{U}} (v\cdot\nabla)v \cdot Av + \int_{\mathcal{U}} fAv \\
\leq & \|f\|_{L^2}^2 + \tfrac{1}{4}\|Av\|_{L^2}^2 + \|v\|_{L^6}\|\nabla v\|_{L^3}\|Av\|_{L^2} \\
\leq & \|f\|_{L^2}^2 + \tfrac{1}{4}\|Av\|_{L^2}^2 + \|Av\|_{L^2}^{\frac{3}{2}}\|\nabla v\|_{L^2}^{\frac{3}{2}} \\
\leq & \tfrac{1}{2}\|Av\|_{L^2}^2 + C\|\nabla v\|_{L^2}^6 + K.
\end{aligned}$$

Hence using comparison lemma we obtain that there exists T^* (independant on i) such that for all $T < T^*$, there exists a constant C (independant on i) with

$$\|v\|_{L^\infty(0,T;H^1)} + \|Av\|_{L^2(0,T;L^2)} \leq C,$$

that is using Proposition 3.1

$$\|v\|_{L^\infty(0,T;H^1)} + \|v\|_{L^2(0,T;H^2)} + \|\pi\|_{L^\infty(0,T;L^2)} + \|\pi\|_{L^2(0,T;H^1)} \leq C. \quad (3.2.4)$$

Remark 3.2. *In the 2-dimensional case we make the following estimate*

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \|Av\|_{L^2}^2 \leq & \int_{\mathcal{U}} (v\cdot\nabla)v \cdot Av + \int_{\mathcal{U}} fAv \\
\leq & \|f\|_{L^2}^2 + \tfrac{1}{4}\|Av\|_{L^2}^2 + \|v\|_{L^4}\|\nabla v\|_{L^4}\|Av\|_{L^2} \\
\leq & \|f\|_{L^2}^2 + \tfrac{1}{4}\|Av\|_{L^2}^2 + \|Av\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2} \\
\leq & \tfrac{1}{2}\|Av\|_{L^2}^2 + C\|\nabla v\|_{L^2}^4 + K
\end{aligned}$$

Using that $\nabla v \in L^2(0,T;L^2)$ we obtain that the solution is global in time through Gronwall Lemma.

Estimates on $\dfrac{\partial v}{\partial t}$: we denote $\dot v = \dfrac{\partial v}{\partial t}$. The function $\dot v$ satisfies the following equation

$$\frac{\partial \dot v}{\partial t} + A\dot v + \Pi_i\Big((\dot v\cdot\nabla)v + (v\cdot\nabla)\dot v\Big) = \Pi_i(\frac{\partial f}{\partial t}) \tag{3.2.5}$$

Multiplying (3.2.5) by $\dot v$ we obtain that

$$\begin{aligned}\frac12\frac{d}{dt}\|\dot v\|_{L^2}^2 + \|\nabla\dot v\|_{L^2}^2 \le\ & \|\nabla v\|_{L^3}\|\dot v\|_{L^6}\|\dot v\|_{L^2} + \|v\|_{L^\infty}\|\dot v\|_{L^2}\|\nabla\dot v\|_{L^2}\\ & +\|\frac{\partial f}{\partial t}\|_{L^2}\|\dot v\|_{L^2}\\ \le\ & (\|v\|_{H^2}+1)\left(\|\dot v\|_{L^2}\|\nabla\dot v\|_{L^2} + \|\dot v\|_{L^2}^2\right) + \|\frac{\partial f}{\partial t}\|_{L^2}^2\\ \le\ & \frac{1}{10}\|\nabla\dot v\|_{L^2}^2 + K(t)\|\dot v\|_{L^2}^2 + C\end{aligned}$$

and by Estimate (3.2.4), it follows $K\in L^1(0,T)$, $T<T^\star$.

We remark now that

$$\dot v(t=0) = -Av_0 - \Pi_i((v_0\cdot\nabla)v_0) - \Pi_i(f_0) \tag{3.2.6}$$

where $f_0 = f(t=0)$. Now with the compatibility conditions (3.2.1) we obtain that $\dot v(t=0)\in\mathbf{V}$. In particular, we should apply Gronwall Lemma with the previous estimate on $\dot v$ to obtain that for all $T<T^*$,

$$\frac{\partial v}{\partial t}\in L^\infty(0,T;L^2(\mathcal{U}))\cap L^2(0,T;H^1(\mathcal{U})).$$

Multiplying (3.2.5) by $A\dot v$ we obtain that

$$\begin{aligned}\frac12\frac{d}{dt}\|\nabla\dot v\|_{L^2}^2 + \|A\dot v\|_{L^2}^2 \le\ & \|A\dot v\|_{L^2}\left(\|\nabla v\|_{L^3}\|\dot v\|_{L^6} + \|\nabla\dot v\|_{L^2}\|v\|_{L^\infty}\right)\\ & +\|\frac{\partial f}{\partial t}\|_{L^2}\|A\dot v\|_{L^2}\\ \le\ & \tfrac12\|A\dot v\|_{L^2}^2 + C(t)\|\nabla\dot v\|_{L^2}^2 + C\end{aligned}$$

where $C(t)\in L^1(0,T)$ with Estimate (3.2.4).

Using Gronwall Lemma, since $\dot v(t=0)\in\mathbf{V}$, we obtain that for all $T<T^*$, there exists a constant C (independant on i) with

$$\|\nabla\dot v\|_{L^\infty(0,T;L^2)} + \|A\dot v\|_{L^2(0,T;L^2)} \le C,$$

and using Proposition (3.1),

$$\|\frac{\partial v}{\partial t}\|_{L^\infty(0,T;H^1)} + \|\frac{\partial v}{\partial t}\|_{L^2(0,T;H^2)} + \|\frac{\partial \pi}{\partial t}\|_{L^\infty(0,T;L^2)} + \|\frac{\partial \pi}{\partial t}\|_{L^2(0,T;H^1)} \leq C. \tag{3.2.7}$$

Estimates on $\frac{\partial^2 v}{\partial t^2}$: we denote $\ddot{v} = \frac{\partial^2 v}{\partial t^2}$. It satisfies

$$\frac{\partial \ddot{v}}{\partial t} + A\ddot{v} + \Pi_i\Big((\ddot{v}\cdot\nabla)v + 2(\dot{v}\cdot\nabla)\dot{v} + \ (v\cdot\nabla)\ddot{v}\Big) = \Pi_i(\frac{\partial^2 f}{\partial t^2}). \tag{3.2.8}$$

We remark that with Equation (3.2.5)

$$\ddot{v}(t=0) = -A(\dot{v}(t=0)) - \Pi_i\left((\dot{v}(t=0)\cdot\nabla)v_0 + (v_0\cdot\nabla)\dot{v}_0 - \frac{\partial f}{\partial t}(t=0)\right),$$

hence with the compatibility conditions (3.2.1),

$$\ddot{v}(t=0) \in \mathbf{V}.$$

Now multiplying (3.2.8) by $\ddot{v}$ we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\ddot{v}\|_{L^2}^2 + \|\nabla\ddot{v}\|_{L^2}^2 \leq \quad & \|\nabla v\|_{L^3}\|\ddot{v}\|_{L^6}\|\ddot{v}\|_{L^2} + \|\dot{v}\|_{L^\infty}\|\dot{v}\|_{H^1}\|\ddot{v}\|_{L^2} \\
& + \|v\|_{L^\infty}\|\ddot{v}\|_{L^2}\|\nabla\ddot{v}\|_{L^2} + \|\frac{\partial^2 f}{\partial t^2}\|_{L^2}\|\ddot{v}\|_{L^2} \\
\leq \quad & \|v\|_{H^2}\|\ddot{v}\|_{L^2}^2 + \|v\|_{H^2}\|\ddot{v}\|_{L^2}\|\nabla\ddot{v}\|_{L^2} + \|\dot{v}\|_{H^2}^2\|\ddot{v}\|_{L^2} \\
& + \|v\|_{H^2}\|\ddot{v}\|_{L^2}\|\nabla\ddot{v}\|_{L^2} + \|\frac{\partial^2 f}{\partial t^2}\|_{L^2}\|\ddot{v}\|_{L^2} \\
\leq \quad & \tfrac{1}{2}\|\nabla\ddot{v}\|_{L^2}^2 + C\left(\|v\|_{H^2}^2 + \|\dot{v}\|_{H^2}^2 + 1\right)\|\ddot{v}\|_{L^2}^2 + C \\
\leq \quad & \tfrac{1}{2}\|\nabla\ddot{v}\|_{L^2}^2 + K(t)\|\ddot{v}\|_{L^2}^2 + C(t)
\end{aligned}$$

using estimate (3.2.7), where $K \in L^1(0,T)$ and $C \in L^\infty(0,T)$.

Using Gronwall Lemma we obtain that for all $T < T^*$, there exists a constant C (independant on i) with

$$\|\ddot{v}\|_{L^\infty(0,T;L^2)} + \|\nabla\ddot{v}\|_{L^2(0,T;L^2)} \leq C.$$

Now we multiply (3.2.8) by $A\ddot{v}$ and we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\ddot{v}\|_{L^2}^2+\|A\ddot{v}\|_{L^2}^2\leq \quad \|A\ddot{v}\|_{L^2}\left(\|\nabla v\|_{L^3}\|\ddot{v}\|_{L^6}+2\|\dot{v}\|_{L^6}\|\nabla\dot{v}\|_{L^3}\right)$$

$$+\|A\ddot{v}\|_{L^2}\|\nabla\ddot{v}\|_{L^2}\|v\|_{L^\infty}$$

$$\leq \quad \tfrac{1}{2}\|A\ddot{v}\|_{L^2}^2+C(t)\|\nabla\ddot{v}\|_{L^2}^2+K,$$

where $C(t)\in L^1(0,T)$ with Estimate (3.2.4).

Using Gronwall Lemma we obtain that for all $T<T^*$, there exists a constant C (independant on i) with

$$\|\nabla\ddot{v}\|_{L^\infty(0,T;L^2)}+\|A\ddot{v}\|_{L^2(0,T;L^2)}\leq C$$

and using Proposition (3.1),

$$\|\frac{\partial^2 v}{\partial t^2}\|_{L^\infty(0,T;H^1)}+\|\frac{\partial^2 v}{\partial t^2}\|_{L^2(0,T;H^2)}+\|\frac{\partial^2 \pi}{\partial t^2}\|_{L^\infty(0,T;L^2)}+\|\frac{\partial^2 \pi}{\partial t^2}\|_{L^2(0,T;H^1)}\leq C. \tag{3.2.9}$$

Conclusion : using estimates (3.2.9) and (3.2.7) and Equation (3.2.5) we obtain that $A\dot{v}$ is bounded in $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ hence using Proposition (3.1), there exists a time T^* such that for $T<T^*$ there exists C such that

$$\|\dot{v}\|_{L^\infty(0,T;H^3)}+\|\dot{v}\|_{L^2(0,T;H^4)}+\|\frac{\partial\pi}{\partial t}\|_{L^\infty(0,T;H^2)}+\|\frac{\partial\pi}{\partial t}\|_{L^2(0,T;H^3)}\leq C.$$

Now using this last bound, Estimate (3.2.4) and Equation (3.2.3), we obtain that Av is in the same spaces than $\dot{v}$ thus for $T<T^*$ there exists C such that

$$\|v\|_{L^\infty(0,T;H^5)}+\|v\|_{L^2(0,T:H^4)}+\|\pi\|_{L^\infty(0,T;H^4)}+\|\pi\|_{L^2(0,T;H^5)}\leq C$$

This estimate independant on i allows us to extract a subsequence of the approximations wich converges to the solution V_0 and there exists T^* such that for all $T < T^*$,

$$\left\{\begin{array}{l} V^0 \in \mathcal{C}^0(0,T;H^5)\cap L^2(0,T;H^6) \\ \\ p^0 \in \mathcal{C}^0(0,T;H^4)\cap L^2(0,T;H^5) \\ \\ \dfrac{\partial V^0}{\partial t} \in \mathcal{C}^0(0,T;H^3)\cap L^2(0,T;H^4) \\ \\ \dfrac{\partial p^0}{\partial t} \in \mathcal{C}^0(0,T;H^2)\cap L^2(0,T;H^3) \end{array}\right. \tag{3.2.10}$$

3.3. **Regularity of $\widetilde{W}^1$.** We consider w^1 an extension of $\dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n}\cdot n)n$ in ω which satisfies, for $T < T^*$

$$w^1 \in L^\infty(0,T;H^4)\cap L^2(0,T;H^5) \text{ and } \frac{\partial w^1}{\partial t} \in L^\infty(0,T;H^2)\cap L^2(0,T;H^3).$$

This extension exists as the trace operator is onto in the considered spaces. We set now

$$\widetilde{W}^1(t,x,z) = w^1(t,x)e^{-z} \text{ in } [0,T^*[\times\omega\times\mathbb{R}_z^+,$$

and we remark that

$$\left\{\begin{array}{l} \widetilde{W}^1 \in \left(L_t^\infty(0,T;H_x^4)\cap L_t^2(0,T;H_x^5)\right)\otimes\mathcal{C}_z^\infty, \\ \\ \dfrac{\partial \widetilde{W}^1}{\partial t} \in \left(L_t^\infty(0,T;H_x^2)\cap L_t^2(0,T;H_x^3)\right)\otimes\mathcal{C}_z^\infty. \end{array}\right. \tag{3.3.1}$$

3.4. **Regularity of V^1.**

Proposition 3.3. *Let v_0 and f as in Theorem 3.2. Let T^* and V^0 given by this theorem. There exists V^1 and p^1 such that*

$$\left\{\begin{array}{l} \dfrac{\partial V^1}{\partial t} - \Delta V^1 + V^0\cdot\nabla V^1 + V^1\cdot\nabla V^0 + \nabla p^1 = 0 \textit{ in } [0,T^*[\times\mathcal{U}, \\ \\ \operatorname{div} V^1 = 0 \textit{ in } [0,T^*[\times\mathcal{U}, \\ \\ V^1 = \dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n}\cdot n)n \textit{ on } [0,T^*[\times\partial\omega, \\ \\ V^1 = 0 \quad \textit{on} \quad [0,T^*[\times\partial\Omega, \end{array}\right. \tag{3.4.1}$$

and satisfying that for all $T < T^*$,

$$\begin{cases} V^1 \in \mathcal{C}^0(0,T;H^2) \cap L^2(0,T;H^3), \\ \dfrac{\partial V^1}{\partial t} \in \mathcal{C}^0(0,T;L^2) \cap L^2(0,T;H^1). \end{cases} \tag{3.4.2}$$

Proof : we will seek V^1 on the form $Z^1 + \Upsilon^1$ where Υ^1 is an extension of $\dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n} \cdot n)n$ on $\mathcal{U}$ satisfying

$$\begin{cases} \Upsilon^1 \in \mathcal{C}^0(0,T;H^3(\mathcal{U})) \cap L^2(0,T;H^4(\mathcal{U})), \\ \dfrac{\partial \Upsilon^1}{\partial t} \in \mathcal{C}^0(0,T;H^1(\mathcal{U})) \cap L^2(0,T;H^2(\mathcal{U})), \\ \dfrac{\partial^2 \Upsilon^1}{\partial t^2} \in L^2(0,T;L^2(\mathcal{U})), \\ -\Delta \Upsilon^1 + \nabla \Pi^1 = 0 \text{ in } \mathcal{U}, \\ \operatorname{div} \Upsilon^1 = 0 \text{ in } \mathcal{U}, \\ \Upsilon^1 = \dfrac{\partial V^0}{\partial n} - (\dfrac{\partial V^0}{\partial n} \cdot n)n \quad \text{on} \quad \partial\omega. \end{cases} \tag{3.4.3}$$

We have to prove the existence of Z^1 solution of

$$\begin{cases} \dfrac{\partial Z^1}{\partial t} - \Delta Z^1 + (V^0 \cdot \nabla) Z^1 + (Z^1 \cdot \nabla) V^0 + \nabla p^1 = Q^1 \\ \operatorname{div} Z^1 = 0 \\ Z^1 = 0 \text{ on } \partial\mathcal{U} \end{cases} \tag{3.4.4}$$

where $Q^1 = -\dfrac{\partial \Upsilon^1}{\partial t} - (V^0 \cdot \nabla)\Upsilon^1 - (\Upsilon^1 \cdot \nabla) V^0$, and where Z^1 satisfies the regularity conditions wanted for V^1 that is

$$\begin{cases} Z^1 \in \mathcal{C}^0(0,T;H^2) \cap L^2(0,T;H^3), \\ \dfrac{\partial Z^1}{\partial t} \in \mathcal{C}^0(0,T;L^2) \cap L^2(0,T;H^1). \end{cases} \tag{3.4.5}$$

for all $T < T^*$.

We remark that for all $T < T^*$,

$$\left\{\begin{array}{l} Q^1 \in \mathcal{C}^0(0,T;H^1(\mathcal{U})) \cap L^2(0,T;H^2(\mathcal{U})) \\ \\ \dfrac{\partial Q^1}{\partial t} \in L^2(0,T;L^2(\mathcal{U})) \end{array}\right. \tag{3.4.6}$$

We have then to build an initial data $Z(t=0)$ for Equation (3.4.4) in order to ensure the desired regularity for Z^1.

We extend Q^1 for $t<0$ setting $Q^1(t) = Q^1(-t)$. We fix $\eta \in \mathcal{C}^\infty([-1,+\infty[)$ such that $\eta(t) = 1$ for $t \geq 0$ and $\eta(t) = 0$ for $t < -\dfrac{1}{2}$. We solve then the following problem, where the initial condition is given for $t = -1$:

$$\left\{\begin{array}{l} \dfrac{\partial Z^1}{\partial t} - \Delta Z^1 + (V^0\cdot\nabla)Z^1 + (Z^1\cdot\nabla)V^0 + \nabla p^1 = \eta(t)Q^1 \text{ for } t \geq -1 \\ \\ \text{div } Z^1 = 0 \text{ on } [-1,T^*[\times\mathcal{U} \\ \\ Z^1 = 0 \text{ on } \partial\mathcal{U} \\ \\ Z^1(-1) = 0 \end{array}\right. \tag{3.4.7}$$

At the initial times, $Z^1_{|t=-1} = 0$, $(\eta Q^1)_{|t=-1} = 0$ and $\left(\dfrac{\partial}{\partial t}(\eta Q^1)\right)_{|t=-1} = 0$. So the compatibility conditions for (3.4.7) are satisfied and we can find a solution Z^1 with the desired regularity conditions. Furthermore, Equation (3.4.7) is linear in Z^1 so the regular solution exists so long as the coefficients of this equations exist, that is on the times interval $[-1,T^*[$.

3.5. **Regularity of q^0 and $\overline{W}^2$.** The pressure q^0 satisfies

$$\left\{\begin{array}{l} \Delta q^0 = 0 \text{ in } \mathbb{R}^+\times\omega, \\ \\ q^0 = -\dfrac{\partial V^0}{\partial n}\cdot n + p^0 \text{ on } \mathbb{R}^+\times\partial\omega. \end{array}\right.$$

With estimates (3.2.10) we obtain that for all $T < T^*$,

$$\left\{\begin{array}{l} q^0 \in L^\infty(0,T;H^4) \cap L^2(0,T;H^5), \\ \\ \dfrac{\partial q^0}{\partial t} \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3). \end{array}\right. \tag{3.5.1}$$

Since $\overline{W}^2 = -\nabla q^0$ we obtain that

$$\begin{cases} \overline{W}^2 \in L^\infty(0,T;H^3) \cap L^2(0,T;H^4), \\ \dfrac{\partial \overline{W}^2}{\partial t} \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2). \end{cases} \tag{3.5.2}$$

3.6. **Regularity of $\widetilde{W}^2$.** Since $\widetilde{W}^2_z \cdot n = -\text{div}\, \widetilde{W}^1$, with estimate (3.3.1) we obtain that

$$\begin{cases} \widetilde{W}^2 \cdot n \in \left(L^\infty_t(0,T;H^2_x) \cap L^2_t(0,T;H^3_x)\right) \otimes \mathcal{C}^\infty_z \\ \dfrac{\partial \widetilde{W}^2 \cdot n}{\partial t} \in \left(L^\infty_t(0,T;L^2_x) \cap L^2_t(0,T;H^1_x)\right) \otimes \mathcal{C}^\infty_z \end{cases} \tag{3.6.1}$$

With this estimate and equation (2.2.14) we obtain that $\widetilde{q^1}$ is in the same spaces than $\widetilde{W}^2 \cdot n$.

Now, with Equation (2.2.15) and with estimates (3.3.1) and (3.4.2) we obtain that for all $T < T^*$,

$$\begin{cases} \widetilde{W}^2 \wedge n \in \left(L^\infty_t(0,T;H^2_x) \cap L^2_t(0,T;H^3_x)\right) \otimes \mathcal{C}^\infty_z \\ \dfrac{\partial \widetilde{W}^2 \wedge n}{\partial t} \in \left(L^\infty_t(0,T;L^2_x) \cap L^2_t(0,T;H^1_x)\right) \otimes \mathcal{C}^\infty_z \end{cases} \tag{3.6.2}$$

3.7. **Regularity of V^2 and p^2.**

Proposition 3.4. *Under hypothesis of Proposition 3.2, there exists V^2 and p^2 such that*

$$\begin{cases} \quad \dfrac{\partial V^2}{\partial t} - \Delta V^2 + V^0 \cdot \nabla V^2 + V^1 \cdot \nabla V^1 \\ \quad +V^2 \cdot \nabla V^0 + \nabla p^2 = 0 \qquad in\ [0,T^*[\times \mathcal{U} \\ \text{div}\, V^2 = 0 \qquad in\ [0,T^*[\times \mathcal{U} \\ V^2(t,x) = W^2(t,x,0)\ \ on\ [0,T^*[\times \partial\omega \\ V^2 = 0\ \ on\ [0,T^*[\times \partial\Omega \end{cases} \tag{3.7.1}$$

and for all $T < T^*$, we have

$$\left\{\begin{array}{l} V^2 \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3) \\ \\ q^2 \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \\ \\ \dfrac{\partial V^2}{\partial t} \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \\ \\ \dfrac{\partial q^2}{\partial t} \in L^2(0,T;L^2) \end{array}\right. \tag{3.7.2}$$

Proof : We consider an extension Υ^2 of $W^2_{|\partial\omega}$ such that

$$\left\{\begin{array}{l} \Upsilon^2 \in L^\infty(0,T;H^1(\mathcal{U})) \cap L^2(0,T;H^2(\mathcal{U})) \\ \\ \dfrac{\partial \Upsilon^2}{\partial t} \in L^2(0,T;L^2(\mathcal{U})) \\ \\ \text{div } \Upsilon^2 = 0 \quad \text{on} \quad [0,T^*[\times\mathcal{U} \\ \\ -\Delta\Upsilon^2 + \nabla\pi^2 = 0 \text{ in } [0,T^*[\times\mathcal{U} \\ \\ \Upsilon^2(t,x) = W^2(t,x,0) \text{ on } [0,T^*[\times\partial\omega \\ \\ \Upsilon^2 = 0 \text{ on } [0,T^*[\times\partial\Omega \end{array}\right.$$

We will seek V^2 on the form $V^2 = \Upsilon^2 + Z^2$ where Z^2 satisfies

$$\left\{\begin{array}{l} \dfrac{\partial Z^2}{\partial t} + AZ^2 + V^0 \cdot \nabla Z^2 + Z^2 \cdot \nabla V^0 = Q^2 \text{ on } [0,T^*[\times\mathcal{U} \\ \\ \text{div } Z^2 = 0 \text{ on } [0,T^*[\times\mathcal{U} \\ \\ Z^2 = 0 \text{ on } [0,T^*[\times\partial\mathcal{U} \end{array}\right. \tag{3.7.3}$$

where

$$Q^2 = -\frac{\partial \Upsilon^2}{\partial t} - V^0 \cdot \nabla\Upsilon^2 - V^1 \cdot \nabla V^1 - \Upsilon^2 \cdot \nabla V^0.$$

We observe that Q^2 is bounded in $L^2(0,T;L^2(\mathcal{U}))$.

As for V^1 the problem is to find an initial data for Z^2 in order to obtain the desired regularity. We proceed as in the proof of Proposition 3.3.

4. Estimate on the remainder term

We introduce $\theta \in \mathcal{C}^\infty(\overline{\omega}; R)$ a cut off function such that $\theta = 1$ in a neighborhood of $\partial\omega$ with supp $\theta \subset \omega_1$.

We define the different terms of the ansatz as in the previous section and we introduce the approximations W^ε of the velocity w^ε in ω and his approximation V^ε in $\mathcal{U}$ defined as follows

$$\begin{cases} W^\varepsilon(t,x) = \varepsilon\theta(x)\widetilde{W}^1(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon^2\overline{W}^2(t,x) + \varepsilon^2\theta(x)\widetilde{W}^2(t,x,\frac{\varphi(x)}{\varepsilon}) \\ \\ V^\varepsilon = V^0(t,x) + \varepsilon V^1(t,x) + \varepsilon^2 V^2(t,x) \end{cases}$$

We set

$$\begin{aligned} w^\varepsilon(t,x) &= W^\varepsilon(t,x) + \varepsilon^{\frac{3}{2}} w^r_\varepsilon(t,x) \\ v^\varepsilon(t,x) &= V^\varepsilon(t,x) + \varepsilon^{\frac{3}{2}} v^r_\varepsilon(t,x) \\ q^\varepsilon(t,x) &= \overline{q^0}(t,x) + \varepsilon\theta(x)\widetilde{q^1} + \varepsilon^{\frac{3}{2}} q^r_\varepsilon(t,x) \\ p^\varepsilon(t,x) &= p^0(t,x) + \varepsilon p^1(t,x) + \varepsilon^{\frac{3}{2}} p^r_\varepsilon(t,x). \end{aligned}$$

4.1. **Equation satisfied by the remainder terms.** We will write the equations satisfied by the remainder terms in order to estimate them.

$$\begin{cases} (1) \quad \begin{aligned} &\frac{\partial w^r_\varepsilon}{\partial t} - \Delta w^r_\varepsilon + (w^r_\varepsilon\cdot\nabla)W^\varepsilon + \varepsilon^{\frac{3}{2}}(w^r_\varepsilon\cdot\nabla)w^r_\varepsilon \\ &\qquad +(W^\varepsilon\cdot\nabla)w^r_\varepsilon + \nabla q^r_\varepsilon + \frac{1}{\varepsilon^2}w^r_\varepsilon = R^\varepsilon_{obst} \end{aligned} & \text{in } \mathbb{R}^+\times\omega \\ \\ (2) \quad \operatorname{div} w^r_\varepsilon = g_\varepsilon & \text{in } \mathbb{R}^+\times\omega \\ \\ (3) \quad \begin{aligned} &\frac{\partial v^r_\varepsilon}{\partial t} - \Delta v^r_\varepsilon + (V^\varepsilon\cdot\nabla)v^r_\varepsilon + (v^r_\varepsilon\cdot\nabla)V^\varepsilon + \nabla p^r_\varepsilon \\ &\qquad +\varepsilon^{\frac{3}{2}}(v^r_\varepsilon\cdot\nabla)v^r_\varepsilon = R^\varepsilon_{flu} - \varepsilon^{\frac{5}{2}}(V^2\cdot\nabla)V^2 \end{aligned} & \text{in } \mathbb{R}^+\times\mathcal{U} \\ \\ (4) \quad \operatorname{div} v^r_\varepsilon = 0 & \text{in } \mathbb{R}^+\times\mathcal{U} \\ \\ (5) \quad v^r_\varepsilon = w^r_\varepsilon & \text{in } \mathbb{R}^+\times\partial\omega \\ \\ (6) \quad -\frac{\partial v^r_\varepsilon}{\partial n} + p^r_\varepsilon n + \frac{\partial w^r_\varepsilon}{\partial n} - q^r_\varepsilon n = R^\varepsilon_{bound} & \text{in } \mathbb{R}^+\times\partial\omega \end{cases} \tag{4.1}$$

where

$$\left\{\begin{aligned}
R^\varepsilon_{obst} = \quad & \frac{1}{\sqrt{\varepsilon}}\left(-\theta\frac{\partial\widetilde{W}^1}{\partial t} - \varepsilon\frac{\partial\overline{W}^2}{\partial t} - \varepsilon\theta\frac{\partial\widetilde{W}^2}{\partial t} + \theta\Delta\widetilde{W}^1 + \right. \\
& \frac{2}{\varepsilon}(\nabla\theta\cdot\nabla\varphi)\widetilde{W^1_z} + \Delta\theta\widetilde{W}^1 + 2\nabla\theta\nabla\widetilde{W}^1 + \varepsilon\Delta\overline{W}^2 \\
& +\theta(2\nabla\varphi\cdot\nabla\widetilde{W}^2_z + \Delta\varphi\widetilde{W}^2_z) + \varepsilon\theta\Delta\widetilde{W}^2 \\
& +\varepsilon\Delta\theta\widetilde{W}^2 + 2\varepsilon\nabla\theta\nabla\widetilde{W}^2 + 2\nabla\theta\nabla\varphi\widetilde{W}^2_z \\
& \left. -\varepsilon(W^\varepsilon\cdot\nabla)W^\varepsilon - \nabla\theta\widetilde{q^1}\right) \\
g_\varepsilon = \frac{1}{\sqrt{\varepsilon}} & \left(-\nabla\theta\widetilde{W}^1 - \varepsilon\nabla\theta\widetilde{W}^2 - \varepsilon\theta\mathrm{div}\,\widetilde{W}^2\right) \\
R^\varepsilon_{flu} = -\varepsilon^{\frac{3}{2}} & \left((V^2\cdot\nabla)V^1 + (V^1\cdot\nabla)V^2\right) \\
R^\varepsilon_{bound} = \quad & \frac{1}{\sqrt{\varepsilon}}\left(-\frac{\partial\widetilde{W}^1}{\partial n} - \widetilde{W}^2_z - \varepsilon\frac{\partial\overline{W}^2}{\partial n} + \widetilde{q}^1 n + \frac{\partial V^1}{\partial n}\right. \\
& \left. -p^1 n + \varepsilon\frac{\partial V^2}{\partial n} - \varepsilon p^2 n - \varepsilon\frac{\partial\widetilde{W^2}}{\partial n}\right)
\end{aligned}\right.$$

Lemma 4.1. *For all $T < T^*$ there exists C such that*

$$\left\{\begin{aligned}
&\|R^\varepsilon_{obst}\|_{L^2(0,T;L^2(\omega))} \leq \frac{C}{\sqrt{\varepsilon}} \\
&\|g_\varepsilon\|_{L^\infty(0,T;L^2(\omega))} \leq C\sqrt{\varepsilon} \\
&\|R^\varepsilon_{flu}\|_{L^2(0,T;L^2(\mathcal{U}))} \leq C \\
&\|R^\varepsilon_{bound}\|_{L^2(0,T;L^2(\partial\omega))} \leq \frac{C}{\sqrt{\varepsilon}}
\end{aligned}\right. \tag{4.2}$$

Proof : these estimates are direct consequences of the estimates satisfied by the different terms of the asymptotic expansion.

We remark that $\nabla\theta = 0$ on a neighbourhood of the boundary and since $\widetilde{W}^1(t,x,z) = w^1(t,x)e^{-z}$, we have $\|\nabla\theta(\cdot)\widetilde{W}^1(t,\cdot,\frac{\varphi(\cdot)}{\varepsilon})\|_{L^2(\omega)} \leq Ke^{-\frac{\eta}{\varepsilon}}$ wich explains the estimates on g_ε and on R^ε_{obs}.

In order to estimate the term $(w^r_\varepsilon \cdot \nabla)w^r_\varepsilon$ we need a divergence free condition.

Lemma 4.2. *There exists $\psi_\varepsilon \in L^2(0,T;H^1_0(\omega))$ for all $T < T^*$ such that*

$$\text{div } \psi_\varepsilon = g_\varepsilon.$$

Furthermore there exists a constant K independant on ε such that

$$\|\psi_\varepsilon\|_{L^2(0,T;H^1_0(\omega))} \leq K\sqrt{\varepsilon}$$

$$\left\|\frac{\partial\psi_\varepsilon}{\partial t}\right\|_{L^2(0,T;H^1_0(\omega))} \leq K\sqrt{\varepsilon}.$$

Proof : the map ψ_ε exists since $\displaystyle\int_\omega g_\varepsilon = 0$

$$\int_\omega g_\varepsilon = \frac{1}{\varepsilon^{\frac{3}{2}}}\int_\omega \text{div } W^\varepsilon = -\frac{1}{\varepsilon^{\frac{3}{2}}}\int_{\partial\omega} W^\varepsilon \cdot n = -\frac{1}{\sqrt{\varepsilon}}\int_{\partial\omega} W^2 \cdot n = 0$$

(see Remark 2.4).

4.2. **Estimate.** We multiply (1) in (4.1) by $w^r_\varepsilon - \psi_\varepsilon$ and we integrate on ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\|w^r_\varepsilon\|^2 + \|\nabla w^r_\varepsilon\|^2 + \frac{1}{\varepsilon^2}\|w^r_\varepsilon\|^2 = I_1 + \ldots + I_{12} \tag{4.3}$$

with

$$I_1 = -\int_{\partial\omega} w^r_\varepsilon\left(\frac{\partial w^r_\varepsilon}{\partial n} - q^r_\varepsilon n\right),$$

$$I_2 = -\int_\omega \left((w_\varepsilon^r \cdot \nabla)W^\varepsilon + (W^\varepsilon \cdot \nabla)w_\varepsilon^r\right) w_\varepsilon^r,$$

$$I_3 = \int_\omega \left((w_\varepsilon^r \cdot \nabla)W^\varepsilon + (W^\varepsilon \cdot \nabla)w_\varepsilon^r\right) \psi_\varepsilon,$$

$$I_4 = -\varepsilon^{\frac{3}{2}} \int_\omega ((w_\varepsilon^r - \psi_\varepsilon) \cdot \nabla)(w_\varepsilon^r - \psi_\varepsilon)(w_\varepsilon^r - \psi_\varepsilon),$$

$$I_5 = -\varepsilon^{\frac{3}{2}} \int_\omega \left((w_\varepsilon^r \cdot \nabla)\psi_\varepsilon \cdot w_\varepsilon^r + (w_\varepsilon^r \cdot \nabla)w_\varepsilon^r \cdot \psi_\varepsilon + (\psi_\varepsilon \cdot \nabla)w_\varepsilon^r \cdot w_\varepsilon^r\right),$$

$$I_6 = \varepsilon^{\frac{3}{2}} \int_\omega \left((w_\varepsilon^r \cdot \nabla)\psi_\varepsilon \cdot \psi_\varepsilon + (\psi_\varepsilon \cdot \nabla)w_\varepsilon^r \cdot \psi_\varepsilon + (\psi_\varepsilon \cdot \nabla)\psi_\varepsilon \cdot w_\varepsilon^r\right),$$

$$I_7 = -\varepsilon^{\frac{3}{2}} \int_\omega (\psi_\varepsilon \cdot \nabla)\psi_\varepsilon \cdot \psi_\varepsilon,$$

$$I_8 = \frac{1}{\varepsilon^2} \int_\omega w_\varepsilon^r \psi_\varepsilon, \quad I_9 = \int_\omega R_{obst}^\varepsilon w_\varepsilon^r,$$

$$I_{10} = -\int_\omega \frac{\partial w_\varepsilon^r}{\partial t} \psi_\varepsilon, \quad I_{11} = \int_\omega \nabla w_\varepsilon^r \nabla \psi_\varepsilon,$$

$$I_{12} = -\int_\omega R_{obst}^\varepsilon \psi^\varepsilon,$$

We multiply (3) in (4.1) by v_ε^r and we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v_\varepsilon^r\|^2 + \|\nabla v_\varepsilon^r\|^2 = J_1 + \ldots + J_5 \tag{4.4}$$

where:

$$J_1 = \int_{\partial\omega} \left(\frac{\partial v_\varepsilon^r}{\partial n} v_\varepsilon^r - p^\varepsilon v_\varepsilon^r \cdot n \right),$$

$$J_2 = -\int_{\mathcal{U}} \Big((V^\varepsilon \cdot \nabla) v_\varepsilon^r \cdot v_\varepsilon^r + (v_\varepsilon^r \cdot \nabla) V^\varepsilon \cdot v_\varepsilon^r \Big)$$

$$J_3 = -\varepsilon^{\frac{3}{2}} \int_{\mathcal{U}} (v_\varepsilon^r \cdot \nabla) v_\varepsilon^r \cdot v_\varepsilon^r,$$

$$J_4 = \int_{\mathcal{U}} R_{flu}^\varepsilon v_\varepsilon^r, \qquad J_5 = \int_{\mathcal{U}} \varepsilon^{\frac{5}{2}} (V^2 \cdot \nabla) V^2 \cdot v_\varepsilon^r.$$

We add (4.3) and (4.4). We estimate the right hand side terms in the following way:

$$\begin{aligned}
I_1 + J_1 = {} & \textstyle\int_{\partial\omega} R_{bound}^\varepsilon w_\varepsilon^r \leq \|R_{bound}^\varepsilon\|_{L^2(\partial\omega)} \|w_\varepsilon^r\|_{L^2(\partial\omega)} \\
\leq {} & \frac{K}{\sqrt{\varepsilon}} \left(\|w_\varepsilon^r\|_{L^2(\omega)}^{\frac{1}{2}} \|\nabla w_\varepsilon^r\|_{L^2(\omega)}^{\frac{1}{2}} + \|w_\varepsilon^r\|_{L^2(\omega)} \right) \\
\leq {} & \frac{K}{\varepsilon} \|w_\varepsilon^r\|_{L^2(\omega)} + \|\nabla w_\varepsilon^r\|_{L^2(\omega)} + \|w_\varepsilon^r\|_{L^2(\omega)} \\
\leq {} & \frac{1}{10\varepsilon^2} \|w_\varepsilon^r\|_{L^2(\omega)}^2 + K + \|\nabla w_\varepsilon^r\|_{L^2(\omega)}^2 + \|w_\varepsilon^r\|_{L^2(\omega)}^2
\end{aligned}$$

$$\begin{aligned}
|I_2| \leq {} & \|w_\varepsilon^r\|_{L^6} \|\nabla W^\varepsilon\|_{L^2} \|w_\varepsilon^r\|_{L^3} + \|W^\varepsilon\|_{L^6} \|\nabla w_\varepsilon^r\|_{L^2} \|w_\varepsilon^r\|_{L^3} \\
\leq {} & \frac{1}{8} \|\nabla w_\varepsilon^r\|_{L^2}^2 + \|W^\varepsilon\|_{H^1}^4 \|w_\varepsilon^r\|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}
|I_3| \leq {} & \|w_\varepsilon^r\|_{L^6} \|\nabla W^\varepsilon\|_{L^2} \|\psi_\varepsilon\|_{L^3} + \|W^\varepsilon\|_{L^6} \|\nabla w_\varepsilon^r\|_{L^2} \|\psi_\varepsilon\|_{L^3} \\
& \leq \frac{1}{8} \|\nabla w_\varepsilon^r\|_{L^2}^2 + \|w_\varepsilon^r\|_{L^2}^2 + \|W^\varepsilon\|_{H^1}^2 \|\psi_\varepsilon\|_{H^1}^2
\end{aligned}$$

$$\begin{aligned} I_4 = & \ -\varepsilon^{\frac{3}{2}} \sum_{i,j} \int_\omega (w_\varepsilon^{r,i} - \psi_\varepsilon^i) \frac{\partial}{\partial x_i} (w^{r,j} - \psi_\varepsilon^j)(w_\varepsilon^{r,j} - \psi_\varepsilon^j) \\ = & \ +\varepsilon^{\frac{3}{2}} \sum_{ij} \int_\omega (w_\varepsilon^{r,j} - \psi_\varepsilon^j) \frac{\partial}{\partial x_i} \left((w_\varepsilon^{r,i} - \psi_\varepsilon^i)(w_\varepsilon^{r,j} - \psi_\varepsilon^j) \right) \\ & \ -\varepsilon^{\frac{3}{2}} \sum_{ij} \int_{\partial\omega} (w_\varepsilon^{r,i} - \psi_\varepsilon^i) n_i (w_\varepsilon^{r,j} - \psi_\varepsilon^j)(w_\varepsilon^{r,j} - \psi_\varepsilon^j) \end{aligned}$$

Using that $\psi_\varepsilon = 0$ on $\partial\omega$ and that div $(w_\varepsilon^r - \psi_\varepsilon) = 0$, we obtain finally that

$$I_4 = \ -\frac{1}{2}\varepsilon^{\frac{3}{2}} \int_{\partial\omega} w_\varepsilon^r \cdot n |w_\varepsilon^r|^2 d\sigma.$$

$$\begin{aligned} |I_5| \leq & \ \varepsilon^{\frac{3}{2}} \|w_\varepsilon^r\|_{L^6} \|\nabla\psi_\varepsilon\|_{L^2} \|w_\varepsilon^r\|_{L^3} + 2\|w_\varepsilon^r\|_{L^3} \|\nabla w_\varepsilon^r\|_{L^2} \|\psi_\varepsilon\|_{L^6} \\ \leq & \ \varepsilon^{\frac{3}{2}} \|\psi_\varepsilon\|_{H^1} \left(\|w_\varepsilon^r\|_{L^2}^2 + \|w_\varepsilon^r\|_{L^2}^{\frac{1}{2}} \|\nabla w_\varepsilon^r\|_{L^2}^{\frac{3}{2}} \right) \\ \leq & \ \frac{1}{8} \|\nabla w_\varepsilon^r\|_{L^2}^2 + C\varepsilon^{\frac{3}{2}} \left(1 + \varepsilon^{\frac{9}{2}} \|\psi_\varepsilon\|_{H^1}^4 \right) \|w_\varepsilon^r\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned} |I_6| \leq & \ \varepsilon^{\frac{3}{2}} \|\nabla\psi_\varepsilon\|_{L^2} \|w_\varepsilon^r\|_{L^6} \|\psi_\varepsilon\|_{L^3} + \varepsilon^{\frac{3}{2}} \|w_\varepsilon^r\|_{L^6} \|\psi_\varepsilon\|_{L^3} \|\nabla\psi_\varepsilon\|_{L^2} \\ \leq & \ \varepsilon^{\frac{3}{2}} \|\psi_\varepsilon\|_{H^1}^2 \|w_\varepsilon^r\|_{L^2} + \varepsilon^{\frac{3}{2}} \|\nabla w_\varepsilon^r\|_{L^2} \|\psi_\varepsilon\|_{H^1}^2 \\ \leq & \ \frac{1}{8} \|\nabla w_\varepsilon^r\|_{L^2}^2 + C\varepsilon^3 \|\psi_\varepsilon\|_{H^1}^4 + \varepsilon^{\frac{3}{2}} \|\psi_\varepsilon\|_{H^1}^2 \|w_\varepsilon^r\|_{L^2} \end{aligned}$$

$$\begin{aligned} |I_7| \leq & \ \varepsilon^{\frac{3}{2}} \|\nabla\psi_\varepsilon\|_{L^2} \|\psi_\varepsilon\|_{L^3} \|\psi_\varepsilon\|_{L^6} \\ \leq & \ \varepsilon^{\frac{3}{2}} \|\psi_\varepsilon\|_{H^1}^3 \end{aligned}$$

$$|I_8| \leq \frac{1}{2\varepsilon^2}\|w_\varepsilon^r\|_{L^2}^2 + \frac{1}{2\varepsilon^2}\|\psi_\varepsilon\|_{L^2}^2$$

$$|I_9| \leq \varepsilon^2\|R_{obst}^\varepsilon\|_{L^2}^2 + \tfrac{1}{\varepsilon^2}\|w_\varepsilon^r\|^2$$

We integrate I_{10} in time from 0 to T and we obtain that

$$\begin{aligned}\int_0^T I_{10} = & \int_\omega w_\varepsilon^r(T)\psi_\varepsilon(T) - \int_\omega w_\varepsilon^r(0)\psi_\varepsilon(0) + \int_0^T\int_\omega w_\varepsilon^r \frac{\partial\psi_\varepsilon}{\partial t} \\ \leq & \frac{1}{10}\|w_\varepsilon^r(T)\|_{L^2} + W\|\psi_\varepsilon(T)\|_{L^2}^2 + K + \int_0^T \|w_\varepsilon^r\|_{L^2}^2 \\ & + \int_0^T \|\frac{\partial\psi_\varepsilon}{\partial t}\|_{L^2}^2\end{aligned}$$

$$|I_{11}| \leq \frac{1}{10}\|\nabla w_\varepsilon^r\|_{L^2}^2 + K\|\psi_\varepsilon\|_{H^1}^2,$$

$$|I_{12}| \leq \|R_{obst}^\varepsilon\|_{L^2}\|\psi_\varepsilon\|_{L^2}$$

$$\begin{aligned}|J_2| \leq & \|V^\varepsilon\|_{L^6}\|\nabla v_\varepsilon^r\|_{L^2}\|v_\varepsilon^r\|_{L^3} + \|\nabla V^\varepsilon\|_{L^2}\|v_\varepsilon^r\|_{L^4}^2 \\ \leq & \|V^\varepsilon\|_{H^1}\|v_\varepsilon^r\|_{H^1}^{\frac{3}{2}}\|v_\varepsilon^r\|_{L^2}^{\frac{1}{2}} \\ \leq & \|V^\varepsilon\|_{H^1}\left(\|v_\varepsilon^r\|_{L^2}^2 + \|\nabla v_\varepsilon^r\|_{L^2}^{\frac{3}{2}}\|v_\varepsilon^r\|_{L^2}^{\frac{1}{2}}\right) \\ \leq & \frac{1}{10}\|\nabla v_\varepsilon^r\|_{L^2}^2 + C(\|V^\varepsilon\|_{H^1}^4 + 1)\|v_\varepsilon^r\|_{L^2}^2\end{aligned}$$

We treat J_3 as I_3 and since $v_\varepsilon^r = w_\varepsilon^r$ on $\partial\omega$, we obtain that

$$J_3 = \varepsilon^{\frac{3}{2}}\frac{1}{2}\int_{\partial\omega}|v_\varepsilon^r|^2(v_\varepsilon^r\cdot n)d\sigma = -I_3$$

$$|J_4| \leq \|R^\varepsilon_{flu}\|^2_{L^2} + \|v^r_\varepsilon\|^2$$

$$\begin{aligned} |J_5| \leq\ & \varepsilon^{\frac{5}{2}} \|V^2\|_{L^3} \|\nabla V^2\|_{L^2} \|v^r_\varepsilon\|_{L^6} \\ \leq\ & \varepsilon^{\frac{5}{2}} \|V^2\|^2_{H^1} \left(\|v^r_\varepsilon\|_{L^2} + \|\nabla v^r_\varepsilon\|_{L^2}\right) \\ \leq\ & \frac{1}{10} \|\nabla v^r_\varepsilon\|^2_{L^2} + C\varepsilon^5 \|V^2\|^4_{H^1} + \|v^r_\varepsilon\|^2_{L^2} \end{aligned}$$

Hence adding all the previous inequality and using Estimates (4.2) we obtain that there exists a function $K \in L^1(0,T;\mathbb{R})$ for all $T < T^*$ such that

$$\frac{d}{dt}\left(\|w^r_\varepsilon\|^2_{L^2} + \|v^r_\varepsilon\|^2_{L^2}\right) + \|\nabla w^r_\varepsilon\|^2_{L^2} + \|\nabla v^r_\varepsilon\|^2_{L^2} + \frac{1}{\varepsilon^2}\|w^r_\varepsilon\|^2_{L^2} \leq K(t)(1 + \|v^r_\varepsilon\|^2_{L^2}). \tag{4.5}$$

With the hypothesis of Theorem 1.1, since

$$\begin{aligned} w^r_\varepsilon(0,x) &= r^\varepsilon(x) - \sqrt{\varepsilon} W^2(0, x, \tfrac{\varphi(x)}{\varepsilon}) \text{ in } \omega, \\ v^r_\varepsilon(0,x) &= r^\varepsilon(x) - \sqrt{\varepsilon} V^2(0,x) \text{ in } \mathcal{U}, \end{aligned}$$

we know that there exists C such that for all $\varepsilon > 0$,

$$\left(\|w^r_\varepsilon(t=0)\|^2_{L^2} + \|v^r_\varepsilon(t=0)\|^2_{L^2}\right) \leq C$$

hence with a classical Gronwall lemma we obtain the desired result.

In order to prove Theorem 1.2, we have to estimate $\varpi^r_\varepsilon = \sqrt{\varepsilon} w^r_\varepsilon$ and $\nu^r_\varepsilon = \sqrt{\varepsilon} v^r_\varepsilon$. Multiplying (4.5) by ε, we obtain that there exists a function $K \in L^1(0,T;\mathbb{R})$ for all $T < T^*$ such that

$$\frac{d}{dt}\left(\|\varpi^r_\varepsilon\|^2_{L^2} + \|\nu^r_\varepsilon\|^2_{L^2}\right) + \|\nabla \varpi^r_\varepsilon\|^2_{L^2} + \|\nabla \nu^r_\varepsilon\|^2_{L^2} + \frac{1}{\varepsilon^2}\|\varpi^r_\varepsilon\|^2_{L^2} \leq K(t)(1 + \|\nu^r_\varepsilon\|^2_{L^2}).$$

Now the initial data satisfy

$$\varpi^r_e(0,x) = r^\varepsilon - W^1(0, x, \frac{\varphi(x)}{\varepsilon}) - \varepsilon W^2(0, x, \frac{\varphi(x)}{\varepsilon}) \text{ in } \omega,$$

$$\nu^r_\varepsilon(0,x) = r^\varepsilon(x) - V^1(0,x) - \varepsilon V^2(0,x) \text{ in } \mathcal{U},$$

that is there exists C such that for all $\varepsilon > 0$,

$$\left(\|\varpi^r_\varepsilon(t=0)\|^2_{L^2} + \|\nu^r_\varepsilon(t=0)\|^2_{L^2}\right) \leq C.$$

Hence with a classical Gronwall lemma the proof of Theorem 1.2 is complete.

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