SEMILINEAR BEHAVIOR FOR TOTALLY LINEARLY DEGENERATE HYPERBOLIC SYSTEMS WITH RELAXATION

G. CARBOU*, B. HANOUZET* AND R. NATALINI†

ABSTRACT. We investigate totally linearly degenerate hyperbolic systems with relaxation. We aim to study their semilinear behavior, which means that the local smooth solutions cannot develop shocks, and the global existence is controlled by the supremum bound of the solution. In this paper we study two specific examples: the *Suliciu*-type and the *Kerr-Debye*-type models. For the Suliciu model, which arises from the numerical approximation of isentropic flows, the semilinear behavior is obtained using pointwise estimates of the gradient. For the Kerr-Debye systems, which arise in nonlinear optics, we show the semilinear behavior via energy methods. For the original Kerr-Debye model, thanks to the special form of the interaction terms, we can show the global existence of smooth solutions.

1. Introduction

We study the behavior of smooth solutions to the Cauchy problem for some hyperbolic operators in one space dimension. We consider $N \times N$ systems which are in the form

$$\partial_t u + A(u)\partial_x u = F(u),\ t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

with the initial condition

$$u(x, 0) = u_0(x),\ x \in \mathbb{R}. \tag{1.2}$$

Here $u = (u_1, \dots, u_N) \in \mathbb{R}^N$, $A(u)$ is a $N \times N$-matrix with smooth coefficients, $F(u)$ is a smooth vector function of the unknown u; u_0 is a sufficiently smooth function. Furthermore, we assume that the operator $\partial_t + A(u)\partial_x$ is strictly hyperbolic, i.e. the N eigenvalues of the matrix $A(u)$ are real and distinct,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u). \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary: 35L65.

Key words and phrases. Dissipative hyperbolic systems, linear degeneration, relaxation systems, semilinear behavior, Suliciu model, Kerr-Debye model.

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In the following we are going to assume always that the C^1-norm of u_0 is bounded:

$$\|u_0\|_{C^1(\mathbb{R})} := \|u_0\|_{L^\infty(\mathbb{R})} + \|u_0'\|_{L^\infty(\mathbb{R})} < +\infty. \tag{1.4}$$

Under these assumptions, it is well known that the Cauchy problem (1.1), (1.2) has a unique local (in time) smooth solution, see for instance [25, 22]. This solution can be globally defined, as for instance in the linear case. However, if the matrix A depends on u or F depends in a nonlinear way from u, singularities can appear in the solution, even for smooth initial data, in a finite time $T^*(u_0)$, the so-called blow-up time. The following results are now classical, see [1, 25]:

i) If $T^*(u_0) < +\infty$, then:

$$\begin{cases} \text{for all } \ t < T^*(u_0), \ \|u(t,\cdot)\|_{C^1(\mathbb{R})} < +\infty \\ \\ \text{and} \quad \sup\limits_{0 \le t < T^*(u_0)} \|u(t,\cdot)\|_{C^1(\mathbb{R})} = +\infty. \end{cases} \tag{1.5}$$

ii) If the system (1.1) is semilinear, i.e.: A does not depend on u, and $T^*(u_0) < +\infty$, then:

$$\begin{cases} \text{for all } \ t < T^*(u_0), \ \|u(t,\cdot)\|_{C^1(\mathbb{R})} < +\infty \\ \\ \text{and} \quad \sup\limits_{0 \le t < T^*(u_0)} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} = +\infty. \end{cases} \tag{1.6}$$

iii) If the system (1.1) is truly quasilinear, i.e.: A depends effectively on u, and $T^*(u_0) < +\infty$, then singularities have a different nature and shock waves can appear. Namely, the following situation is allowed:

$$\begin{cases} \text{for all } \ t < T^*(u_0), \ \|u(t,\cdot)\|_{C^1(\mathbb{R})} < +\infty, \\ \\ \sup\limits_{0 \le t < T^*(u_0)} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty \\ \\ \text{and} \quad \sup\limits_{0 \le t < T^*(u_0)} \|\partial_x u(t,\cdot)\|_{L^\infty(\mathbb{R})} = +\infty. \end{cases} \tag{1.7}$$

However, in the quasilinear case, (1.7) is not true for every system.

Definition 1.1. *We say the system* (1.1) *has a semilinear behavior if, for every smooth initial datum which satisfies* (1.4) *and such that* $T^*(u_0) < +\infty$*, we have that* (1.5) *implies* (1.6)*.*

Therefore, for a system with a semilinear behavior, shock waves cannot appear. Actually, for such a system, if for a local smooth solution, defined on an interval $[0, T[$, we have

$$\sup_{0 \le t < T} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty,$$

then

$$\sup_{0 \le t < T} \|\partial_x u(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty.$$

Let us now introduce the right and left eigenvectors of $A(u)$:

$$\begin{aligned} A(u)r_i(u) &= \lambda_i(u)r_i(u), \\ {}^tA(u)l_i(u) &= \lambda_i(u)l_i(u),\ i = 1, \dots, N. \end{aligned} \tag{1.8}$$

They depend smoothly on u and they are normalized such that

$${}^tl_i(u)r_j(u) = \delta_{ij},\ i, j = 1, , \dots, N,$$

where δ_{ij} is the standard Kronecker's symbol.

Following the classical definitions, first introduced by P.D. Lax [21], the i-characteristic field is genuinely nonlinear at $u \in \mathbb{R}^N$ if

$$\lambda_i'(u)r_i(u) \neq 0. \tag{1.9}$$

A characteristic field which is not genuinely nonlinear for all $u \in \mathbb{R}^N$ is called linearly degenerate. If this is the case for the i-field, then

$$\lambda_i'(u)r_i(u) \equiv 0. \tag{1.10}$$

Definition 1.2. *The system* (1.1) *is called totally linearly degenerate (TLD) if all of the characteristic fields of the matrix $A(u)$ are linearly degenerate.*

Consider the following problem, already proposed by Majda [25] and more recently by Brenier [7]: have the TLD systems the semilinear behavior?

In the following, we are going to investigate this problem for some relaxation approximation models to quasilinear hyperbolic systems (for an introduction to this topic see for instance [26, 5] and references therein). These models have the form (1.1). Even if most of the examples of relaxation approximations are written as semilinear systems, which trivially verify the conjecture, this is not the most general case. Recently some quasilinear relaxation approximations, which verify the TLD property, have been proposed as quite effective approximations for various hyperbolic systems, see [4, 5, 13]. These models yield numerical schemes such that the solution of the corresponding Riemann problem is quite simple, since only contact discontinuities are allowed. In this class, the most interesting example is given by the *Suliciu*-type relaxation model, which will be investigated in Section 3. Another interesting and more physically motivated model, is the Kerr-Debye relaxation system, see [9, 10, 11] and references therein, which arises in nonlinear optics and will be investigated in Section 4.

Let us now present a short review of the state of the art for the general case of TLD systems of the form (1.1). For $N = 2$ the situation is mostly clear, since in that case the systems are diagonalizable by Riemann invariants. In [28], it is proved that a 2×2 strictly hyperbolic TLD system has the semilinear behavior. Otherwise, if system (1.1) is homogeneous, namely $F \equiv 0$, and one of the two eigenvalues is genuinely nonlinear in one point, there exist C^∞ initial data with compact support, such that the corresponding solutions have shocks in finite time, see [25, 1]. Let us also point out that, according to a counterexample in [27], shocks can appear even for TLD 2×2 systems, if the strictly hyperbolicity assumption fails.

The situation for $N \geq 3$ is not yet completely understood. In the case of *homogeneous* diagonal TLD systems, the results obtained in [28] imply the global existence of smooth solutions for all initial data, so $T^*(u_0) = +\infty$ and there is nothing to prove. For the general (non diagonal) homogeneous TLD case, many results are known about the global existence of solution for small initial data, see [8, 20, 14] and references therein. However, from an example in [18], it is known that finite time blow-up of solutions can occur for some (suitably large) initial data. So, at least for these initial data, it is still possible to address the problem of the semilinear behavior.

For systems with a non vanishing source, both in the TLD and in the general case, all kinds of behavior are possible, since the source term can be sufficiently dissipative to avoid the formation of singularities and to yield global existence of smooth solutions for small initial data, see for instance [15]. On the other hand, for some choices of the source term, smooth solutions can blow up for all initial data.

In this paper we aim to investigate the semilinear behavior for some specific models with relaxation we mentioned before: the *Suliciu*-type and the *Kerr-Debye*-type models. These models are both written as strictly hyperbolic TLD models on a open domain in $\mathbb{R}^3$, with a partially dissipative source term.

The plan of our paper is as follows.

In Section 2, first we investigate the properties due to the linear degeneracy using the John's decomposition [19], which yield a fast conclusion on some quite academic examples. We also present the Suliciu and Kerr-Debye models.

The following section is devoted to the study of the Suliciu model. The semilinear behavior is obtained since the system is *rich* according to the definition in [29, 30]: there exists a regular change of variable which makes the system diagonal and the differential part has a conservative form. Therefore we can apply a general result: rich strictly hyperbolic TLD systems have the semilinear behavior. Let us remark that recently this paticular result has been independently obtained in [24].

The Kerr-Debye system is not rich, and so it does not fit in the previous framework. In the last section, we extend the previous results of [10], to deal with a more general class of TLD systems and for general source terms, by showing the semilinear behavior via energy methods. For the original Kerr-Debye model, thanks to the special form of the interaction terms, we can show the global existence of smooth solutions.

2. The John's decomposition and some examples

2.1. **The John's formula.** The John's formula, see [19] and also [17, 20], is a key ingredient for the study of singularities of systems of type (1.1). Using this formula it is possible to highlight the rôle of linear degeneration phenomena. To obtain the formula, we decompose the spatial gradient of u on the right eigenvectors of $A(u)$

$$\partial_x u = \sum_{j=1}^{N} p_j r_j(u), \text{ with } p_j = {}^t l_j \partial_x u. \tag{2.1}$$

Therefore, (1.1) reads

$$\partial_t u + \sum_{j=1}^{N} p_j \lambda_j(u) r_j(u) = F(u). \tag{2.2}$$

Differentiating (2.1) with respect to t and using (2.2) to evaluate $\partial_t u$, we obtain

$$\begin{aligned} \partial^2_{xt} u = \; & \sum_{j=1}^{N} \partial_t p_j r_j(u) - \sum_{j,k=1}^{N} p_j p_k \lambda_k(u) r'_j(u) r_k(u) \\ & + \sum_{j=1}^{N} p_j r'_j(u) F(u). \end{aligned} \tag{2.3}$$

On the other hand, we differentiate (2.2) with respect to x, to find

$$\begin{aligned} \partial^2_{tx} u = \; & - \sum_{j=1}^{N} \lambda_j(u) \partial_x p_j r_j(u) - \sum_{j=1}^{N} \lambda'_j(u) (\sum_{k=1}^{N} p_k r_k(u)) p_j r_j(u) \\ & - \sum_{j=1}^{N} \lambda_j(u) p_j r'_j(u) (\sum_{k=1}^{N} p_k r_k(u)) + F'(u) (\sum_{k=1}^{N} p_k r_k(u)). \end{aligned} \tag{2.4}$$

Finally, taking the scalar product (2.2) and (2.3) by the left eigenvalue $l_i(u)$, we find the John's formula:

$$\begin{aligned} \partial_t p_i + \lambda_i(u) \partial_x p_i \; & = - \sum_{k=1}^{N} p_i p_k \lambda'_i(u) r_k(u) \\ & + \sum_{j,k=1}^{N} (\lambda_k(u) - \lambda_j(u)) p_j p_k {}^t l_i(u) r'_j(u) r_k(u) \\ & + \sum_{k=1}^{N} p_k {}^t l_i(u) \left(F'(u) r_k(u) - r'_k(u) F(u) \right), \; i = 1, \dots, N. \end{aligned} \tag{2.5}$$

For homogeneous systems the last term vanishes. The first two terms are quadratic in $p = {}^t(p_1, \dots, p_N)$, with variable coefficients depending on u. The $F(u)$'s contribution is concentrated in the third term, which is linear in p, with coefficients depending on u. If the system (1.1) is diagonal, the decomposition reduces to

$$\partial_t p_i + \lambda_i(u) \partial_x p_i = - \sum_{k=1}^{N} p_i p_k \partial_{u_k} \lambda_i(u) + \sum_{k=1}^{N} p_k \partial_{u_k} F_i(u). \tag{2.6}$$

When the system (1.1) is TLD, there is no squared term in (2.5) or (2.6), i.e.: no term of the form p_i^2. It is well-know that this property plays a fundamental rôle in the analysis of semilinear hyperbolic problems with quadratic interactions, see for instance [33, 2]. We are going to see that in some simple examples, this is enough to conclude for the semilinear behavior.

2.2. **Some examples.** First, let us consider two homogeneous 2×2 TLD systems introduced by T.T. Li and F.G. Liu, which show a C^1 blow-up of solutions for some smooth initial data, see [23] and also [20]. Let us consider a system of the form

$$\partial_t u + A(u)\partial_x u = 0 \tag{2.7}$$

We take, for the first example,

$$A(u) = \begin{pmatrix} -1 & 0 & 0 \\ -e^{u_2} & 0 & 0 \\ -2e^{-u_2} & 0 & 1 \end{pmatrix}. \tag{2.8}$$

This system is strictly hyperbolic and TLD, since the eigenvalues are given by

$$\lambda_1 = -1 < \lambda_2 = 0 < \lambda_3 = +1. \tag{2.9}$$

The right and left eigenvalues are given, respectively, by

(2.10)

$$\begin{aligned} &r_1(u) = {}^t(1, e^{u_2}, e^{-u_2}), \quad r_2(u) = {}^t(0,1,0), \qquad r_3(u) = {}^t(0,0,1), \\ &l_1(u) = {}^t(1,0,0), \qquad\quad l_2(u) = {}^t(-e^{u_2},1,0), \quad l_3(u) = {}^t(-e^{-u_2},0,1). \end{aligned}$$

The corresponding John's decomposition is

$$\begin{cases} \partial_t p_1 - \partial_x p_1 &= 0, \\ \partial_t p_2 &= e^{u_2} p_1 p_2, \\ \partial_t p_3 + \partial_x p_3 &= -e^{-u_2} p_1 p_2. \end{cases} \tag{2.11}$$

The second example uses the matrix

$$A(u) = \begin{pmatrix} -1 & 0 & 0 \\ -(1+u_2^2) & 0 & 0 \\ u_2 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

The eigenvalues are still given by

$$\lambda_1 = -1 < \lambda_2 = 0 < \lambda_3 = +1, \tag{2.13}$$

and the right and left eigenvalues are given, respectively, by

(2.14)

$$\begin{aligned} &r_1(u) = {}^t(1, 1+u_2^2, -\tfrac{1}{2}u_2), \quad r_2(u) = {}^t(0,1,0), \qquad r_3(u) = {}^t(0,0,1), \\ &l_1(u) = {}^t(1,0,0), \qquad\quad l_2(u) = {}^t(-(1+u_2^2),1,0), \quad l_3(u) = {}^t(\tfrac{1}{2}u_2,0,1). \end{aligned}$$

The corresponding John's decomposition is now given by

$$\left\{\begin{array}{rcl}\partial_t p_1 - \partial_x p_1 & = & 0, \\ \\ \partial_t p_2 & = & 2u_2 p_1 p_2, \\ \\ \partial_t p_3 + \partial_x p_3 & = & -\frac{1}{2} u_2 p_1 p_2.\end{array}\right. \tag{2.15}$$

For these two examples it is easy to establish the semilinear behavior. More precisely, let $T > 0$ be such that $u_2 \in L^\infty([0,T[\times\mathbb{R})$, so that also $r_1(u) \in L^\infty([0,T[\times\mathbb{R})$. Therefore, by some straightforward computations in (2.11) or in (2.15), it is easy to see that $p_1, p_2,\ p_3 \in L^\infty([0,T[\times\mathbb{R})$ and then $\partial_x u \in L^\infty([0,T[\times\mathbb{R})$. On the other hand, it is possible to show, see [23], that the L^∞-norm of the considered smooth solution blows up in T^*.

Finally we consider a system introduced by A. Jeffrey [18]. The matrix $A(u)$ in system (2.7) is

$$A(u) = \begin{pmatrix} -\cosh(2u_2) & 0 & -\sinh(2u_2) \\ \\ \cosh(u_2) & 0 & \sinh(u_2) \\ \\ \sinh(2u_2) & 0 & \cosh(2u_2) \end{pmatrix}. \tag{2.16}$$

The eigenvalues are still given by

$$\lambda_1 = -1 < \lambda_2 = 0 < \lambda_3 = +1, \tag{2.17}$$

and the right and left eigenvalues are given, respectively, by

$$\begin{aligned} &r_1(u) = {}^t(-\cosh(u_2), 1, \sinh(u_2)), \quad r_2(u) = {}^t(0,1,0), \\ &r_3(u) = {}^t(-\sinh(u_2), 0, \cosh(u_2)), \\ &l_1(u) = {}^t(-\cosh(u_2), 0, \sinh(u_2)), \quad l_2(u) = {}^t(\cosh(u_2), 1, \sinh(u_2)), \\ &l_3(u) = {}^t(\sinh(u_2), 0, \cosh(u_2)). \end{aligned} \tag{2.18}$$

The corresponding John's decomposition is

$$\left\{\begin{array}{rcl}\partial_t p_1 - \partial_x p_1 & = & -2p_1p_3 - p_2p_3, \\ \\ \partial_t p_2 & = & 2p_1p_3 + p_2p_3, \\ \\ \partial_t p_3 + \partial_x p_3 & = & p_1p_2.\end{array}\right. \tag{2.19}$$

This system is semilinear and the right-hand side is a quadratic constant coefficients form. Therefore, we can use the Tartar's result in [33], which show the existence of global solutions for (2.19) for small initial data in $L^1(\mathbb{R})$ (for p). For this system, however, Jeffrey has shown in [18] the blow-up of smooth solution in finite time, at least for some special (large) initial data.

Here we want to show, using the methods introduced in [2], the existence of blow-up solutions to system (2.19) and then to obtain by a different method, the break-down of solutions to the Jeffrey's model. We look for a solution $\varphi(t,x)$ to (2.19) in the form of a polarized traveling profile

$$\varphi(t,x) = \psi(x-ct)^t(\lambda_1, \lambda_2, \lambda_3).$$

Take $a \neq 0$, $c \neq 0, +1, -1$, and $\psi(\xi) = \dfrac{a}{\xi^* - \xi}$ for some fixed value ξ^*. The function

$$\varphi(t,x) = \psi(x-ct)^t(\pm\frac{c}{a}, \mp\frac{c+1}{a}, \frac{c(c+1)}{a(c-1)}) \tag{2.20}$$

is a solution to (2.19) out of the set $\{(x,t),\ x-ct = \xi^*\}$. To yield an actual solution to (2.19) corresponding to a given Cauchy datum $p(0,x) = p_0(x)$, we use the finite speed of propagation. Choose for instance

$$\begin{cases} c > 2,\ \xi^* = -2, \\ \\ p_0 \in C_0^\infty(]-2,+2[),\ p_0(x) = \psi(x)\lambda \text{ for } |x| \leq 1. \end{cases} \tag{2.21}$$

Then, the unique solution $p(t,x)$ to (2.19) with initial condition p_0 is such that

$$p(t,x) = \varphi(t,x)$$

for $t \geq 0$, $t \leq 1+x$, $t \leq 1-x$, and $t \leq \frac{x+2}{c}$. Therefore, p blows up at $T^* \leq \tilde{T} := \frac{1}{c-1}$.

Next, we construct $u_0 \in (C_0^\infty(]-2,+2[))^3$ such that

$$\partial_x u_0 = p_{01} r_1(u_0) + p_{02} r_2(u_0) + p_{03} r_3(u_0). \tag{2.22}$$

With this object we can modify p_0 on $]-2,2[\setminus[-1,1]$ preserving (2.21) such that $\displaystyle\int_{\mathbb{R}} (p_{01}(\xi) + p_{02}(\xi))d\xi = 0$ so that

$$u_{02}(x) = \int_{-\infty}^{x} (p_{01}(\xi) + p_{02}(\xi))d\xi \in \mathcal{C}_0^\infty(\mathbb{R}).$$

We fix now p_{01} and p_{03} such that

$$\int_{\mathbb{R}} (\cosh u_{02}(\xi) p_{01}(\xi) + \sinh u_{02}(\xi) p_{03}(\xi))d\xi = 0,$$

and

$$\int_{\mathbb{R}} (\sinh u_{02}(\xi) p_{01}(\xi) + \cosh u_{02}(\xi) p_{03}(\xi))d\xi = 0,$$

so that u_0 defined integrating (2.22) is compactely supported in $[0,2]$.

Hence, the smooth solution of the Cauchy problem for (2.7) with $A(u)$ given by (2.16), blows up in $T^* \leq \tilde{T}$. Let us notice however that, by this argument, we do not know if also u blows up in T^*. Then, the problem of the semilinear behavior for the Jeffrey's model stays unsolved.

2.3. **The Suliciu model.** The original model proposed by Suliciu in [32], was a semilinear approximation to the following one dimensional p-system arising in elasticity

$$\begin{cases} \partial_t u_1 - \partial_x u_2 &= 0, \\ \partial_t u_2 - \partial_x (p(u_1)) &= 0, \; p'(u_1) > 0. \end{cases}$$

The viscoelastic Suliciu approximation is

$$\begin{cases} \partial_t u_1 - \partial_x u_2 &= 0, \\ \partial_t u_2 - \partial_x v &= 0, \\ \partial_t v - \mu \partial_x u_2 &= \frac{1}{\varepsilon}(p(u_1) - v), \end{cases}$$

where μ is a positive constant and $\varepsilon << 1$ is the relaxation parameter. Many authors have investigated the convergence properties of this model when $\varepsilon \to 0$, see [12] and references therein. More recently a similar model has been proposed for the approximation of the system of isentropic gas dynamics in Eulerian coordinates

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) &= 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) &= 0. \end{cases} \tag{2.23}$$

Here, $\rho \geq 0$ is the density of the gas, u its fluid velocity and the pressure function $p = p(\rho)$ satisfies $p'(\rho) > 0$. In [5] and [13], the following Suliciu relaxation approximation was introduced:

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) &= 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + \pi) &= 0, \\ \partial_t (\rho \pi) + \partial_x (\rho u \pi + c^2 u) &= \frac{1}{\varepsilon} \rho (p(\rho) - \pi), \end{cases} \tag{2.24}$$

for a constant $c > 0$ to be fixed later.

Set $U = (\rho, u, \pi)$ and $\Omega = \{U \in \mathbb{R}^3; \rho > 0\}$. It is easy to see that the system (2.24) can be written in the standard form

$$\partial_t U + A(U) \partial_x U = F(U), \tag{2.25}$$

with A defined in Ω and given by

$$A(U) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & c^2/\rho & u \end{pmatrix} \tag{2.26}$$

and

$$F(U) = {}^t(0, 0, \frac{1}{\varepsilon}(p(\rho) - \pi)). \tag{2.27}$$

Notice that, when the source term for (2.24) is completely general, namely it is given by

$$f(U) = {}^t(f_1(U), f_2(U), f_3(U)),$$

the interaction in (2.25) reads

$$F(U) = {}^t(f_1(U), \frac{1}{\rho}(f_2(U) - uf_1(U)), \frac{1}{\rho}(f_3(U)) - \pi f_1(U)).$$

This system (2.25) is strictly hyperbolic and TLD in Ω. Indeed, its eigenvalues are given by

$$\lambda_1(U) = u - \frac{c}{\rho} < \lambda_2(U) = u < \lambda_3(U) = u + \frac{c}{\rho}, \tag{2.28}$$

and the right and left eigenvectors are given, respectively, by

$$r_1(U) = {}^t(1, -\frac{c}{\rho^2}, \frac{c^2}{\rho^2}),\ r_2(U) = {}^t(1,0,0),\ r_3(U) = {}^t(1, \frac{c}{\rho^2}, \frac{c^2}{\rho^2}) \tag{2.29}$$

and

$$l_1(U) = {}^t(0, -\frac{\rho^2}{2c}, \frac{\rho^2}{2c^2}),\ l_2(U) = {}^t(1, 0, -\frac{\rho^2}{c^2}),\ l_3(U) = {}^t(0, \frac{\rho^2}{2c}, \frac{\rho^2}{2c^2}). \tag{2.30}$$

Let

$$G(U,p) = \sum_{k=1}^{3} p_k \left(F'(U) r_k(U) - r_k'(U) F(U) \right).$$

The John's decomposition of system (2.26) is

$$\begin{cases} \partial_t p_1 + \lambda_1(U)\partial_x p_1 &= -\frac{3c}{\rho^2} p_1 p_2 - \frac{6c}{\rho^2} p_1 p_3 + {}^t l_1(U) G, \\ \partial_t p_2 + \lambda_2(U)\partial_x p_2 &= \frac{c}{\rho^2}(p_1 p_2 + p_2 p_3) + {}^t l_2(U) G, \\ \partial_t p_3 + \lambda_3(U)\partial_x p_3 &= \frac{6c}{\rho^2} p_1 p_3 + \frac{3c}{\rho^2} p_2 p_3 + {}^t l_3(U) G. \end{cases} \tag{2.31}$$

It is not easy to deduce the semilinear behavior directly from (2.31). To show this property we are going to use in Section 3 the definition of *richness* of a systems. According to [30], see also [29], a strictly hyperbolic system is *rich* if it has a conservative form and it is diagonalizable along its Riemann invariants.

Proposition 2.1. *The Suliciu model* (2.25), (2.26) *is a rich system in* Ω.

Proof. The Riemann invariants for the Suliciu model are

$$w_1 = \pi - cu,\ w_2 = \frac{1}{\rho} + \frac{\pi}{c^2},\ w_3 = \pi + cu. \tag{2.32}$$

We have also

$$\begin{array}{lll} w_1' r_1 = \frac{2c^2}{\rho^2}, & w_1' r_2 = 0, & w_1' r_3 = 0; \\ w_2' r_1 = 0, & w_2' r_2 = -\frac{1}{\rho^2}, & w_2' r_3 = 0; \\ w_3' r_1 = 0, & w_3' r_2 = 0, & w_3' r_3 = \frac{2c^2}{\rho^2}. \end{array}$$

Therefore w_1 is a 2,3-Riemann invariant, w_2 is a 1,3-Riemann invariant, w_3 is a 1,2-Riemann invariant. The map (2.32) defines a diffeomorphism from Ω to $\Omega_1 := \{W \in \mathbb{R}^3; 2c^2 w_2 - w_1 - w_3 > 0\}$, and it holds

$$\rho = 2c^2(2c^2 w_2 - w_1 - w_3)^{-1}, \; u = \frac{1}{2c}(w_3 - w_1), \; \pi = \frac{1}{2}(w_1 + w_3). \tag{2.33}$$

In the new unknown $W = (w_1, w_2, w_3)$, the system has the diagonal form

$$\partial_t W + \Lambda(W)\partial_x W = F_1(W), \tag{2.34}$$

where

$$\begin{aligned} \Lambda(W) &= \begin{pmatrix} \lambda_1(W) & 0 & 0 \\ 0 & \lambda_2(W) & 0 \\ 0 & 0 & \lambda_3(W) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{c}(w_3 - c^2 w_2) & 0 & 0 \\ 0 & \frac{1}{2c}(w_3 - w_1) & 0 \\ 0 & 0 & -\frac{1}{c}(w_1 - c^2 w_3) \end{pmatrix}. \end{aligned} \tag{2.35}$$

The system is strictly hyperbolic in Ω_1 and, thanks to (2.34), also rich. □

2.4. **The Kerr-Debye model.** The Kerr-Debye model is a relaxation approximation of the Kerr model in nonlinear optics [34]. Recall that the propagation of the electromagnetic waves is described by the Maxwell equations

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t B + \text{curl } E = 0, \\ \text{div } D = \text{div } B = 0. \end{cases}$$

The Kerr model describes an instantaneous response of the medium, where the constitutive relations read

$$B = \mu_0 H, \; D = \varepsilon_0(1 + \varepsilon_r |E|^2)E.$$

The Kerr-Debye model describes a delayed response of the medium, by the constitutive relations

$$B = \mu_0 H, \; D = \varepsilon_0(1 + \chi)E,$$

where χ solves the equation

$$\partial_t \chi = \frac{1}{\tau}(\varepsilon_r |E|^2 - \chi),$$

the constant $\tau > 0$ being a delay time. The analytical convergence of the Kerr-Debye to the Kerr model has been investigated in [16] for the Cauchy problem and in [9, 10, 11] for the initial-boundary value problem. Following

[9], we deal with the following one dimensional version of the Kerr-Debye model

$$\left\{\begin{array}{rcl}\partial_t d + \partial_x h &=& 0,\\ \partial_t h + \partial_x e &=& 0,\\ \partial_t \chi &=& \dfrac{1}{\varepsilon}(e^2-\chi),\end{array}\right. \tag{2.36}$$

with $d = (1+\chi)e$. If the initial condition

$$(d,h,\chi)(0,x) = (d_0,h_0,\chi_0)(x) \tag{2.37}$$

is such that $\chi_0 \geq 0$, then for every positive time, where the solution is defined, we still have $\chi(t,x) \geq 0$, and we can replace e by $(1+\chi)^{-1}d$ in system (2.36). So, setting $u = (d,h,\chi)$ and $\Omega = \{u \in \mathbb{R}^3; \chi \geq -1\}$, we can rewrite the system (2.36) as

$$\partial_t u + A(u)\partial_x u = F(u), \tag{2.38}$$

where $A(u)$ is given in Ω by

$$A(u) = \begin{pmatrix} 0 & 1 & 0 \\ (1+\chi)^{-1} & 0 & -(1+\chi)^{-2}d \\ 0 & 0 & 0 \end{pmatrix} \tag{2.39}$$

and

$$F(u) = {}^t(0,0,\frac{1}{\varepsilon}((1+\chi)^{-2}d - \chi)). \tag{2.40}$$

This system is strictly hyperbolic and TLD since its eigenvalues are given by

$$\lambda_1(u) = -(1+\chi)^{-\frac{1}{2}} < \lambda_2(u) = 0 < \lambda_3(u) = (1+\chi)^{-\frac{1}{2}}, \tag{2.41}$$

and the right and left eigenvectors are given, respectively, by

$$\begin{aligned} &r_1(u) = {}^t(1,-(1+\chi)^{-\frac{1}{2}},0), \;\; r_2(u) = {}^t((1+\chi)^{-1}d,0,1),\\ &r_3(u) = {}^t(1,(1+\chi)^{-\frac{1}{2}},0),\end{aligned} \tag{2.42}$$

and

$$\begin{aligned} &l_1(u) = \tfrac{1}{2}{}^t(1,-(1+\chi)^{1/2},-(1+\chi)^{-1}d), \;\; l_2(u) = {}^t(0,0,1),\\ &l_3(u) = \tfrac{1}{2}{}^t(1,(1+\chi)^{1/2},-(1+\chi)^{-1}d).\end{aligned} \tag{2.43}$$

Let G be given by

$$G(U,p) = \sum_{k=1}^{3} p_k \left(F'(U)r_k(U) - r_k'(U)F(U)\right).$$

The John's formula for the Kerr-Debye model is given by
(2.44)
$$\begin{cases} \partial_t p_1 + \lambda_1(U)\partial_x p_1 = & -\frac{5}{4}(1+\chi)^{-\frac{3}{2}} p_1 p_2 + \frac{1}{4}(1+\chi)^{-\frac{3}{2}} p_2 p_3 + {}^t l_1(U) G, \\ \partial_t p_2 = & {}^t l_2(U) G, \\ \partial_t p_3 + \lambda_3(U)\partial_x p_3 = & -\frac{1}{4}(1+\chi)^{-\frac{3}{2}} p_1 p_2 + \frac{5}{4}(1+\chi)^{-\frac{3}{2}} p_2 p_3 + {}^t l_3(U) G. \end{cases}$$

Remark 2.1. *Unlike the Suliciu model, the Kerr-Debye model is not a rich system. There are no functions which are at the same time the Riemann invariants for the first two characteristic fields. In Section 4, we establish the semilinear behavior for a generalized version of the Kerr-Debye system, using energy estimates. Moreover, for the original Kerr-Debye system, we are also able to prove global existence for smooth initial data.*

3. The Suliciu model

3.1. **Semilinear behavior of rich systems.** Consider the $N \times N$ system (1.1) and assume it is strictly hyperbolic and rich. So, such a system can be written in a diagonal form as

$$\partial_t W + \Lambda(W)\partial_x W = G(W), \tag{3.1}$$

where $\Lambda(W) = diag(\lambda_1(W), \ldots, \lambda_N(W))$, and

$$\lambda_1(W) < \lambda_2(W) < \cdots < \lambda_N(W).$$

Moreover, according to [30, 29], the following relations are always verified (set $\partial_i := \partial_{w_i}$):

$$\partial_k \frac{\partial_i \lambda_j}{\lambda_i - \lambda_j} = \partial_i \frac{\partial_k \lambda_j}{\lambda_k - \lambda_j}, \; i \neq j \neq k. \tag{3.2}$$

From these relations there exist N smooth functions $\alpha_j(W)$ such that:

$$\partial_i \alpha_j = \frac{\partial_i \lambda_j}{\lambda_i - \lambda_j}, \; i \neq j. \tag{3.3}$$

If moreover we assume that the system is TLD, then, in the diagonal form (3.1) we have

$$\partial_i \lambda_i = 0, \; i = 1, \ldots, N. \tag{3.4}$$

For this kind of systems the John's decomposition is specially simple and effective. Thanks to (3.3), we can prove the following result.

Theorem 3.1. *A strictly hyperbolic system with source term, which is both rich and TLD, has the semilinear behavior.*

Proof. We consider the Cauchy problem for the system (3.1), with a smooth initial condition W_0, which is bounded in C^1. We assume that the local smooth solution is defined and bounded on $[0, T[\times\mathbb{R}$:

$$\text{there exists } C > 0 \text{ such that: } |W(t,x)| \leq C, \; 0 \leq t < T, \; x \in \mathbb{R}. \tag{3.5}$$

To show the semilinear behavior it suffices to show that $p = \partial_x W$ is also bounded on the same strip $[0, T[\times\mathbb{R}$. Since the system is TLD, using (3.4), we can show that $p = {}^t(p_1, \ldots, p_N) = {}^t(\partial_x w_1, \ldots, \partial_x w_N)$ is a solution to the Cauchy problem

(3.6)

$$\partial_t p_i + \lambda_i(W)\partial_x p_i + \left(\sum_{j\neq i} \partial_j \lambda_i(W) p_j\right) p_i = \sum_k \partial_k G_i(W) p_k, \ i = 1, \ldots, N,$$

$$p(0, x) = \partial_x W_0(x). \tag{3.7}$$

Now, for $j \neq i$, we have

$$p_j = \big(\lambda_i - \lambda_j\big)^{-1}\big(\partial_t w_j + \lambda_i(W)\partial_x w_j - G_j(W)\big)\,.$$

So, using (3.3) we obtain

$$\begin{aligned}\left(\sum_{j\neq i} \partial_j \lambda_i(W) p_j\right) &= -\sum_{j\neq i} \partial_j \alpha_i(W)(\partial_t w_j + \lambda_i(W)\partial_x w_j - G_j(W))) \\ &= -(\partial_t + \lambda_i(W)\partial_x)\alpha_i(W) + \sum_j \partial_j \alpha_i(W) G_j(W).\end{aligned}$$

Inserting this equality in (3.6), we obtain

(3.8)

$$\begin{aligned}(\partial_t + \lambda_i(W)\partial_x) p_i \quad &-[(\partial_t + \lambda_i(W)\partial_x)\alpha_i(W)]p_i \\ &= -\left(\sum_j \partial_j \alpha_i(W) G_j(W)\right) p_i + \sum_k \partial_k G_i(W) p_k.\end{aligned}$$

So, setting $q_i := e^{-\alpha_i(W)} p_i$, the function q is the solution to the Cauchy problem

(3.9)

$$\begin{aligned}\partial_t q_i + \lambda_i(W)\partial_x q_i \quad &= -\left(\sum_j \partial_j \alpha_i(W) G_j(W)\right) q_i \\ &+ e^{-\alpha_i(W)}\left(\sum_k \partial_k G_i(W) e^{\alpha_k(W)} q_k\right), \ i = 1, \ldots, N,\end{aligned}$$

$$q_i(0, x) = e^{-\alpha_i(W_0)}\partial_x w_{0i}(x), \ i = 1, \ldots, N. \tag{3.10}$$

The system (3.9) is a diagonal linear system with smooth and bounded coefficients on the strip $[0, T[\times\mathbb{R}$. Therefore the function q, and then also $p = \partial_x W$, is bounded on the same strip as required.

□

Remark 3.1. *For $N = 2$, we recover the results in* [28]*, since in this case the conditions* (3.2) *are empty and so trivially verified. On the other hand, for $N \geq 3$, these conditions appear to be quite restrictive.*

Remark 3.2. *In Theorem 3.1, the assumptions are taken globally on* $\mathbb{R}^N$*. However, in many cases, it should be possible to restrict the analysis to an open bounded domain of* $\mathbb{R}^N$*.*

3.2. **Semilinear behavior of the Suliciu model.** Let us focus now on the system (2.24), but for a generic source term. In the unknown $U = (\rho, u, \pi)$, we have (2.25), where A is given by (2.26) and F is a smooth interaction term. We write the system for the Riemann invariants $W = {}^t(w_1, w_2, w_3)$ given by (2.32), which is

$$\partial_t W + \Lambda(W)\partial_x W = G(W), \tag{3.11}$$

where $\Lambda(W)$ is given by (2.35). This system is strictly hyperbolic in $\Omega_1 = \{W \in \mathbb{R}^3; 2c^2 w_2 - w_1 - w_3 > 0\}$, since we have

$$\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2 = \frac{c}{\rho} = \frac{1}{2c}(2c^2 w_2 - w_1 - w_3).$$

It is easy to see that the system is rich and TLD in the same domain and the functions α_j, for $j = 1, 2, 3$ are explicitly given by

$$\alpha_1 = \alpha_2 = \alpha_3 = -\log(2c^2 w_2 - w_1 - w_3). \tag{3.12}$$

Therefore, we can write the system (3.9) and come back to the original variables if

$$2c^2 w_2 - w_1 - w_3 = \frac{2c^2}{\rho} > 0. \tag{3.13}$$

Let us also observe that, for the original variables, the matrix $A(U)$ is just defined and strictly hyperbolic in the domain $\Omega = \{U \in \mathbb{R}^3; \rho > 0\}$. Now, following [25], the loss of regularity for the local smooth solutions can be stated in the following more precise form: If $T^*(U_0) < +\infty$, then

i) $\sup_{0 \le t < T^*} \left(\|\partial_t U\|_{L^\infty(\mathbb{R})} + \|\partial_x U\|_{L^\infty(\mathbb{R})}\right) = +\infty$
or
ii) for every compact set $\mathcal{K} \subset\subset \Omega$, $U(t)$ escapes from $\mathcal{K}$ as $t \nearrow T^*$.

To establish the semilinear behavior, we have to avoid the singularity coming out from the vanishing of the density ρ. Hence, we are going to assume that our local smooth solutions of Cauchy problem for equation (2.25), is defined and bounded on the strip $[0, T[\times\mathbb{R}$ and moreover

$$\exists \bar{\rho} > 0 \text{ such that } \rho(t, x) \ge \bar{\rho},\ 0 \le t < T,\ x \in \mathbb{R}. \tag{3.14}$$

Therefore, the function W given by (2.32), satisfies (3.5); actually $W(t, x)$ is in a compact set of Ω_1 for $0 \le t < T,\ x \in \mathbb{R}$. Then, thanks to (3.9), $\partial_x W$ is bounded in $[0, T[\times\mathbb{R}$, and the same is true for $\partial_x U$. So, we have proved the following result.

Theorem 3.2. *For the Suliciu model* (2.25)*, with* $A(U)$ *given by* (2.27) *and* F *being a smooth source term, let* U_0 *be an initial data such that* $T^*(U_0) <$

$+\infty$. *Assume that there exists* $\bar{\rho} > 0$ *such that* $\rho(t,x) \geq \bar{\rho}$ *in* $[0, T^*[\times\mathbb{R}$. *Then*

$$\sup_{0 \leq t < T*} \|U(t,\cdot)\|_{L^\infty(\mathbb{R})} = +\infty.$$

According to Theorem 3.2, the smooth solutions to Suliciu model cannot develop shocks, as far as the density ρ is strictly positive. However, it is still possible to wonder about the existence of global solutions, and in particular when the source term is given by the relaxation term (2.27) and, for some fixed interval I in $\mathbb{R}$, the subcharacteristic condition

$$\forall \xi \in I, \ \xi^2 p'(\xi) < c^2, \tag{3.15}$$

holds. This condition has been introduced in [4] to guarantee the dissipativity of the system (see also the next Subsection 3.3). Nevertheless, even under these conditions, it is possible to show the blow-up of smooth solutions in finite time.

More precisely, if we take the pressure law

$$p(\rho) = -\frac{c_0^2}{\rho}, \tag{3.16}$$

with the interaction term given by (2.27), the subcharacteristic condition (3.15) holds on $I =]0, +\infty[$ for $c_0^2 < c^2$. Following an example given by F. Bouchut [6], we are going to show the existence of a smooth solution for this system, such that $\rho(t,x) \geq \bar{\rho} > 0$, and which nevertheless blows up in finite time.

let us rewrite system (2.24) in the Lagrangian coordinates, still denoted by (t,x). We have

$$\left\{\begin{array}{rcl} \partial_t(\frac{1}{\rho}) - \partial_x u &=& 0, \\ \partial_t u + \partial_x \pi &=& 0, \\ \partial_t \pi + c^2 \partial_x u &=& \dfrac{1}{\varepsilon}(p(\rho) - \pi), \end{array}\right. \tag{3.17}$$

When $p(\rho)$ is given by (3.16), this system is linear in the unknowns $(\frac{1}{\rho}, u, \pi)$. We are going to find a solution to (3.17) such that $\frac{1}{\rho}$ is bounded, but vanishes in finite time, which then implies that $\rho \to +\infty$.

Let us look for a solution of the form

$${}^t(\frac{1}{\rho}, u, \pi) = e^{ikx}\varphi(t) + {}^t(\frac{1}{\rho_0}, 0, -\frac{c_0^2}{\rho_0}). \tag{3.18}$$

This yields

$$\varphi' + \begin{pmatrix} 0 & -ik & 0 \\ 0 & 0 & ik \\ \frac{c_0^2}{\varepsilon} & ic^2 k & \frac{1}{\varepsilon} \end{pmatrix} \varphi = 0. \tag{3.19}$$

Choosing $c_0 = \frac{c}{3}$, $k^2c^2 = \frac{1}{3\varepsilon^2}$, $\lambda = \frac{1}{3\varepsilon}$ is a triple eigenvalue of the matrix of the system (3.19). Therefore, for every fixed constant $\beta > 0$, the vector

$$\begin{cases} \dfrac{1}{\rho(t,x)} = \dfrac{1}{\rho_0} + i\beta k e^{ikx} e^{-\frac{t}{3\varepsilon}} t(1 + \dfrac{t}{3\varepsilon}), \\ \\ u(t,x) = \beta e^{ikx} e^{-\frac{t}{3\varepsilon}} (1 + \dfrac{t}{3\varepsilon} - \dfrac{t^2}{9\varepsilon^2}), \\ \\ \pi(t,x) = -\dfrac{c_0^2}{\rho_0} + i\beta c^2 k e^{ikx} e^{-\frac{t}{3\varepsilon}} t(-1 + \dfrac{t}{9\varepsilon}), \end{cases} \tag{3.20}$$

is a solution to (3.17) for the initial condition ${}^t(\dfrac{1}{\rho_0}, \beta e^{ikx}, -\dfrac{c_0^2}{\rho_0})$. Fixing $\rho_0 > 0$, it is possible to take β large enough to obtain that the real part of $\frac{1}{\rho}$ vanishes in finite time.

3.3. **Other properties of the Suliciu model.** Next we aim to show that the Suliciu model (2.24) fits in the general framework of the partially dissipative hyperbolic systems with a strictly convex entropy function, which have been recently investigated in [15] and [3]. Therefore, following [13], we modify the source term (assuming that the subcharacteristic condition (3.15) is always verified) in order to construct a regular entropy function. We have that the function $h(\xi) = p(\xi) + \frac{c^2}{\xi}$ is invertible for $\xi \in I$ and so we can set, for $\pi + \frac{c^2}{\rho} \in h(I)$,

$$\hat{\rho} = h^{-1}(\pi + \frac{c^2}{\rho}). \tag{3.21}$$

Notice that, ρ and $\hat{\rho}$ are connected by the relation

$$\pi + \frac{c^2}{\rho} = p(\hat{\rho}) + \frac{c^2}{\hat{\rho}}. \tag{3.22}$$

In (2.24), we replace the source term $\frac{1}{\varepsilon}\rho(p(\rho) - \pi)$ by $\frac{1}{\varepsilon}\rho(p(\hat{\rho}) - \pi)$, so that the system (2.25) reads now

$$\partial_t U + A(U)\partial_x U = F(U), \tag{3.23}$$

with $A(U)$ given by (2.26) and

$$F(U) = {}^t(0, 0, \frac{1}{\varepsilon}(p(\hat{\rho}) - \pi)). \tag{3.24}$$

This new system has the same properties we have studied in Subsection 3.2, and shares with the original Suliciu model also the equilibrium manifold. This follows by considering that the equilibrium manifold is given by $F(U) = 0$, which is equivalent to $p(\hat{\rho}) = \pi$. Therefore, using (3.22), we have $\frac{c^2}{\rho} = \frac{c^2}{\hat{\rho}}$, which implies $\hat{\rho} = \rho$.

Fixing the relaxation parameter ε, we can show the global existence of smooth solutions at least for initial data which are small perturbations of constant equilibrium states.

Theorem 3.3. *Let $\bar{U} = (\bar{\rho} > 0, \bar{u}, \bar{\pi} = p(\bar{\rho}))$ a constant state belonging to the equilibrium manifold of the system* (3.23)*, with $A(U)$ given by* (2.26) *and $F(U)$ by* (3.24)*. Let U_0 be a smooth perturbation of $\bar{U}$. There exists $\delta > 0$ such that, if $\|U_0 - \bar{U}\|_{H^2(\mathbb{R})} \leq \delta$, then there exists a global smooth solution U to system* (3.23) *corresponding to the initial condition U_0 and*

$$U - \bar{U} \in C^0([0, +\infty[; H^2(\mathbb{R})) \cap C^1([0, +\infty[; H^1(\mathbb{R})).$$

Proof. The Theorem follows from Theorem 1 in [15], by proving that the system (3.23) has *i)* a strictly dissipative entropy according to Definition 2 in [15] and *ii)* the Shizuta-Kawashima condition [31] holds.

i) Let us rewrite (3.23) for the conservative variables $U : (\rho, v, w) := (\rho, \rho u, \rho\pi)$. Let φ be the function defined by

$$\varphi'(Y) = -\frac{p(\hat{\rho}(Y))}{c^2}. \tag{3.25}$$

First we show that the function $\mathcal{E}$, given by

$$\mathcal{E}(U) = \frac{v^2}{2\rho} + \frac{w^2}{2c^2\rho} + \rho\varphi\left(\frac{c^2 + w}{\rho}\right), \tag{3.26}$$

is a strictly convex entropy for (3.23) and

$$\partial_t \mathcal{E}(U) + \partial_x (\rho u \mathcal{E}(U) + \pi u) = -\frac{\rho}{\varepsilon c^2} (\pi - p(\hat{\rho}))^2. \tag{3.27}$$

Set $X = \frac{u^2}{2} + \frac{\pi}{2c^2}$ and $Y = \frac{c^2}{2} + \pi$. Then $\mathcal{E}(U) = \rho(X + \varphi(Y))$ and we have

$$\partial_t \mathcal{E}(U) + \partial_x (\rho u \mathcal{E}(U) + \pi u) = -\frac{\rho}{\varepsilon c^2}(\pi - p(\hat{\rho}))(\pi + c^2\varphi'(Y)),$$

Then (3.25) implies (3.27). Concerning the strict convexity, let us notice that, if the function φ is convex in Y, then $\mathcal{E}$ is also strictly convex. Now, thanks to the subcharacteristic condition (3.15), we find that

$$\varphi''(Y) = -\frac{p'(\hat{\rho}(Y))}{c^2\left(p'(\hat{\rho}(Y)) - \frac{c^2}{(\hat{\rho}(Y))^2}\right)} > 0.$$

We also have that the entropy function $\mathcal{E}$ is dissipative, since

$$\begin{aligned} &(\mathcal{E}'(\rho, v, w) - \mathcal{E}'(\bar{\rho}, \bar{v}, \bar{w}))\, F(\rho, v, w) \\ &= \mathcal{E}'(\rho, v, w) F(\rho, v, w) = -\tfrac{\rho}{\varepsilon c^2} (\pi - p(\hat{\rho}))^2 \leq 0. \end{aligned} \tag{3.28}$$

Finally, we show that $\mathcal{E}$ is a strictly dissipative entropy. Following [15], let us introduce the entropy variable

$$W := \mathcal{E}'(\rho, v, w) = (U_1, U_2, V). \tag{3.29}$$

So the condition holds, since

$$F(W) = {}^t(0, 0, -\frac{c^2}{\varepsilon} V). \tag{3.30}$$

ii) To check the Shizuta-Kawashima condition, we have to verify that the eigenvalues of $A(\bar{U})$ are not belonging to the kernel of $F'(\bar{U})$, where F is given by (3.24).

The right eigenvectors r_i are given by (2.29), so, from

$$F'(\bar{U})r_1 = F'(\bar{U})r_3 = {}^t(0, 0, -\frac{c^2}{\varepsilon\bar{\rho}^2}) \neq 0,$$

and

$$F'(\bar{U})r_2 = {}^t(0, 0, -\frac{c^2}{\varepsilon\bar{\rho}^2}p'(\bar{\rho})\hat{\rho}'(\bar{y})) \neq 0,$$

we deduce that the Shizuta-Kawashima condition holds. □

Remark 3.3. *Thanks to Theorem 1 in* [15], *we can estimate the entropy variable* (3.29) *as follows (for* $\varepsilon = 1$*):*

(3.31)

$$\sup_{0\leq t<+\infty} \|U - \bar{U}\|^2_{H^2(\mathbb{R})} + \int_0^\infty \|U_1(\tau)\|^2_{H^1(\mathbb{R})} + \|U_2(\tau)\|^2_{H^1(\mathbb{R})} + \|V(\tau)\|^2_{H^2(\mathbb{R})} d\tau \leq C\|U_0 - \bar{U}_0\|^2_{H^2(\mathbb{R})},$$

for some positive constant C*. Also, it is possible to describe the asymptotic behavior for large times of the smooth global solutions, see* [3] *for more details.*

4. The Kerr-Debye Model

4.1. **Semilinear behavior for a generalized Kerr-Debye model.**

In [10] we proved the semilinear behavior for the Kerr-Debye system in the one-dimensional case. The proof is based on a careful choice of variables: we rewrite Kerr-Debye system in the variables $U = (e, h, \chi)$, and we obtain

$$(4.1) \qquad \begin{cases} (1+\chi)\partial_t e + \partial_x h &= -e\partial_t\chi = -\frac{1}{\varepsilon}e(e^2 - \chi), \\ \partial_t h + \partial_x e &= 0, \\ \partial_t\chi &= \frac{1}{\varepsilon}(e^2 - \chi). \end{cases}$$

Here, we study the following generalization of the Kerr-Debye system

$$(4.2) \qquad \begin{cases} A_0(\chi)\partial_t u + A_1\partial_x u &= \varphi(v), \\ \partial_t\chi &= \psi(v), \end{cases}$$

with the initial condition

$$(4.3) \qquad (u(0, x), \chi(0, x)) = (u_0(x), \chi_0(x)) := v_0(x).$$

We make the following assumptions. The unknown $v = (u, \chi)$ takes its values in $\mathbb{R}^{n-r} \times \mathbb{R}^r$. $A_0 \in \mathcal{C}^\infty(\mathbb{R}^r; \mathcal{M}_{n-r}(\mathbb{R}))$, and for all $\chi \in \mathbb{R}^r$, $A_0(\chi)$ is a symmetric positive definite $(n-r) \times (n-r)$-matrix such that

$$(4.4) \qquad \exists\alpha > 0, \forall\chi \in \mathbb{R}^r, \forall\xi \in \mathbb{R}^{n-r}, A_0(\chi)\xi \cdot \xi \geq \alpha\|\xi\|^2.$$

A_1 is a symmetric invertible $(n-r) \times (n-r)$-matrix. The function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^{n-r})$ with $\varphi(0) = 0$ and $\psi \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^r)$ with $\psi(0) = 0$. We denote $\Phi = (\varphi, \psi)$. The initial condition $v_0 \in H^2(\mathbb{R})$.

Remark 4.1. *As the original Kerr-Debye system, the system (4.2) is totally linearly degenerate.*

We have the following result.

Theorem 4.1. *The system (4.2) has the semilinear behavior.*

Proof. Let v be the regular maximal solution of the Cauchy problem (4.2). Let us assume that its lifespan T^* is finite. From the continuation principle in [25], we know that

$$\|v\|_{L^\infty([0,T^*[\times\mathbb{R})} + \|\partial_t v\|_{L^\infty([0,T^*[\times\mathbb{R})} + \|\partial_x v\|_{L^\infty([0,T^*[\times\mathbb{R})} = +\infty. \tag{4.5}$$

We will prove that if

$$\|v\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq K, \tag{4.6}$$

then

$$\|\partial_t u\|_{L^\infty([0,T^*[\times\mathbb{R})} + \|\partial_x u\|_{L^\infty([0,T^*[\times\mathbb{R})} < +\infty,$$

which contradicts Majda's result and shows the semilinear behavior.

The L^2 estimate. Taking the inner product of (4.2) with v we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (A_0(\chi)u\cdot u + |\chi|^2)dx = \int_{\mathbb{R}} \Phi(v)\cdot v dx + \frac{1}{2}\int_{\mathbb{R}} \partial_t(A_0(\chi))u\cdot u dx.$$

Since $\Phi(0) = 0$, with (4.6), there exists a constant K such that

$$\left|\int_{\mathbb{R}} \Phi(v)\cdot v dx\right| \leq K\|v\|^2_{L^2(\mathbb{R})}.$$

In addition, since $\partial_t\chi = \psi(v)$, there exists a constant K such that

$$\|\partial_t\chi\|_{L^\infty(\mathbb{R})} \leq K. \tag{4.7}$$

As $\partial_t(A_0(\chi)) = A_0'(\chi)(\partial_t\chi)$, using (4.6) we obtain

$$\|\partial_t(A_0(\chi))\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq K. \tag{4.8}$$

Therefore, there exists K such that

$$\left|\int_{\mathbb{R}} \partial_t(A_0(\chi))u\cdot u dx\right| \leq K\|v\|^2_{L^2(\mathbb{R})},$$

and we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (A_0(\chi)u\cdot u + |\chi|^2)dx \leq K\|v\|^2_{L^2(\mathbb{R})}.$$

Integrating in time and using (4.4) we conclude by Gronwall's Lemma that there exists a constant C such that

$$\|v\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C. \tag{4.9}$$

The H^1-estimate. We differentiate the system (4.2) with respect to t and we obtain

(4.10)

$$\begin{pmatrix} A_0(\chi)\partial_{tt}u \\ \\ \partial_{tt}\chi \end{pmatrix} + \begin{pmatrix} A_1\partial_x\partial_t u \\ \\ 0 \end{pmatrix} = -\begin{pmatrix} (\partial_t(A_0(\chi))\partial_t u \\ \\ 0 \end{pmatrix} + \Phi'(v)\partial_t v,$$

with the initial data obtained by the equation (4.2)

$$\partial_t u(0,x) = (A_0(\chi_0))^{-1}(\varphi(v_0) - A_1\partial_x u_0),\ \partial_t\chi(0,x) = \psi(v_0).$$

Taking the inner product of (4.10) with $\partial_t v$ we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(A_0(\chi)\partial_t u\cdot\partial_t u + |\partial_t\chi|^2)dx = \ & -\frac{1}{2}\int_{\mathbb{R}}\partial_t(A_0(\chi))\partial_t u\cdot\partial_t u dx \\ & + \int_{\mathbb{R}}\Phi'(v)(\partial_t v)\cdot\partial_t v.\end{aligned}$$

Using (4.8), (4.6) and Gronwall's Lemma, we can see there exists a constant C such that

$$\|\partial_t v\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C. \tag{4.11}$$

Next, from (4.2), we have

$$\partial_x u = (A_1)^{-1}(\varphi(v) - A_0(\chi)\partial_t u).$$

Hence, from (4.11), there exists C such that

$$\|\partial_x u\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C. \tag{4.12}$$

In addition, derivating $(4.2)_2$ with respect to x we have

$$\partial_t\partial_x\chi = \partial_2\psi(v)(\partial_x\chi) + \partial_1\psi(v)(\partial_x u).$$

Integrating in time from 0 to t, we obtain

$$\partial_x\chi(t,x) = \partial_x\chi_0(x) + \int_0^t \partial_2\psi(v)(\partial_x\chi) + \partial_1\psi(v)(\partial_x u).$$

Then we deduce

$$\begin{aligned}\|\partial_x\chi(t,.)\|_{L^2(\mathbb{R})} \leq \|\partial_x\chi_0\|_{L^2(\mathbb{R})} \quad & + \textstyle\int_0^t \|\partial_2\psi(v)\|_{L^\infty(\mathbb{R})}\|\partial_x\chi\|_{L^2(\mathbb{R})} \\ & + \|\partial_1\psi(v)\|_{L^\infty(\mathbb{R})}\|\partial_x u\|_{L^2(\mathbb{R})},\end{aligned}$$

and so, using (4.6), (4.11), (4.12) and Gronwall's Lemma, we find

$$\|\partial_x\chi\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C.$$

Therefore, we have obtained that there exists a constant C such that

$$\|v\|_{L^\infty([0,T^*[;H^1(\mathbb{R}))} + \|\partial_t v\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C. \tag{4.13}$$

The H^2-estimate. We first remark that

$$\partial_{tt}\chi = \psi'(v)(\partial_t v)$$

so, using (4.6) and (4.13), we can see that there exists C such that

$$\|\partial_{tt}\chi\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} \leq C. \tag{4.14}$$

We differentiate (4.10) with respect to t and we obtain

(4.15)

$$\begin{pmatrix} A_0(\chi)\partial_{ttt}u \\ \\ \partial_{ttt}\chi \end{pmatrix} + \begin{pmatrix} A_1\partial_x\partial_{tt}u \\ \\ 0 \end{pmatrix} = -2\begin{pmatrix} (\partial_t(A_0(\chi))\partial_{tt}u \\ \\ 0 \end{pmatrix}$$

$$+\begin{pmatrix} (\partial_{tt}(A_0(\chi))\partial_t u \\ \\ 0 \end{pmatrix} + \Phi''(v)(\partial_t v, \partial_t v) + \Phi'(v)(\partial_{tt}v)$$

Taking the inner product of this equation with $\partial_{tt}v$ we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(A_0(\chi)\partial_{tt}u\cdot\partial_{tt}u + |\partial_{tt}\chi|^2)dx = I_1 + \ldots + I_5,$$

where

$$I_1 = -\frac{3}{2}\int_{\mathbb{R}}\partial_t(A_0(\chi))\partial_{tt}u\cdot\partial_{tt}u\,dx,$$

$$I_2 = -\int_{\mathbb{R}}A_0''(\chi)(\partial_t\chi,\partial_t\chi)(\partial_t u)\cdot\partial_{tt}u\,dx,$$

$$I_3 = -\int_{\mathbb{R}}A_0'(\chi)(\partial_{tt}\chi)(\partial_t u)\cdot\partial_{tt}u\,dx,$$

$$I_4 = \int_{\mathbb{R}}\Phi''(v)(\partial_t v,\partial_t v)\cdot\partial_{tt}v\,dx,$$

$$I_5 = \int_{\mathbb{R}}\Phi'(v)(\partial_{tt}v)\cdot\partial_{tt}v\,dx.$$

Now, using (4.8) we find

$$|I_1| \le K\|\partial_t u\|^2_{L^2(\mathbb{R})}.$$

Next, from (4.6) and (4.7), we have

$$|I_2| \le K\|\partial_t u\|_{L^2(\mathbb{R})}\|\partial_{tt}u\|_{L^2(\mathbb{R})},$$

and from (4.6) and (4.14), we have

$$|I_3| \le K\|\partial_{tt}u\|_{L^2(\mathbb{R})}.$$

By (4.13), we know that v is bounded in $L^\infty((0,T^*)\times\mathbb{R})$. So

$$|I_4| \le K\|\partial_t v\|_{L^\infty(\mathbb{R})}\|\partial_t v\|_{L^2(\mathbb{R})}\|\partial_{tt}v\|_{L^2(\mathbb{R})}.$$

Since v is bounded in $L^\infty((0,T^*)\times\mathbb{R})$, we obtain

$$|I_5| \le K\|\partial_{tt}v\|^2_{L^2(\mathbb{R})}.$$

From (4.10) we have

$$\partial_x\partial_t u = (A_1)^{-1}(-A_0(\chi)\partial_{tt}u - \partial_t(A_0(\chi))\partial_t u + \varphi'(v)(\partial_t v)). \tag{4.16}$$

By (4.8) and (4.11) we find, adding up the previous estimates,

$$\frac{d}{dt}\int_{\mathbb{R}}(A_0(\chi)\partial_{tt}u\cdot\partial_{tt}u + |\partial_{tt}\chi|^2)dx \le C(1 + \|\partial_{tt}v\|^2_{L^2(\mathbb{R})}),$$

and by Gronwall's Lemma we deduce

$$\|\partial_{tt} v\|_{L^\infty(0,T^*;L^2(\mathbb{R}))} \le C. \tag{4.17}$$

Next, using (4.16), we find

$$\|\partial_t u\|_{L^\infty(0,T^*;H^1(\mathbb{R}))} \le C$$

and, since $\partial_x \partial_t \chi = \psi'(v)(\partial_x v)$, we have

$$\|\partial_t \chi\|_{L^\infty(0,T^*;H^1(\mathbb{R}))} \le C.$$

Now, differentiating (4.2) with respect to x, we obtain

$$\partial_{xx} u = (A_1)^{-1}(\varphi'(v)(\partial_x v) - A_0'(\chi)(\partial_x \chi)\partial_t u)$$

and the following estimate follows

$$\|u\|_{L^\infty(0,T^*;H^2(\mathbb{R}))} \le C.$$

In addition

$$\begin{aligned}\partial_t \partial_{xx}\chi = \quad & \partial_1 \psi(v)\partial_{xx}\chi + 2\partial_{12}^2\psi(v)(\partial_x u, \partial_x \chi) \\ & +\partial_1^2\psi(v)(\partial_x u, \partial_x u) + \partial_2^2\psi(v)(\partial_x \chi, \partial_x \chi).\end{aligned}$$

Integrating this equation, we obtain by Gronwall's Lemma,

$$\|\partial_{xx}\chi\|_{L^\infty(0,T^*;L^2(\mathbb{R}))} \le C.$$

Therefore we have proved that, provided (4.6), there exists a constant C such that

$$\|v\|_{L^\infty(0,T^*;H^2(\mathbb{R}))} + \|\partial_t v\|_{L^\infty(0,T^*;H^1(\mathbb{R}))} + \|\partial_{tt} v\|_{L^\infty(0,T^*;L^2(\mathbb{R}))} \le C$$

and, by Sobolev inequalities, we have

$$\|\partial_x v\|_{L^\infty([0,T^*[\times\mathbb{R})} + \|\partial_t v\|_{L^\infty([0,T^*[\times\mathbb{R})} \le C,$$

which contradicts (4.5). $\square$

For the Kerr-Debye system (4.1), we have $u = (e,h)$, $v = U = (e,h,\chi)$ and

$$A_0(\chi) = \begin{pmatrix} 1+\chi & 0 \\ 0 & 1 \end{pmatrix}.$$

We assume that the initial data $\chi_0 \ge 0$. Then from the last equation in (4.1), χ remains positive, and the condition (4.4) is satisfied on the domain under consideration. So we can adapt the proof of Theorem 4.1 and we obtain the following result.

Corollary 4.1. *The Kerr-Debye system (4.1) has the semilinear behavior.*

4.2. **Global solutions for the Kerr-Debye system.** In the previous section, in order to obtain the semilinear behavior for the Kerr-Debye system (4.1), we only use the structure of the system (4.2); we did not make use of the fact that the system (4.1) admits a strictly convex partially dissipative entropy given by

$$\mathcal{E}(U) = \frac{1}{2}(1+\chi)e^2 + \frac{1}{2}h^2 + \frac{1}{4}\chi^2, \tag{4.18}$$

satisfying

$$\partial_t \mathcal{E}(U) + \partial_x(eh) = -\frac{1}{2\varepsilon}(\chi - e^2)^2. \tag{4.19}$$

We rewrite the Kerr-Debye system using the variables $U = (e, h, \chi)$:

$$\left\{ \begin{array}{rcl} (1+\chi)\partial_t e + e\partial_t \chi + \partial_x h &=& 0, \\ \\ \partial_t h + \partial_x e &=& 0, \\ \\ \partial_t \chi &=& \dfrac{1}{\varepsilon}(e^2 - \chi). \end{array} \right. \tag{4.20}$$

with the initial condition $U(0, x) = U_0 \in H^2(\mathbb{R})$.

Theorem 4.2. *The Cauchy problem (4.20) with the initial data U_0, such that $\chi_0 \geq 0$, has a global smooth solution.*

Proof. let us assume that the lifespan T^* of the regular solution U is finite. Then, from Corollary 4.1, we know that $\|U\|_{L^\infty([0,T^*[\times\mathbb{R})} = +\infty$. On the other side, we will obtain by variational estimates that $U = (e, h, \chi)$ is bounded in $L^\infty([0, T^*[\times\mathbb{R})$, so proving the result by contradiction.

The L^2 estimate. Integrating (4.19) on $\mathbb{R}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}((1+\chi)e^2 + h^2 + \frac{1}{2}\chi^2)dx + \frac{\varepsilon}{2}\int_{\mathbb{R}} |\partial_t\chi|^2 \, dx = 0.$$

Then there exists a constant C such that

$$\|U\|_{L^\infty(0,T^*;L^2(\mathbb{R}))} \leq C. \tag{4.21}$$

The H^1 estimate. We differentiate (4.20) with respect to t and we have

$$\left\{ \begin{array}{rcl} (1+\chi)\partial_{tt} e + 2\partial_t e\partial_t \chi + e\partial_{tt}\chi + \partial_x\partial_t h &=& 0, \\ \\ \partial_{tt} h + \partial_x \partial_t e &=& 0, \\ \\ \partial_{tt} \chi &=& \dfrac{1}{\varepsilon}(2e\partial_t e - \partial_t\chi), \end{array} \right. \tag{4.22}$$

where the initial data on $\partial_t U(0, x)$ is given by (4.20).

Taking the inner product of $(4.22)_1$ with $\partial_t e$ and of $(4.22)_2$ with $\partial_t h$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}((1+\chi)(\partial_t e)^2 + (\partial_t h)^2)dx + \frac{3}{2}\int_{\mathbb{R}} \partial_t\chi(\partial_t e)^2 \, dx + \int_{\mathbb{R}} \partial_{tt}\chi e \partial_t e = 0.$$

From $(4.20)_3$ we have

$$\int_{\mathbb{R}} \partial_t\chi(\partial_t e)^2\, dx = \frac{1}{\varepsilon}\int_{\mathbb{R}} e^2(\partial_t e)^2\, dx - \frac{1}{\varepsilon}\int_{\mathbb{R}} \chi(\partial_t e)^2\, dx$$

as well as from $(4.22)_3$ we have

$$\int_{\mathbb{R}} \partial_{tt}\chi e\partial_t e = \frac{\varepsilon}{2}\int_{\mathbb{R}} |\partial_{tt}\chi|^2 dx + \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}} |\partial_t\chi|^2 dx.$$

Consequently we deduce

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left((1+\chi)(\partial_t e)^2 + (\partial_t h)^2 + \frac{1}{2}|\partial_t\chi|^2\right) dx + \frac{3}{2\varepsilon}\int_{\mathbb{R}} e^2(\partial_t e)^2\, dx$$
$$+\frac{\varepsilon}{2}\int_{\mathbb{R}} |\partial_{tt}\chi|^2 dx = \frac{3}{2\varepsilon}\int_{\mathbb{R}} \chi(\partial_t e)^2\, dx$$

and thus, by Gronwall's Lemma, we obtain that there exists a constant C such that

$$\|\partial_t U\|_{L^\infty(0,T^*;L^2(\mathbb{R}))} \leq C. \tag{4.23}$$

Now, from $(4.20)_2$ we have $\partial_x e = -\partial_t h$ and so there exists a constant C such that

$$\|e\|_{L^\infty(0,T^*;H^1(\mathbb{R}))} \leq C. \tag{4.24}$$

Then, by Sobolev inequalities, we conclude that

$$\|e\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq C. \tag{4.25}$$

Next, solving $(4.20)_3$ we have

$$\chi(t,x) = \chi_0(x)\exp(-\frac{t}{\varepsilon}) + \int_0^t \exp(-\frac{t-s}{\varepsilon})(e(s,x))^2\, ds.$$

Therefore, from (4.25), we obtain

$$\|\chi\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq C. \tag{4.26}$$

In the same way, from $(4.20)_1$, we have

$$\partial_x h = -(1+\chi)\partial_t e - e\partial_t\chi,$$

so using (4.24), (4.26) and (4.23), we find

$$\|h\|_{L^\infty(0,T^*;H^1(\mathbb{R}))} \leq C, \tag{4.27}$$

and again by Sobolev inequalities

$$\|h\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq C. \tag{4.28}$$

Therefore we have proved that there exists a constant C such that

$$\|U\|_{L^\infty([0,T^*[\times\mathbb{R})} \leq C,$$

and we obtain a contradiction. Hence $T^* = +\infty$. □

4.3. **Stability for a constant equilibrium state.** We can generalize Theorem 4.2 for a perturbation U of a constant equilibrium state $\overline{U} = (\bar{e}, \bar{h}, \bar{\chi} = (\bar{e})^2)$: if the initial data U^0 satisfies that $U^0 - \overline{U}$ is in $H^2(\mathbb{R})$ with $\chi^0 + \bar{\chi} \geq 0$, then the solution of the Kerr-Debye system with initial data U^0 is global in time and we have

$$U - \overline{U} \in \mathcal{C}^0(\mathbb{R}^+; H^2(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^1(\mathbb{R})).$$

So for this result, the smallness condition on $\|U^0 - \overline{U}\|_{H^2(\mathbb{R})}$ in [15] is relaxed.

Actually, more general stability results can be obtained in the framework of [15, 3]: the function $\mathcal{E}$ given by (4.18), is a strictly dissipative entropy as in [15] and the Shizuta-Kawashima condition holds if and only if $\bar{e} \neq 0$. So in this case, we obtain the estimates similar to (3.31). However, if $\bar{e} = 0$, the stability problem remains open.

Acknowledgements. The authors would like to thank C. Berthon and F. Bouchut for fruitfull discussions about the Suliciu model.

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