Relaxed model for the hysteresis in micromagnetism

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Abstract : in this paper we study a model of ferromagnetic material with hysteresis effects. The magnetic moment behaviour is described by the non-linear Landau-Lifschitz equation with an additional term modelling the hysteresis. This term takes the form of a maximal monotone operator acting on the time derivative of the magnetic moment. In our model, it is approximated via a relaxing heat equation. For this relaxed model we prove local existence of regular solutions.

1 Introduction

The hysteresis properties of the ferromagnetic materials are a very wide domain in physics (see E. Della Torre [4]). The Preisach model describing the magnetic hysteresis is obtained by a phenomenological approach (see [6]). It is explained from the mathematical point of view by A. Visintin in [8]. With a physical approach, W. F. Brown developed in [2] the micromagnetism theory. The model described by Landau and Lifschitz in [5] is the following. The magnetic moment u is a unitary vector field linking the magnetic field and the magnetic induction by the relation $B = H + u$. The variations of u are described by the Landau-Lifschitz equation:

$$\frac{\partial u}{\partial t} = -u \wedge \mathcal{H}_{eff} - u \wedge (u \wedge \mathcal{H}_{eff}), \tag{1.1}$$

where the effective field is given by $\mathcal{H}_{eff} = \Delta u + h_d(u) + H_a + \Psi(u)$, and the demagnetizing field $h_d(u)$ is solution of the magnetostatic equations

$$\text{div } (h_d(u) + u) = 0 \text{ and curl } h_d(u) = 0, \tag{1.2}$$

where H_a is an applied magnetic field and where $\Psi(u)$ is an anisotropic term.

Micromagnetic modeling and Preisach modeling are two complementary approaches but the links between these two models are not clear. Using a two time-scales asymptotic method, J. Starynkévitch [7] gives a first answer to bring to the fore the hysteresis in Landau-Lifschitz model. We study here a model due to M. Effendiev. The hysteresis effect in Landau-Lifschitz equation is reinforced by an additional term in the effective field. This term is described with the maximal monotone operator β defined as follows

$$\beta(\xi) = \begin{cases} \dfrac{\xi}{|\xi|} \text{ si } \xi \neq 0, \\ \\ B(0,1) \text{ si } \xi = 0. \end{cases} \tag{1.3}$$

In this model the effective field is given by:

$$\mathcal{H}_{eff} = \Delta u + h_d(u) + H_a + \Psi(u) - \beta(\frac{\partial u}{\partial t}). \tag{1.4}$$

The existence of regular solutions for the system (1.1)-(1.4) is open. We propose here a relaxation model for this system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \wedge (\Delta u + h_d(u) + H_a + \Psi(u) - v) - u \wedge (u \wedge (\Delta u + h_d(u) + H_a + \Psi(u) - v)) \\ \\ \dfrac{\partial v}{\partial t} = \Delta v + \dfrac{1}{\varepsilon}(\beta(\dfrac{\partial u}{\partial t}) - v) \\ \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega, \\ \\ u(t=0) = u_0 \text{ and } v(t=0) = v_0 \text{ on } \Omega. \end{cases} \tag{1.5}$$

We prove an existence result of strong solutions for this relaxed system for $\varepsilon > 0$ fixed.
We assume that the initial data satisfies the following conditions:

$$\begin{cases} u_0 \in H^2(\Omega) \text{ and } v_0 \in H^1(\Omega), \\ \\ |u_0| = 1 \text{ on } \Omega, \\ \\ \dfrac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.6}$$

For regular solutions, the equation (1.5) with initial data satisfying (1.6) is equivalent to the following system (see [3]):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = u|\nabla u|^2 + u \wedge \Delta u + u \wedge (h_d(u) + H_a + \Psi(u) - v) \\ \qquad\qquad -u \wedge (u \wedge (h_d(u) + H_a + \Psi(u) - v)) \\ \\ \dfrac{\partial v}{\partial t} = \Delta v + \dfrac{1}{\varepsilon}(\beta(\dfrac{\partial u}{\partial t}) - v) \\ \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega, \\ \\ u(t=0) = u_0 \text{ and } v(t=0) = v_0 \text{ on } \Omega. \end{cases} \tag{1.7}$$

Indeed if (u, v) is a regular solution of (1.5) then the punctual norm of u is preserved and so $|u| = 1$. Then we have $\Delta|u|^2 = 0 = u \cdot \Delta u + |\nabla u|^2$.
So $u \wedge (u \wedge \Delta u) = (u \cdot \Delta u)u - |u|^2 \Delta u = -\Delta u - u|\nabla u|^2$.
In addition if (u, v) is a regular solution of (1.7) then $|u|^2$ satisfies a parabolic equation which unique solution is $|u|^2 \equiv 1$. Then the previous computation remains valid and (u, v) satisfies (1.5).

Our main result is the following theorem:

Theorem 1.1 *We fix $\varepsilon > 0$. Let (u_0, v_0) satisfying (1.6). Then there exists $T^* > 0$, there exists (u, v) solution of (1.7) such that for all $T < T^*$,*

$$u \in \mathcal{C}^0(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \quad v \in \mathcal{C}^0(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

In the following section we recall technical lemmas about equivalent norms in the H^p spaces, about the demagnetizing field h_d and about the maximal monotone operator β.
In the last section we prove Theorem 1.1.

2 Technical lemmas

2.1 Estimates tolls

The results of this subsection are proved in [3].

Lemma 2.1 *Let Ω be a bounded regular open set. There exists a constant C such that for all $u \in H^2(\Omega)$ satisfying $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, we have*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.1}$$

$$\|\nabla u\|_{H^1(\Omega)} \leq C \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.2}$$

and for $u \in H^3(\Omega)$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$,

$$\|\nabla u\|_{H^2(\Omega)} \leq C \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} + \|\nabla\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{2.3}$$

Using Lemma 2.1 and the classical interpolation inequality, we rewrite Sobolev and Gagliardo-Nirenberg inequalities on the following form:

Lemma 2.2 *Let Ω be a regular bounded domain of $\mathbb{R}^3$. There exists a constant C such that for all $u \in H^2(\Omega)$ such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$,*

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.4}$$

$$\|\nabla u\|_{L^6(\Omega)} \leq C \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.5}$$

$$\|\nabla u\|^2_{L^4(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} \left(\|u\|^2_{L^2} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.6}$$

and for all $u \in H^3(\Omega)$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$,

$$\|D^2 u\|_{L^3(\Omega)} \leq C \left(\left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} + \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{4}} \|\nabla\Delta u\|^{\frac{1}{2}}_{L^2(\Omega)} \right). \tag{2.7}$$

2.2 Demagnetizing field

We consider the operator $u \mapsto h_d(u)$ defined by (1.2). It satisfies

$$\begin{cases} h_d(u) \in L^2(\mathbb{R}^3), & \\ \text{curl } h_d(u) = 0 & \text{in } \mathbb{R}^3, \\ \text{div } \left(h_d(u) + \bar{u} \right) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\bar{u}$ is the extension of u by zero outside $\overline{\Omega}$.
We observe that $u \mapsto -h_d(u)$ is the orthogonal projection of $\bar{u}$ on the vector fields of gradients in $L^2(\mathbb{R}^3)$. We prove in [3] the following estimates concerning the operator h_d:

Lemma 2.3 *Let $p \in]1, +\infty[$. Then, if u belongs to $W^{1,p}(\Omega)$ (resp. $W^{2,p}(\Omega)$), the restriction of $H(u)$ to Ω belongs to $W^{1,p}(\Omega)$ (resp. $W^{2,p}(\Omega)$) and there exists a constant C such that*

$$\|h_d(u)\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad 1 < p < +\infty. \tag{2.8}$$

$$\|h_d(u)\|_{W^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{2.9}$$

and

$$\|h_d(u)\|_{W^{2,p}(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)}. \tag{2.10}$$

2.3 Maximal monotone operators tools

We remark that β is a maximal monotone operator. We recall usefull results proved in [1]. The first proposition is about the approximation of β by a continuous operator:

Proposition 2.1 *For $\lambda > 0$ we define β_λ by*

$$\beta_\lambda(\xi) = \begin{cases} \dfrac{\xi}{|\xi|} \text{ for } |\xi| \geq \lambda \\ \dfrac{\xi}{\lambda} \text{ for } |\xi| \leq \lambda. \end{cases}$$

*Then if ξ_λ tends to ξ uniformly on $[0, T] \times \Omega$ then extracting a subsequence, $\beta_\lambda(\xi_\lambda)$ tends to $\beta(\xi)$ in L^∞ weak *.*

In order to take the limit in a maximal monotone operator, we have the following lemma:

Proposition 2.2 *If A is a maximal monotone operator, if $y_n \in A(x_n)$, if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, if lim sup $< x_n|y_n > \leq < x|y >$, then $y \in A(x)$ and $< x_n|y_n > \longrightarrow < x|y >$.*

3 Proof of Theorem 1.1

3.1 First Step : Galerkin Approximation

We denote by V_n the finite dimension space built on the n first eigen-functions of $-\Delta + Id$ with domain $D(A) = \left\{ u \in H^2(\Omega), \ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$, and by $\mathbf{P}_n$ the orthogonal projection from $L^2(\Omega)$ on V_n.
We first solve the Galerkine approximation for system (1.7). We fix n and we want to build (u_n, v_n) the solution of the following approximate problem:

$$\begin{cases} u_n \in \mathcal{C}^1([0, T_n[; V_n), \quad v_n \in \mathcal{C}^1([0, T_n[; V_n) \\[2ex] \dfrac{\partial u_n}{\partial t} - \Delta u_n = \mathbf{P}_n \left(u_n |\nabla u_n|^2 + \ u_n \wedge \Delta u_n + u_n \wedge (h_d(u_n) + H_a + \Psi(u_n) - v_n) \right) \\ \qquad -\mathbf{P}_n \left(u_n \wedge \ (u_n \wedge (h_d(u_n) + H_a + \Psi(u_n) - v_n)) \right) \\[2ex] \dfrac{\partial v_n}{\partial t} - \dfrac{1}{\varepsilon} \mathbf{P}_n \left(\beta(\dfrac{\partial u_n}{\partial t}) \right) - \Delta v_n = -\dfrac{1}{\varepsilon} v_n \\[2ex] u_n(t=0) = \mathbf{P}_n(u_0), \quad v_n(t=0) = \mathbf{P}_n(v_0) \end{cases} \tag{3.1}$$

In order to solve this problem and to take into account the specificity of the maximal monotone operator β, we consider the approximation β_λ of β, described in the previous section, and we solve the following equation:

$$\begin{cases} u_n^\lambda \in \mathcal{C}^1([0,T_n^\lambda[;V_n), \quad v_n^\lambda \in \mathcal{C}^1([0,T_n^\lambda[;V_n) \\ \dfrac{\partial u_n^\lambda}{\partial t} - \Delta u_n^\lambda = \mathbf{P}_n\left(u_n^\lambda|\nabla u_n^\lambda|^2 + u_n^\lambda \wedge \Delta u_n^\lambda + u_n^\lambda \wedge (h_d(u_n^\lambda) + H_a + \Psi(u_n^\lambda) - v_n^\lambda)\right) \\ \qquad -\mathbf{P}_n\left(u_n^\lambda \wedge (u_n^\lambda \wedge (h_d(u_n^\lambda) + H_a + \Psi(u_n^\lambda) - v))\right) \\ \dfrac{\partial v_n^\lambda}{\partial t} - \dfrac{1}{\varepsilon}\mathbf{P}_n\left(\beta_\lambda(\dfrac{\partial u_n^\lambda}{\partial t})\right) - \Delta v_n^\lambda = -\dfrac{1}{\varepsilon}v_n^\lambda \\ u_n^\lambda(t=0) = \mathbf{P}_n(u_0), \quad v_n^\lambda(t=0) = \mathbf{P}_n(v_0) \end{cases} \tag{3.2}$$

This equation can be written on the following form:

$$\begin{cases} u_n^\lambda \in \mathcal{C}^1([0,T_n^\lambda[;V_n), \quad v_n^\lambda \in \mathcal{C}^1([0,T_n^\lambda[;V_n) \\ \dfrac{\partial u_n^\lambda}{\partial t} = F_n(u_n^\lambda, v_n^\lambda) \\ \dfrac{\partial v_n^\lambda}{\partial t} - \dfrac{1}{\varepsilon}\mathbf{P}_n\left(\beta_\lambda(\dfrac{\partial u_n^\lambda}{\partial t})\right) = G(v_n^\lambda) \\ u_n^\lambda(t=0) = \mathbf{P}_n(u_0), \quad v_n^\lambda(t=0) = \mathbf{P}_n(v_0) \end{cases} \tag{3.3}$$

where $F_n : V_n \times V_n \longrightarrow V_n$ and $G : V_n \longrightarrow V_n$ are smooth. Since we can replace the second equation by

$$\frac{\partial v_n^\lambda}{\partial t} = \frac{1}{\varepsilon}\mathbf{P}_n(\beta_\lambda(F_n(u_n^\lambda, v_n^\lambda))) + G(v_n^\lambda) \tag{3.4}$$

for a fixed λ we can apply the Cauchy-Lisfchitz theorem on the finite dimensional space $V_n \times V_n$: there exists a unique solution for equation (3.3) defined on the maximal interval $[0, T_n^\lambda[$.
Since $\|\beta_\lambda(\xi)\|_{L^\infty(\Omega)} \leq 1$, there exists K depending only on n such that for all $w \in V_n$ we have:

$$\|\mathbf{P}_n(\beta_\lambda(w))\|_{V_n} \leq K.$$

Since G is linear, we can obtain from (3.4) that there exists a constant C depending on n such that for all $t \leq T_n^\lambda$ we have:

$$\|v_n^\lambda\|_{V_n} \leq Ce^{Ct}.$$

Now, there exists a constant K depending on n such that for $(u,v) \in V_n \times V_n$ we have

$$\|F_n(u,v)\|_{V_n} \leq K_n'(\|u\|_{V_n}^4 + \|v\|_{V_n}^2).$$

By comparison lemma we then obtain that there exists a time $T^n > 0$ such that for all $\lambda > 0$, $T_n^\lambda \geq T^n$, and there exists a constant K_n such that for all λ,

$$\|u_n^\lambda\|_{L^\infty(0,T^n)} + \|v_n^\lambda\|_{L^\infty(0,T^n)} \leq K_n. \tag{3.5}$$

Using (3.5) in (3.3), we obtain a bound for $\dfrac{\partial u_\lambda^n}{\partial t}$ and $\dfrac{\partial v_\lambda^n}{\partial t}$, and derivating the first equation of (3.3) with respect to t, we obtain a bound of $\dfrac{\partial^2 u_\lambda^n}{\partial t^2}$. Thus there exists a constant K such that for all λ,

$$\|u_n^\lambda\|_{L^\infty(0,T^n)} + \|\frac{\partial u_n^\lambda}{\partial t}\|_{L^\infty(0,T^n)} + \|\frac{\partial^2 u_n^\lambda}{\partial t^2}\|_{L^\infty(0,T^n)} + \|v_n^\lambda\|_{L^\infty(0,T^n)} + \|\frac{\partial v_n^\lambda}{\partial t}\|_{L^\infty(0,T^n)} \leq K_n. \quad (3.6)$$

For a fixed n we take the limit when λ tends to zero. From (3.6) we obtain that there exists u_n and v_n such that

$$u_n^\lambda \longrightarrow u_n \text{ in } L^\infty(0,T_n)$$

$$\frac{\partial u_n^\lambda}{\partial t} \longrightarrow \frac{\partial u_n}{\partial t} \text{ in } L^\infty(0,T_n)$$

$$v_n^\lambda \longrightarrow v_n \text{ in } L^\infty(0,T_n)$$

In addition using Proposition 2.1, we have that $\beta^\lambda(\dfrac{\partial u_n^\lambda}{\partial t})$ tends to w_n and $w_n \in \beta(\dfrac{\partial u_n}{\partial t})$. Furthermore we can take the limit when λ tends to zero in Equation (3.2) and we obtain that there exist $T_n > 0$, $u_n \in \mathcal{C}^1([0,T_n[;V_n)$ and $v_n \in \mathcal{C}^1([0,T_n[;V_n)$ satisfying (3.1).

3.2 Estimates for u_n and v_n

Taking the inner product in $L^2(\Omega)$ of the first equation in (3.1) with u_n we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2_{L^2(\Omega)}\right) + \|\nabla u_n\|^2_{L^2(\Omega)} \leq \|u_n\|^2_{L^\infty(\Omega)}\|\nabla u_n\|^2_{L^2(\Omega)}. \quad (3.7)$$

Taking the inner product in $L^2(\Omega)$ of the second equation in (3.1) with v_n we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|v_n\|^2_{L^2(\Omega)}\right) + \|\nabla v_n\|^2_{L^2(\Omega)} \leq K(1 + \|v_n\|^2_{L^2(\Omega)}), \quad (3.8)$$

since $\left\|\beta(\dfrac{\partial u_n}{\partial t})\right\|_{L^2(\Omega)} \leq K$.

We take the inner product in $L^2(\Omega)$ of the second equation in (3.1) with Δv_n. Integrating by part the right hand side, and absorbing $\|\Delta v_n\|_{L^2(\Omega)}$ using that $\left\|\beta(\dfrac{\partial u_n}{\partial t})\right\|_{L^2(\Omega)} \leq K$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla v_n\|^2_{L^2(\Omega)}\right) + \|\Delta v_n\|^2_{L^2(\Omega)} \leq K(1 + \|\nabla v_n\|^2_{L^2(\Omega)}). \quad (3.9)$$

We take the inner product in $L^2(\Omega)$ of the second equation in (3.1) with $\Delta^2 u_n$. We obtain that:

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta u_n(t)\|^2_{L^2(\Omega)}\right) + \|\nabla \Delta u_n(t)\|^2_{L^2(\Omega)} = I_1 + I_2 + I_3 + I_4 + I_5$$

with

$$I_1 = \int_\Omega \nabla \left(|\nabla u_n|^2 u_n\right) \nabla\Delta u_n dx,$$

$$I_2 = \int_\Omega \nabla \Big(u_n \wedge \Delta u_n \Big) \nabla\Delta u_n dx,$$

$$I_3 = \int_\Omega \nabla \Big(u_n \wedge h_d\,(u_n) - \; u_n \wedge (u_n \wedge h_d(u_n))\Big) \nabla\Delta u_n dx,$$

$$I_4 = \int_\Omega \nabla \Big(u_n \wedge (H_a + \Psi(u_n)) - \; u_n \wedge (u_n \wedge (H_a + \Psi(u_n)))\Big) \nabla\Delta u_n dx,$$

$$I_5 = \int_\Omega \nabla \Big(u_n \wedge v_n - \; u_n \wedge (u_n \wedge v_n)\Big) \cdot \nabla\Delta u_n.$$

We bound separately each term.

• Estimate on I_1

$$\begin{aligned} |I_1| \leq \;& \int_\Omega |\nabla u_n|^3 |\nabla\Delta u_n| dx + \int_\Omega |D^2 u_n||\nabla u_n||u_n||\nabla\Delta u_n| dx, \\ \leq \;& \|\nabla u_n\|^3_{L^6(\Omega)} \|\nabla\Delta u_n\|_{L^2(\Omega)} + \|u_n\|_{L^\infty(\Omega)} \|D^2 u_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla\Delta u_n\|_{L^2(\Omega)} \end{aligned}$$

hence using the Sobolev embeding and Lemmas 2.1 and 2.2 we obtain that there exists a constant K independant of n such that

$$\begin{aligned} |I_1| \leq K \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{3}{2}} \|\nabla\Delta u_n\|_{L^2(\Omega)} \\ + K \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{5}{4}} \|\nabla\Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)}. \end{aligned} \tag{3.10}$$

• Estimate on I_2

By Sobolev embeddings and interpolation, we obtain that

$$\begin{aligned} |I_2| \leq \;& \|\nabla u_n\|_{L^6(\Omega)} \|\Delta u_n\|_{L^3(\Omega)}\| \, \|\nabla\Delta u_n\|_{L^2(\Omega)} \\ \leq \;& K \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right) \|\nabla\Delta u_n\|_{L^2(\Omega)} + \\ & K \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{3}{4}} \|\nabla\Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)}. \end{aligned} \tag{3.11}$$

• Estimate on I_3

We have

$$|I_3| \leq (1 + \|u_n\|_{L^\infty(\Omega)}) \Big(\|\nabla u_n\|_{L^2(\Omega)} \|h_d(u_n)\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Omega)} \|\nabla h_d(u_n)\|_{L^2(\Omega)}\Big) \|\nabla\Delta u_n\|_{L^2(\Omega)},$$

and using Lemmas 2.1, 2.2 and 2.3 we obtain that there exists a constant K such that

$$|I_3| \leq K \left(1 + \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)\right) \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right) \|\nabla\Delta u_n\|_{L^2(\Omega)}. \tag{3.12}$$

• Estimate on I_4

From the linearity of Ψ we obtain that there exists a constant K such that

$$|I_4| \leq K\left(1+\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)\right)\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)\|\nabla\Delta u_n\|_{L^2(\Omega)} \qquad (3.13)$$

• Estimate on I_5

We have

$$|I_5| \leq (\|u_n\|_{L^\infty(\Omega)}+\|u_n\|^2_{L^\infty(\Omega)})(\|v_n\|_{L^2(\Omega)}+\|\nabla v_n\|_{L^2(\Omega)})\|\nabla\Delta u_n\|_{L^2(\Omega)}$$

thus there exists a constant K such that

$$|I_5| \leq K(1+\|u_n\|^2_{L^\infty(\Omega)})(\|v_n\|_{L^2(\Omega)}+\|\nabla v_n\|_{L^2(\Omega)})\|\nabla\Delta u_n\|_{L^2(\Omega)}\,. \qquad (3.14)$$

Using Gronwall lemma with the estimates (3.9) and (3.8) we obtain that for all T there exists a constant $C(T)$ such that for all n

$$\|v_n\|_{L^\infty(0,T;H^1(\Omega))}+\|v_n\|_{L^2(0,T;H^2(\Omega))} \leq C(T) \qquad (3.15)$$

Thus plugging this estimate on (3.14), adding up estimates (3.7), (3.10), (3.11), (3.12), (3.13) and (3.14), for all T there exists a constant $C(T)$ such that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)+\|\nabla\Delta u_n\|^2_{L^2(\Omega)} &\leq C(T)\,\|\nabla\Delta u_n\|_{L^2(\Omega)}\\ &+C(T)\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)^2\|\nabla\Delta u_n\|_{L^2(\Omega)}\\ &+K\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{3}{4}}\|\nabla\Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)}\end{aligned}$$

and after absorption of $\|\nabla\Delta u_n\|_{L^2(\Omega)}$ in the right hand side term we obtain that for all T there exists a constant $C(T)$ such that

$$\frac{d}{dt}\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)+\|\nabla\Delta u_n\|^2_{L^2(\Omega)} \leq C(T)\left(1+\left(\|u_n\|^2_{L^2(\Omega)}+\|\Delta u_n\|^2_{L^2(\Omega)}\right)^3\right) \qquad (3.16)$$

We consider the solution of the following ordinary differential equation :

$$\begin{cases}\dfrac{d}{dt}\xi = C(T)(1+\xi^3)\\[2ex] \xi(0)=\left(\|u_0\|^2_{L^2(\Omega)}+\|\Delta u_0\|^2_{L^2(\Omega)}\right)\end{cases}$$

Since for all n, $\left(\|\mathbf{P}_n(u_0)\|^2_{L^2(\Omega)}+\|\Delta\mathbf{P}_n(u_0)\|^2_{L^2(\Omega)}\right) \leq \left(\|u_0\|^2_{L^2(\Omega)}+\|\Delta u_0\|^2_{L^2(\Omega)}\right)$ we obtain that for all t and for all n, we have:

$$\left(\|u_n(t)\|^2_{L^2(\Omega)}+\|\Delta u_n(t)\|^2_{L^2(\Omega)}\right) \leq \xi(t),$$

and if we denote by T^* the lifespan of ξ, for all $T<T^*$, for all n, we have:

$$\|u_n\|_{L^\infty(0,T;H^2(\Omega))}+\|u_n\|_{L^2(0,T;H^3(\Omega))}+\|v_n\|_{L^\infty(0,T;H^1(\Omega))}+\|v_n\|_{L^2(0,T;H^2(\Omega))} \leq C(T) \qquad (3.17)$$

In addition using the equation (3.1) we obtain a bound for $\dfrac{\partial u_n}{\partial t}$ and $\dfrac{\partial v_n}{\partial t}$:

$$\|\frac{\partial u_n}{\partial t}\|_{L^2(0,T;H^1(\Omega))}+\|\frac{\partial v_n}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \leq C(T). \qquad (3.18)$$

3.3 Limit when n tends to $+\infty$

From (3.17) we obtain a uniform bound for u_n in $L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))$ and using the first equation of (3.1) we obtain a uniform bound for $\dfrac{\partial u_n}{\partial t}$ in $L^2(0,T;H^1(\Omega))$. Thus we can extract a subsequence such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } L^\infty(0,T;H^2(\Omega)) \text{ weak}* \\ u_n \rightharpoonup u \text{ in } L^2(0,T;H^3(\Omega)) \text{ weak} \\ \dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ in } L^2(0,T;H^1(\Omega)) \text{ weak} \end{cases}$$

In addition, concerning v_n we have by (3.17) a uniform bound in $L^\infty(0,T;H^1\Omega))\cap L^2(0,T;H^2(\Omega))$ and using the second equation of (3.1) we obtain a uniform bound for $\dfrac{\partial v_n}{\partial t}$ in $L^2(0,T;L^2(\Omega))$. Thus we can extract a subsequence such that

$$\begin{cases} v_n \rightharpoonup u \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak}* \\ v_n \rightharpoonup u \text{ in } L^2(0,T;H^2(\Omega)) \text{ weak} \\ \dfrac{\partial v_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ in } L^2(0,T;L^2(\Omega)) \text{ weak} \end{cases}$$

Since $\mathbf{P}_n(\beta(\frac{\partial u_n}{\partial t}))$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ we can assume that

$$\mathbf{P}_n(\beta(\frac{\partial u_n}{\partial t})) \rightharpoonup w \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak}*$$

Taking the limit in (3.1) we obtain that u, v and w satisfy the following system on the time interval $[0,T^*[$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = u|\nabla u|^2 + u \wedge \Delta u + u \wedge (h_d(u) + H_a + \Psi(u) - v) \\ \qquad -u \wedge (u \wedge (h_d(u) + H_a + \Psi(u) - v)) \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\varepsilon} w - \Delta v = -\dfrac{1}{\varepsilon} v \\ u(t=0) = u_0, \quad v(t=0) = v_0 \end{cases} \tag{3.19}$$

It remains to prove that $w \in \beta(\frac{\partial u}{\partial t})$. We will prove that $\dfrac{\partial u_n}{\partial t}$ tends to $\dfrac{\partial u}{\partial t}$ strongly in $L^2(0,T\times\Omega)$. Then we will apply Proposition 2.2: since $< \dfrac{\partial u_n}{\partial t}|\beta(\dfrac{\partial u_n}{\partial t}) > \longrightarrow < \dfrac{\partial u}{\partial t}|w >$, then $w \in \beta(\dfrac{\partial u}{\partial t})$.

We know that $\dfrac{\partial u_n}{\partial t}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. In order to obtain compactness for $\dfrac{\partial u_n}{\partial t}$, we seek a bound on $\dfrac{\partial^2 u_n}{\partial t^2}$. We have :

$$\frac{\partial^2 u_n}{\partial t^2} = T_1 + \ldots + T_7$$

where

$$T_1 = \Delta\frac{\partial u_n}{\partial t}$$

$$T_2 = \mathbf{P}_n\left(u_n \wedge \Delta\frac{\partial u_n}{\partial t}\right)$$

$$T_3 = \mathbf{P}_n\left(\frac{\partial u_n}{\partial t}|\nabla u_n|^2 + \frac{\partial u_n}{\partial t} \wedge \Delta u_n\right)$$

$$T_4 = \mathbf{P}_n\left(2u_n\nabla u_n\nabla\frac{\partial u_n}{\partial t}\right)$$

$$T_5 = \mathbf{P}_n\left(\frac{\partial u_n}{\partial t} \wedge (H(u_n) - v_n) - \frac{\partial u_n}{\partial t} \wedge (u_n \wedge (H(u_n) - v_n)) - u_n \wedge (\frac{\partial u_n}{\partial t} \wedge (H(u_n) - v_n)\right)$$

where $H(u_n) = h_d(u_n) + H_a + \Psi(u_n)$

$$T_6 = \mathbf{P}_n\left(u_n \wedge H(\frac{\partial u_n}{\partial t}) - u_n \wedge (u_n \wedge H(\frac{\partial u_n}{\partial t})\right)$$

$$T_7 = \mathbf{P}_n\left(u_n \wedge \frac{\partial v_n}{\partial t} - u_n \wedge (u_n \wedge \frac{\partial v_n}{\partial t})\right)$$

From (3.17) and (3.18) we estimate each term on the following way:

- $\|T_1\|_{L^2(0,T;H^{-1}(\Omega))} \leq K$
- We estimate the H^{-1} norm of T_2 by duality arguments: for $\varphi \in \mathcal{C}^1([0,T[;H_0^1(\Omega))$ we have

$$\begin{aligned}< \mathbf{P}_n(u_n \wedge \Delta\frac{\partial u_n}{\partial t})|\varphi >= &\ - < \Delta\frac{\partial u_n}{\partial t}|u_n \wedge \mathbf{P}_n(\varphi) > \\ = &\ < \nabla\frac{\partial u_n}{\partial t}|\nabla u_n \wedge \mathbf{P}_n(\varphi) > + < \nabla\frac{\partial u_n}{\partial t}|u_n \wedge \nabla\mathbf{P}_n(\varphi) >\end{aligned}$$

We integrate in time and we obtain that

$$\begin{aligned}\left|\int_0^T < T_2|\varphi >\right| \leq &\ \|\mathbf{P}_n(\nabla u_n \wedge \nabla\frac{\partial u_n}{\partial t})\|_{L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))}\|\varphi\|_{L^4(0,T;H_0^1(\Omega))} \\ &+\|u_n \wedge \nabla\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\|\nabla\mathbf{P}_n(\varphi)\|_{L^2(0,T;L^2(\Omega))} \\ \leq &\ \|\nabla u_n\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega))}\|\nabla\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\|\varphi\|_{L^4(0,T;H_0^1(\Omega))} \\ &+\|u_n\|_{L^\infty(0,T\times\Omega)}\|\nabla\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\|\nabla\varphi\|_{L^2(0,T;L^2(\Omega))}\end{aligned}$$

Hence

$$\|T_2\|_{L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))} \leq K$$

- we have
$$\|\frac{\partial u_n}{\partial t}|\nabla u_n|^2\|_{L^2(0,T;L^2(\Omega))} \leq \|\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^6(\Omega))}\|\nabla u_n\|_{L^\infty(0,T;L^6(\Omega))}$$
$$\leq \|\frac{\partial u_n}{\partial t}\|_{L^2(0,T;H^1(\Omega))}\|\nabla u_n\|_{L^\infty(0,T;H^1(\Omega))}.$$
In addition
$$\|\frac{\partial u_n}{\partial t}\wedge\Delta u_n\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \leq \|\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^6(\Omega))}\|u_n\|_{L^\infty(0,T;H^2(\Omega))}.$$
Hence
$$\|T_3\|_{L^2(0,T;H^{-1}(\Omega))} \leq K.$$

- We have $\|\nabla u_n\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega))} \leq K$ by interpolation theorem. Hence, since for all $p < +\infty$, $L^4(0,T;H^{\frac{3}{2}}(\Omega)) \subset L^4(0,T;L^p(\Omega))$, we have that for all $\eta > 0$,
$$\begin{aligned}\|T_4\|_{L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))} &\leq \|T_4\|_{L^{\frac{4}{3}}(0,T;L^{2-\eta}(\Omega))}\\ &\leq \|u_n\|_{L^\infty(0,T\times\Omega)}\|\nabla u_n\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega))}\|\nabla\frac{\partial u_n}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\\ &\leq K.\end{aligned}$$

- $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0,T;L^6(\Omega))$, $H(u_n) - v_n$ is bounded in $L^\infty(0,T;L^6(\Omega))$, and u_n is bounded in $L^\infty(0,T\times\Omega)$. Hence T_5 is bounded in $L^2(0,T;L^3(\Omega))$, so there exists a constant K such that
$$\|T_5\|_{L^2(0,T;H^{-1}(\Omega))} \leq K.$$

- $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0,T;L^2(\Omega))$ hence by property of the operator h_d (see Proposition 2.3), since u_n is bounded in $L^\infty(0,T\times\Omega)$,
$$\|T_6\|_{L^2(0,T;L^2(\Omega))} \leq K.$$

- $\frac{\partial v_n}{\partial t}$ is bounded in $L^2(0,T;L^6(\Omega))$, therefore since u_n is bounded in $L^\infty(0,T\times\Omega)$,
$$\|T_7\|_{L^2(0,T;L^2(\Omega))} \leq K.$$

Therefore we obtain that there exists a constant K independant of n such that
$$\|\frac{\partial^2 u_n}{\partial t^2}\|_{L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))} \leq K.$$
Now $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0,T;H^1(\Omega))$. So by Simon's lemma,
$$\frac{\partial u_n}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \text{ in } L^2(0,T;L^2(\Omega)) \text{ strong.}$$
We have $w_n = \beta(\frac{\partial u_n}{\partial t}) \rightharpoonup w$ in $L^2(0,T;L^2(\Omega)$. So
$$< w_n|\frac{\partial u_n}{\partial t} > \longrightarrow < w|\frac{\partial u}{\partial t} > .$$
Hence by Proposition 2.2, $w \in \beta(\frac{\partial u}{\partial t})$, which concludes the proof of Theorem 1.1.

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