Ferromagnetic Nanowires

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1. Modelization
2. Walls in infinite nanowires
3. Finite nanowires

1. Modelization

3d Model:

Magnetic moment: $u : \Omega \subset I\!R^3 \to I\!R^3, \quad |u| = 1$

$$B = H + \bar{u}$$

Landau-Lifschitz Equation:

$$\frac{\partial u}{\partial t} = -u \times H_e - u \times (u \times H_e)$$

$$H_e = \varepsilon^2 \Delta u + H_d + H_a.$$

1. Modelization

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Exchange field

3d Model:

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$$\frac{\partial u}{\partial t} = -u \times H_e - u \times (u \times H_e)$$

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Applied field

1. Modelization

3d Model:

Magnetic moment: $u : \Omega \subset I\!R^3 \to I\!R^3, \quad |u| = 1$

$$B = H + \bar{u}$$

Landau-Lifschitz Equation:

$$\frac{\partial u}{\partial t} = -u \times H_e - u \times (u \times H_e)$$

$$H_e = \varepsilon^2 \Delta u + H_d + H_a.$$

Demagnetizing field:

$$\begin{cases} \text{curl } H_d = 0 \text{ in } I\!R^3, \\ \\ \text{div } (H_d + \bar{u}) = 0 \text{ in } I\!R^3 \quad \text{(Law of Faraday)} \end{cases}$$

1. Modelization

Infinite Nanowire

Diameter of the wire 2η :

$$\Omega_\eta = \mathbb{R} \times B(0, \eta)$$

Diameter small compared to the exchange lenght:

$$\eta \longrightarrow 0$$

D. Sanchez, *Behaviour of the Landau-Lifschitz equation in a ferromagnetic wire*, to appear in Math. Methods Appl. Sci.

1. Modelization

Infinite Nanowire

- wire $\sim \mathbb{R}e_1$
- $H_d(u) \sim -u_2 e_2 - u_3 e_3$

$$\mathcal{E}_d(u) = \frac{1}{2}\int_{\mathbb{R}} (|u_2|^2 + |u_3|^2)$$

- applied field: $H_a = \delta e_1$.

1. Modelization

Infinite Nanowire

$$\begin{cases} u : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow S^2 \\ \dfrac{\partial u}{\partial t} = -u \times h_\delta(u) - u \times (u \times h_\delta(u)) \\ h_\delta(u) = \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1 \end{cases}$$

$$\mathcal{E}_\delta = \frac{\varepsilon^2}{2} \int_{\mathbb{R}} |\frac{\partial u}{\partial x}|^2 + \frac{1}{2} \int_{\mathbb{R}} (|u_2|^2 + |u_3|^2) - \delta \int_{\mathbb{R}} u_1$$

Infinite Nanowire (after rescaling)

$$\begin{cases} u : I\!\!R_t^+ \times I\!\!R_x \longrightarrow S^2 \\ \dfrac{\partial u}{\partial t} = -u \times h_\delta(u) - u \times (u \times h_\delta(u)) \\ h_\delta(u) = \dfrac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1 \end{cases}$$

$$\mathcal{E}_\delta = \frac{1}{2}\int_{I\!\!R} |\frac{\partial u}{\partial x}|^2 + \frac{1}{2}\int_{I\!\!R} (|u_2|^2 + |u_3|^2) - \delta \int_{I\!\!R} u_1$$

Finite Nanowire

The wire :

$$\Omega_\eta = [0, L] \times B(0, \eta)$$

Diameter is small compared to the exchange lenght and the lenght of the wire:

$$\eta \to 0$$

Finite Nanowire

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$$\Omega_\eta = [0, L] \times B(0, \eta)$$

Diameter is small compared to the exchange lenght and the lenght of the wire:

$$\eta \longrightarrow 0$$

Wire $\sim [0, L]e_1$

Equivalent demagnetizing energy:

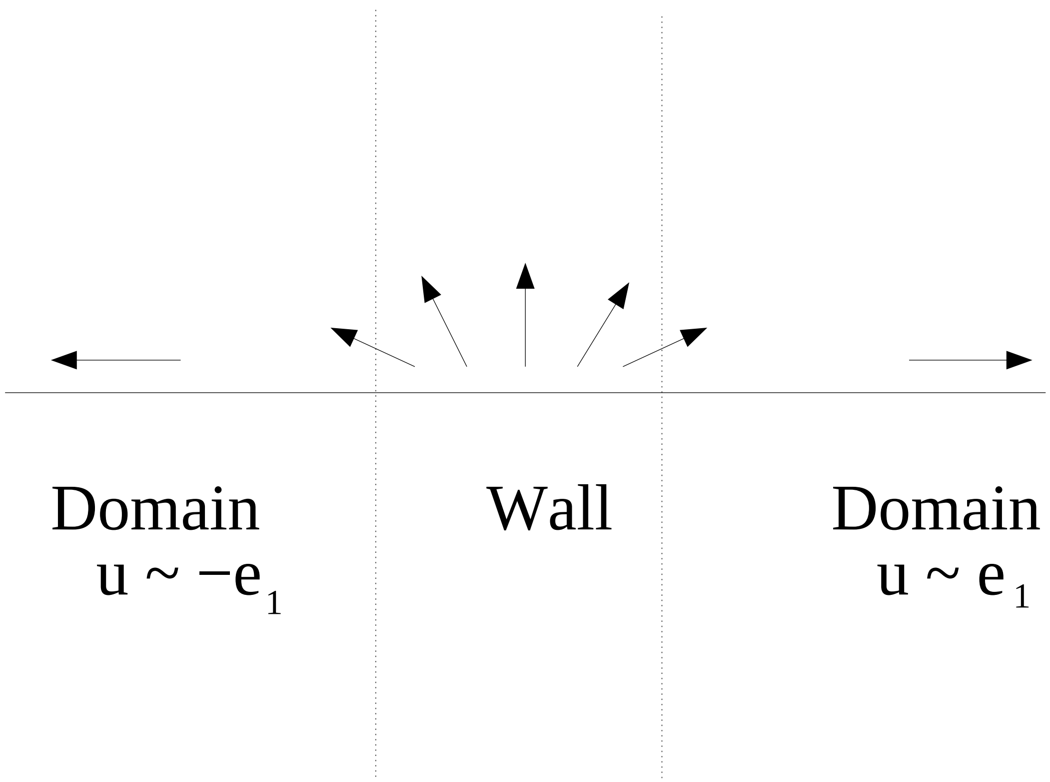
$$\int_{[0,L]} (|u_2|^2 + |u_3|^2)$$

Finite Nanowire

$$
\begin{cases}
u : I\!\!R_t^+ \times [0, \frac{L}{\varepsilon}]_x \longrightarrow S^2 \\[2ex]
\dfrac{\partial u}{\partial t} = -u \times h_\delta(u) - u \times (u \times h_\delta(u)) \\[2ex]
h_\delta(u) = \dfrac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1 \\[2ex]
\dfrac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = \dfrac{L}{\varepsilon}
\end{cases}
$$

2. Walls in infinite nanowires

Static walls: $\delta = 0$



Static walls:

$$U_0(t,x) = M_0(x) = \begin{pmatrix} \operatorname{th} x \\ 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix}$$

Wall with an applied field:

$\delta \neq 0 \quad \Rightarrow$ translation-rotation of the wall

$$U_\delta(t,x) = R_{\delta t}(M_0(x+\delta t))$$

where

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

2. Walls in infinite nanowires

Stability of the wall configuration ?

Controlability of the wall position ?

2. Walls in infinite nanowires: Stability

$$\frac{\partial u}{\partial t} = -u \times h_\delta(u) - u \times (u \times h_\delta(u)) \tag{1}$$

where $h_\delta(u) = \dfrac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1$

Solution for $\delta = 0$:

$$U_0(t, x) = M_0(x) = \begin{pmatrix} \operatorname{th} x \\ 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix}.$$

Solution for $\delta \neq 0$

$$U_\delta(t, x) = R_{\delta t}(M_0(x + \delta t))$$

2. Walls in infinite nanowires: Stability

Theoreme 1. Stability.
If $|\delta| < \delta_0$, the solution U_δ is stable for (1) and asymptotically stable modulo a translation-rotation.

If $\|u(t=0,x) - U_\delta(t=0,x)\|_{H^2}$ is small, there exists σ_∞ and θ_∞ such that

$$\|u(t,x) - R_{\theta_\infty}(U_\delta(t, x+\sigma_\infty))\|_{H^2} \to 0$$

G. Carbou, S. Labbé, *Stability for static walls in ferromagnetic nanowires,* Discrete Contin. Dyn. Syst. Ser. B **6** (2006)

2. Walls in infinite nanowires: Stability

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u)(u-\theta)$$

T. Kapitula, *Multidimensional stability of planar travelling waves*, Trans. Amer. Math. Soc., **349** (1997).

New difficulties :

- non linear constraint $|u| = 1$
- invariance by rotation
- Landau-Lifschitz is quasi-linear

2. Walls in infinite nanowires: Stability

First difficulty: non linear constraint

The perturbations must satisfy the constraint $|u| = 1$

The admissible perturbations are described in an adapted mobile frame:
For $\delta = 0$, $(M_0(x), M_1(x), M_2)$

$$M_1(x) = \begin{pmatrix} \dfrac{1}{\operatorname{ch} x} \\ 0 \\ -\operatorname{th} x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2 + \sqrt{1 - r_1^2 - r_2^2}M_0(x).$$

2. Walls in infinite nanowires: Stability

First difficulty: non linear constraint

u solution to (1) $\Leftrightarrow r = (r_1, r_2)$ solution to (2)

$$\frac{\partial r}{\partial t} = (\mathcal{L} + \delta l)r + G(r)(\frac{\partial^2 r}{\partial x^2}) + H(x, r, \frac{\partial r}{\partial x}) \tag{2}$$

- $\mathcal{L} = JL$
- $J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$
- $L = -\frac{\partial^2}{\partial x^2} + 2\mathrm{th}^2 x - 1$
- $l = \frac{\partial}{\partial x} + \mathrm{th}\, x$

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u solution to (1) $\Leftrightarrow r = (r_1, r_2)$ solution to (2)

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U_δ **stable for (1)** $\Longleftrightarrow$ **0 stable for (2)**

No more non linear constraint: r takes its values in $\mathbb{R}^2$

Second difficulty: invariance by rotation-translation

If $\Lambda = (\theta, \sigma)$

$$M_\Lambda(x) = R_\theta(M_0(x - \sigma))$$

In the mobile frame

$$R_\Lambda(x) = \begin{pmatrix} M_\Lambda(x) \cdot M_1(x) \\ M_\Lambda(x) \cdot M_2 \end{pmatrix}$$

2-parameters family of solutions $\Rightarrow$ 0 is a double eigenvalue for the linearized

2. Walls in infinite nanowires: Stability

Second difficulty: invariance by rotation-translation

$$\mathcal{L}r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Lr_1 \\ Lr_2 \end{pmatrix}$$

$$L = -\frac{\partial^2}{\partial x^2} + 2\mathrm{th}^2 x - 1$$

2. Walls in infinite nanowires: Stability

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- $L = l^\star \circ l$ where $l = \dfrac{\partial}{\partial x} + \mathrm{th}\, x \Rightarrow L \geq 0.$

2. Walls in infinite nanowires: Stability

Second difficulty: invariance by rotation-translation

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- $L = l^\star \circ l$ where $l = \dfrac{\partial}{\partial x} + \mathrm{th}\, x \Rightarrow L \geq 0.$
- $L(\dfrac{1}{\mathrm{ch}\, x}) = 0 \Rightarrow 0$ is the first eigenvalue of L

Ker $\mathcal{L}$=Vect$\left\{ \begin{pmatrix} \frac{1}{\mathrm{ch}\, x} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\mathrm{ch}\, x} \end{pmatrix} \right\}$

2. Walls in infinite nanowires: Stability

Second difficulty: invariance by rotation-translation

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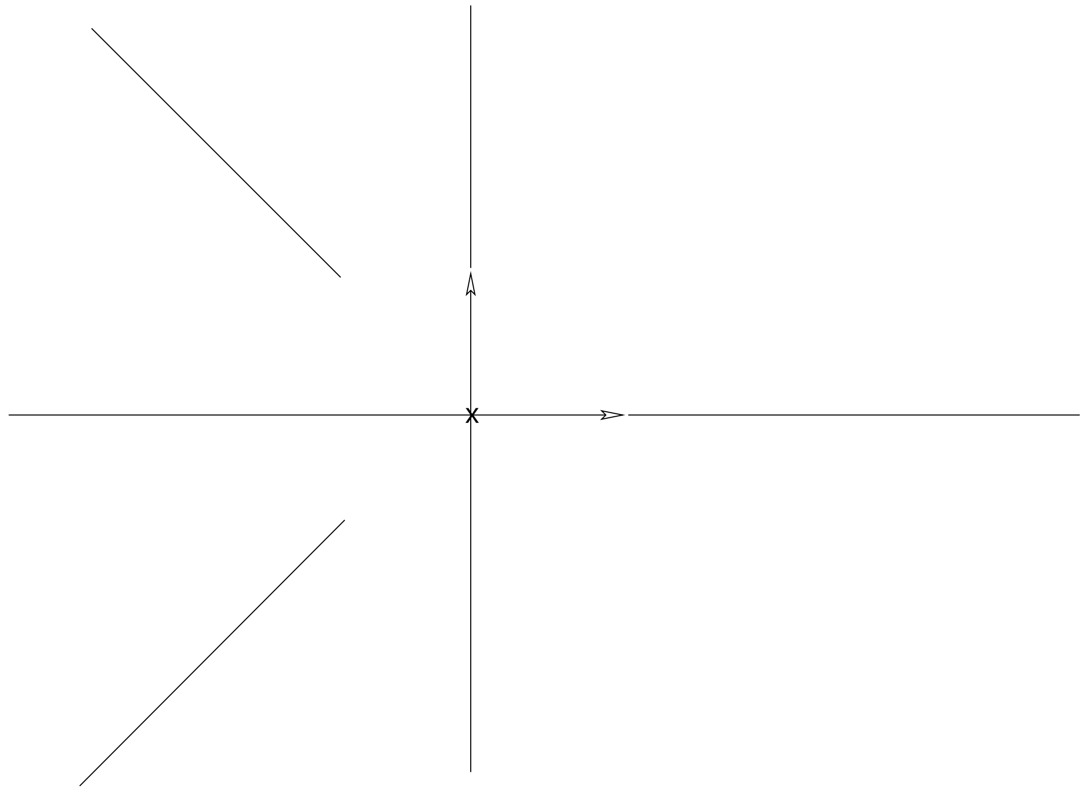
$$L = -\frac{\partial^2}{\partial x^2} + 2\mathrm{th}^2 x - 1$$

- $L = l^\star \circ l$ where $l = \dfrac{\partial}{\partial x} + \mathrm{th}\, x \Rightarrow L \geq 0.$
- $L(\dfrac{1}{\mathrm{ch}\, x}) = 0 \Rightarrow 0$ is the first eigenvalue of L
- Ess. Spec. $L = [1, +\infty[$

$l \circ l^\star = -\dfrac{\partial^2}{\partial x^2} + 1 \Rightarrow$ no other eigenvalues.

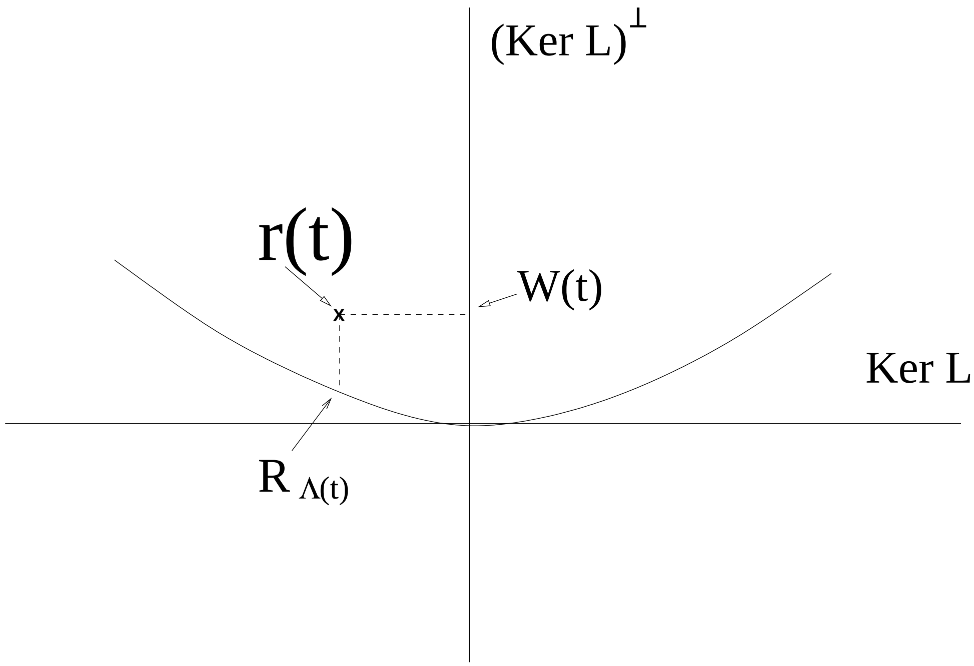
Second difficulty: invariance by rotation-translation

Spectrum of $\mathcal{L}$



2. Walls in infinite nanowires: Stability

Second difficulty: change of variables



$$r(t,x) = W(t,x) + R_{\Lambda(t)}(x)$$

- $\forall\ t, W(t,.) \in \mathcal{E} = (\text{Ker } \mathcal{L})^\perp$
- $\Lambda : \mathbb{R}_t^+ \longrightarrow \mathbb{R}_\theta \times \mathbb{R}_\sigma$

2. Walls in infinite nanowires: Stability

Second difficulty: change of variables

r solution to (2) $\Leftrightarrow$ (W, Λ) solution to (3)

$$\begin{aligned} \frac{\partial W}{\partial t} &= \mathcal{L}W + \mathcal{R}(\delta, x, \Lambda, W, \frac{\partial W}{\partial x}, \frac{\partial^2 W}{\partial x^2}) \\ \frac{d\Lambda}{dt} &= \mathcal{M}(\Lambda, W, \frac{\partial W}{\partial x}) \end{aligned} \tag{3}$$

2. Walls in infinite nanowires: Stability

Second difficulty: change of variables

If $W(t=0)$ and $\Lambda(t=0)$ are small then

1. $\|W(t)\|_{H^2}$ and Λ remain small,
2. $\|W\|_{H^2} \to 0$,
3. $\Lambda(t) \to \Lambda_\infty$.

Variational estimates for W

On $\mathcal{E}$, $\|LW\|_{L^2} \sim \|W\|_{H^2}$ and $\|L^{\frac{3}{2}}W\|_{L^2} \sim \|W\|_{H^3}$

Multiplying by $J^2\mathcal{L}^2W$

$$\frac{d}{dt}\|LW\|_{L^2}^2 + \|L^{\frac{3}{2}}W\|_{L^2}^2 \left(1 - \; K(|\Lambda| + |\delta| + \|W\|_{H^2})\right) \leq 0$$

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Multiplying by $J^2\mathcal{L}^2 W$

$$\frac{d}{dt}\|LW\|_{L^2}^2 + \|L^{\frac{3}{2}}W\|_{L^2}^2 \left(1 - \ K(|\Lambda| + |\delta| + \|W\|_{H^2})\right) \leq 0$$

While $|\Lambda| + |\delta| + \|W\|_{H^2} \leq \dfrac{1}{2K}$

$$\frac{d}{dt}\|LW\|_{L^2}^2 + \frac{1}{2}\|L^{\frac{3}{2}}W\|_{L^2}^2 \leq 0$$

So

$$\|LW(t)\|_{L^2}^2 \leq \|LW(0)\|_{L^2}^2 e^{-2\alpha t}$$

Estimate on Λ

$$\frac{d\Lambda}{dt} = \mathcal{M}(\Lambda, W, \frac{\partial W}{\partial x})$$

$$|\mathcal{M}(\Lambda, W, \frac{\partial W}{\partial x})| \leq C\,(|\Lambda| + \|W\|_{H^1})\,\|W\|_{H^1}$$

Integrating in time: while $|\Lambda| + |\delta| + \|W\|_{H^2} \leq \dfrac{1}{2K}$

$$|\Lambda(t)| \leq |\Lambda_0| + C\|W(0)\|_{H^2} e^{-\alpha t}$$

Conclusion

While $|\Lambda| + |\delta| + \|W\|_{H^2} \leq \dfrac{1}{2K}$

$$\|W(t)\|_{H^2} \leq C\|W_0\|_{H^2} e^{-\alpha t}$$

$$|\Lambda(t)| \leq |\Lambda_0| + C\|W_0\|_{H^2} e^{-\alpha t}$$

If $|\delta|$ is small, if $|\Lambda_0|$ and $\|W_0\|_{H^2}$ are small, $\|W(t)\|_{H^2}$ and Λ remain small.

Conclusion

While $|\Lambda| + |\delta| + \|W\|_{H^2} \leq \dfrac{1}{2K}$

$$\|W(t)\|_{H^2} \leq C\|W_0\|_{H^2} e^{-\alpha t}$$

$$|\Lambda(t)| \leq |\Lambda_0| + C\|W_0\|_{H^2} e^{-\alpha t}$$

If $|\delta|$ is small, if $|\Lambda_0|$ and $\|W_0\|_{H^2}$ are small, $\|W(t)\|_{H^2}$ and Λ remain small.

$$\frac{d\Lambda}{dt} = \mathcal{M}(\Lambda, W, \frac{\partial W}{\partial x})$$

$\Rightarrow \dfrac{d\Lambda}{dt}$ is integrable on $I\!R^+$, so Λ has a limit when $t \to +\infty$.

2. Walls in infinite nanowires: Stability

Conclusion

If $|\delta|$ is small, if $|\Lambda_0|$ and $\|W_0\|_{H^2}$ are small,

- $\|W(t)\|_{H^2}$ **and** Λ **remain small**
- $\|W(t)\|_{H^2} \to 0$
- $\Lambda(t) \to \Lambda_\infty$

2. Walls in infinite nanowires: Controlability

Can we control the position of the wall with the applied field ?

2. Walls in infinite nanowires: Controlability

$$u^{\delta,\theta,\sigma}(t,x) = R_{\delta t+\theta}(M_0(x+\delta t-\sigma))$$

We fix (δ_1, σ_1), et (δ_2, σ_2)

Theorem 2. Controlability. If δ_1 and δ_2 are small, for all $\varepsilon > 0$, there exists a final time T, there exists a control $\delta(.) \in L^\infty(\mathbb{R}^+)$ such that if u is the solution to (1) associated to δ with

$$\|u(0,.) - u^{\delta_1,\theta_1,\sigma_1}(0,.)\|_{H^2} \leq \varepsilon$$

then there exists θ_2 such that $\|u(T,.) - u^{\delta_2,\theta_2,\sigma_2}(T,.)\|_{H^2} \leq \varepsilon$.
In addition $\|u(t,.) - u^{\delta_2,\theta'_2,\sigma'_2}(t,.)\|_{H^2} \to 0$ when $t \to +\infty$ with $|\theta'_2 - \theta_2| + |\sigma'_2 - \sigma_2| \leq \varepsilon$.

G. Carbou, S. Labbé, E. Trélat, *Control of Travelling Walls in a Ferromagnetic Nanowire*, Discrete Contin. Dyn. Syst. Ser. S, **1** (2008), no. 1, 51–59.

2. Walls in infinite nanowires: Controlability

The control is given by

$$\delta(t) = \begin{cases} \delta_2 - \dfrac{\sigma_2 - \sigma_1}{T} \text{ for } 0 \leq t \leq T \\ \\ \delta_2 \text{ for } t \geq T \end{cases}$$

For the stability: $\delta(t)$ must remain small.

$\Rightarrow T$ must be great enough to have a sufficiently small control.

3. Finite Nanowires

$$\begin{cases} u : \mathbb{R}_t^+ \times [0, \frac{L}{\varepsilon}]_x \longrightarrow S^2 \\ \dfrac{\partial u}{\partial t} = -u \times h_\delta(u) - u \times (u \times h_\delta(u)) \\ h_\delta(u) = \dfrac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1 \\ \dfrac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = \dfrac{L}{\varepsilon} \end{cases}$$

Wall profiles

For sufficiently long wires, existence of wall steady state profiles

$$U_0 = \begin{pmatrix} \sin\theta_0 \\ \cos\theta_0 \\ 0 \end{pmatrix} \text{ where}$$

$$\theta_0'' + 2\sin\theta_0\cos\theta_0 = 0$$

$$\theta_0'(0) = \theta_0'(L/\varepsilon) = 0$$

Wall profiles

For sufficiently long wires, existence of wall steady state profiles

They are not stable

$$\partial_t r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{L}^1(r_1) \\ \mathcal{L}^2(r_2) \end{pmatrix}$$

$\mathcal{L}^2 = -\partial_{xx} + g_0$ where $g_0 = -(\partial_x \theta_0)^2 + \sin^2 \theta_0$.

$\mathcal{L}^2 \geq 0$ with 0 simple eigenvalue.

$\mathcal{L}^1 = \mathcal{L}^2 - K$ where $K > 0$.

Are these wall profiles stabilizable by the applied magnetic field ?

3. Finite Nanowires

Are these wall profiles stabilizable by the applied magnetic field ?

Description of the switching ?

Wall profiles

For sufficiently long wires, existence of wall steady state profiles

They are not stable

Stabilizable by the applied field:

$$\delta = -\frac{1}{L}\int_0^L u_1$$

Stability of constant states

Constant solutions

$u = e_1$ stable is and only if $\delta > -1$

$u = -e_1$ stable is and only if $\delta < 1$

Explanation of the hysteresis, but we don't describe the switching